M661: Mathematical Theory of Finite Elements. Mid-Term, March 11, 2007.

Question 1: Let V be a real Hilbert space equipped with the scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$. Let U be a nonempty, closed, and convex subset of V.

1. Let $f \in V$. Show that there is a unique u in V such that $\|f-u\|_V = \min_{v\in V}\|f-v\|_V$. (*Hint*: Consider a minimizing sequence and show that it is a Cauchy sequence.)
2. Show that u is the solution to the above minimization problem if and only if $(f-u, v-u)_V \le 0$ for all $v \in U$.
3. Let a be a continuous, symmetric, and V-coercive bilinear form. Let L be a continuous linear form on V. Set $J(v) = \frac{1}{2}a(v,v) - L(v)$. Show that there is a unique $u \in V$ such that $J(u) = \min_{v\in V} J(v)$ and that u is a minimizing solution if and only if $a(u, v-u) \ge L(v-u)$ for all $v \in U$.

Question 2: Let Ω be a smooth (say with Lipschitz boundary) bounded domain in $\mathbb{R}^2$, let $f \in L^2(\Omega)$, and let $\sigma \in \mathbb{R}$. Show that if $|\sigma| < 1$, the following problem is well-posed:

$$\begin{cases} \text{Seek } u \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega \big[\partial_x u \partial_x v + \sigma(\partial_x u \partial_y v + \partial_y u \partial_x v) + \partial_y u \partial_y v\big] = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega). \end{cases}$$

Question 3: Consider the domain Ω whose definition in polar coordinates is $\Omega = \{(r,\theta);\, 0 < r < 1,\, \frac{\pi}{\alpha} < \theta < 0\}$ with $\alpha < -\frac{1}{2}$. Let $\partial\Omega_1 = \{(r,\theta);\, r = 1,\, \frac{\pi}{\alpha} < \theta < 0\}$ and $\partial\Omega_2 = \partial\Omega\backslash\partial\Omega_1$. Consider the following problem: $-\Delta u = 0$ in Ω, $u = \sin(\alpha\theta)$ on $\partial\Omega_1$, and $u = 0$ on $\partial\Omega_2$.

1. Let $\varphi_1 = r^\alpha \sin(\alpha\theta)$ and $\varphi_2 = r^{-\alpha}\sin(\alpha\theta)$. Prove that φ_1 and φ_2 solve the above problem. (*Hint*: In polar coordinates, $\Delta\varphi = \frac{1}{r}\partial_r(r\partial_r\varphi) + \frac{1}{r^2}\partial_{\theta\theta}\varphi$.)
2. Prove that φ_1 and φ_2 are in $L^2(\Omega)$ if $-1 < \alpha < -\frac{1}{2}$.
3. Consider the following problem: Seek $u \in H^1(\Omega)$ such that $u = \sin(\alpha\theta)$ on $\partial\Omega_1$, $u = 0$ on $\partial\Omega_2$, and $\int_\Omega \nabla u \cdot \nabla v = 0$ for all $v \in H_0^1(\Omega)$. Prove that φ_2 solves this problem, but φ_1 does not. Comment.

Question 4: Let K be a rectangle and denote by $\{a_1, a_2, a_3, a_4\}$ the midpoints of its sides. Set $P = \mathbb{Q}_1$ and $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ such that $\sigma_i(p) = p(a_i)$, $1 \le i \le 4$. Show that $\{K, P, S\}$ is not a finite element, i.e., unisolvence does not hold.

Question 5: Let $\mathcal{I}_h^1$ be the one-dimensional $\mathbb{P}_1$ Lagrange interpolation operator defined on a mesh of the interval $[0,1]$.

1. Prove that for all h and $v \in \mathcal{C}^0(\overline{\Omega})$, $\|\mathcal{I}_h^1 v\|_{\mathcal{C}^0(\overline{\Omega})} \le \|v\|_{\mathcal{C}^0(\overline{\Omega})}$.
2. Prove that for all h and $v \in \mathcal{C}^1(\overline{\Omega})$, $\|v - \mathcal{I}_h^1 v\|_{\mathcal{C}^0(\overline{\Omega})} \le h\|v\|_{\mathcal{C}^1(\overline{\Omega})}$. (*Hint*: Use the mean-value theorem.)