1

Finite Element Interpolation

This chapter introduces the concept of finite elements along with the corresponding interpolation techniques. As an introductory example, we study how to interpolate functions in one dimension. Finite elements are then defined in arbitrary dimension, and numerous examples of scalar- and vector-valued finite elements are presented. Next, the concepts underlying the construction of meshes, approximation spaces, and interpolation operators are thoroughly investigated. The last sections of this chapter are devoted to the analysis of interpolation errors and inverse inequalities.

1.1 One-Dimensional Interpolation

The scope of this section is the interpolation theory of functions defined on an interval $]a, b[$. For an integer $k \geq 0$, $\mathbb{P}_k$ denotes the space of the polynomials in one variable, with real coefficients and of degree at most k.

1.1.1 The mesh

A mesh of $\Omega =]a, b[$ is an indexed collection of intervals with non-zero measure $\{I_i = [x_{1,i}, x_{2,i}]\}_{0\leq i\leq N}$ forming a partition of Ω, i.e.,

$$\overline{\Omega} = \bigcup_{i=0}^{N} I_i \qquad \text{and} \qquad \overset{\circ}{I}_i \cap \overset{\circ}{I}_j = \emptyset \qquad \text{for } \ i \neq j. \tag{1.1}$$

The simplest way to construct a mesh is to take $(N+2)$ points in $\overline{\Omega}$ such that

$$a = x_0 < x_1 < ... < x_N < x_{N+1} = b, \tag{1.2}$$

and to set $x_{1,i} = x_i$ and $x_{2,i} = x_{i+1}$ for $0 \leq i \leq N$. The points in the set $\{x_0, \dots, x_{N+1}\}$ are called the *vertices* of the mesh. The mesh may have a variable step size

$$h_i = x_{i+1} - x_i, \qquad 0 \le i \le N,$$

and we set

$$h = \max_{0 \le i \le N} h_i.$$

In the sequel, the intervals I_i are also called *elements* (or *cells*) and the mesh is denoted by $\mathcal{T}_h = \{I_i\}_{0 \le i \le N}$. The subscript h refers to the refinement level.

1.1.2 The $\mathbb{P}_1$ Lagrange finite element

Consider the vector space of continuous, piecewise linear functions

$$P_h^1 = \{v_h \in \mathcal{C}^0(\overline{\Omega}); \forall i \in \{0, \dots, N\}, v_{h|I_i} \in \mathbb{P}_1\}. \tag{1.3}$$

This space can be used in conjunction with Galerkin methods to approximate one-dimensional PDEs; see, e.g., Chapters 2 and 3. For this reason, P_h^1 is called an *approximation space*. Introduce the functions $\{\varphi_0, \dots, \varphi_{N+1}\}$ defined elementwise as follows: For $i \in \{0, \dots, N+1\}$,

$$\varphi_i(x) = \begin{cases} \frac{1}{h_{i-1}}(x - x_{i-1}) & \text{if } x \in I_{i-1}, \\ \frac{1}{h_i}(x_{i+1} - x) & \text{if } x \in I_i, \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

with obvious modifications if $i = 0$ or $N+1$. Clearly, $\varphi_i \in P_h^1$. These functions are often called "hat functions" in reference to the shape of their graph; see Figure 1.1.

Proposition 1.1. *The set $\{\varphi_0, \dots, \varphi_{N+1}\}$ is a basis for P_h^1.*

Proof. The proof relies on the fact that $\varphi_i(x_j) = \delta_{ij}$, the Kronecker symbol, for $0 \le i, j \le N+1$. Let $(\alpha_0, \dots, \alpha_{N+1})^T \in \mathbb{R}^{N+2}$ and assume that the continuous function $w = \sum_{i=0}^{N+1} \alpha_i \varphi_i$ vanishes identically in Ω. Then, for $0 \le i \le N+1$, $\alpha_i = w(x_i) = 0$; hence, the set $\{\varphi_0, \dots, \varphi_{N+1}\}$ is linearly independent. Furthermore, for all $v_h \in P_h^1$, it is clear that $v_h = \sum_{i=0}^{N+1} v_h(x_i)\varphi_i$ since, on each element I_i, the functions v_h and $\sum_{i=0}^{N+1} v_h(x_i)\varphi_i$ are affine and coincide at two points, namely x_i and x_{i+1}. $\square$

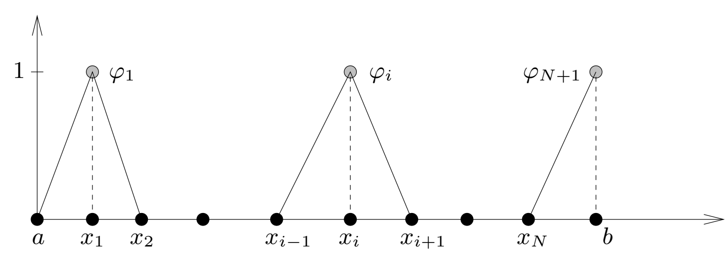


Fig. 1.1. One-dimensional hat functions.

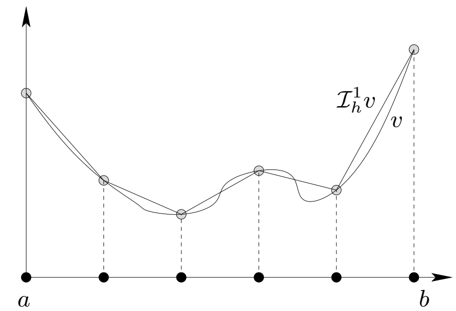


Fig. 1.2. Interpolation by continuous, piecewise linear functions.

Definition 1.2. *Choose a basis* $\{\gamma_0, \ldots, \gamma_{N+1}\}$ *for* $\mathcal{L}(P_h^1; \mathbb{R})$*; henceforth, the linear forms in this basis are called the* global degrees of freedom *in* P_h^1*. The functions in the dual basis are called the* global shape functions *in* P_h^1*.*

For $i \in \{0, \ldots, N+1\}$, choose the linear form

$$\gamma_i : \mathcal{C}^0(\overline{\Omega}) \ni v \longmapsto \gamma_i(v) = v(x_i) \in \mathbb{R}. \tag{1.5}$$

The proof of Proposition 1.1 shows that a function $v_h \in P_h^1$ is uniquely defined by the $(N+2)$-uplet $(v_h(x_i))_{0 \le i \le N+1}$. In other words, $\{\gamma_0, \ldots, \gamma_{N+1}\}$ is a basis for $\mathcal{L}(P_h^1; \mathbb{R})$. Choosing the linear forms (1.5) as the global degrees of freedom in P_h^1, the global shape functions are the functions $\{\varphi_0, \ldots, \varphi_{N+1}\}$ defined in (1.4) since $\gamma_i(\varphi_j) = \delta_{ij}$, $0 \le i, j \le N+1$.

Consider the so-called *interpolation operator*

$$\mathcal{I}_h^1 : \mathcal{C}^0(\overline{\Omega}) \ni v \longmapsto \sum_{i=0}^{N+1} \gamma_i(v) \varphi_i \in P_h^1. \tag{1.6}$$

For a function $v \in \mathcal{C}^0(\overline{\Omega})$, $\mathcal{I}_h^1 v$ is the unique continuous, piecewise linear function that takes the same value as v at all the mesh vertices; see Figure 1.2. The function $\mathcal{I}_h^1 v$ is called the *Lagrange interpolant* of v of degree 1. Note that the approximation space P_h^1 is the codomain of $\mathcal{I}_h^1$.

When approximating PDEs using finite elements, it is important to investigate the properties of $\mathcal{I}_h^1$ in Sobolev spaces; see Appendix B. In particular, recall that for an integer $m \ge 1$, $H^m(\Omega)$ denotes the space of square-integrable functions over Ω whose distributional derivatives up to order m are square-integrable. We use the following notation: $\|v\|_{0,\Omega} = \|v\|_{L^2(\Omega)}$, $|v|_{1,\Omega} = \|v'\|_{0,\Omega}$, $\|v\|_{1,\Omega} = (\|v\|_{0,\Omega}^2 + \|v'\|_{0,\Omega}^2)^{\frac{1}{2}}$, $|v|_{2,\Omega} = \|v''\|_{0,\Omega}$, etc.

Lemma 1.3. $P_h^1 \subset H^1(\Omega)$.

Proof. Let $v_h \in P_h^1$. Clearly, $v_h \in L^2(\Omega)$. Furthermore, owing to the continuity of v_h, its first-order distributional derivative is the piecewise constant function w_h such that

$$\forall I_i \in \mathcal{T}_h, \quad w_{h|I_i} = \frac{v_h(x_{i+1}) - v_h(x_i)}{h_i}. \tag{1.7}$$

Clearly, $w_h \in L^2(\Omega)$; hence, $v_h \in H^1(\Omega)$. □

Proposition 1.4. *$\mathcal{I}_h^1$ is a linear continuous mapping from $H^1(\Omega)$ to $H^1(\Omega)$, and $\|\mathcal{I}_h^1\|_{\mathcal{L}(H^1(\Omega);H^1(\Omega))}$ is uniformly bounded with respect to h.*

Proof. (1) In one dimension, a function in $H^1(\Omega)$ is continuous. Indeed, for $v \in H^1(\Omega)$ and $x, y \in \overline{\Omega}$,

$$|v(y) - v(x)| \leq \int_x^y |v'(s)| \, \mathrm{d}s \leq |y - x|^{\frac{1}{2}} |v|_{1,\Omega}, \tag{1.8}$$

owing to the Cauchy–Schwarz inequality (this can be justified rigorously by a density argument). Furthermore, taking x to be a point where $|v|$ reaches its minimum over $\overline{\Omega}$, the above inequality implies

$$\|v\|_{L^\infty(\Omega)} \leq |b - a|^{-\frac{1}{2}} \|v\|_{0,\Omega} + |b - a|^{\frac{1}{2}} |v|_{1,\Omega}, \tag{1.9}$$

since $|v(x)| \leq |b-a|^{-\frac{1}{2}} \|v\|_{0,\Omega}$. Therefore, $\mathcal{I}_h^1 v$ is well-defined for $v \in H^1(\Omega)$. Moreover, Lemma 1.3 implies $\mathcal{I}_h^1 v \in H^1(\Omega)$; hence, $\mathcal{I}_h^1$ maps $H^1(\Omega)$ to $H^1(\Omega)$.
(2) Let $I_i \in \mathcal{T}_h$ for $0 \leq i \leq N$. Owing to (1.7), $(\mathcal{I}_h^1 v)'_{|I_i} = h_i^{-1}(v(x_{i+1}) - v(x_i))$; hence, using (1.8) yields the estimate $|\mathcal{I}_h^1 v|_{1,I_i} \leq |v|_{1,I_i}$. Therefore, $|\mathcal{I}_h^1 v|_{1,\Omega} \leq |v|_{1,\Omega}$. Moreover, since $\|\mathcal{I}_h^1 v\|_{0,\Omega} \leq |b-a|^{\frac{1}{2}} \|\mathcal{I}_h^1 v\|_{L^\infty(\Omega)}$ and $\|\mathcal{I}_h^1 v\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)}$, we deduce from (1.9) that $\|\mathcal{I}_h^1 v\|_{0,\Omega} \leq c \, \|v\|_{1,\Omega}$ where c is independent of h (assuming h bounded). The conclusion follows readily. □

Proposition 1.5. *For all h and $v \in H^2(\Omega)$,*

$$\|v - \mathcal{I}_h^1 v\|_{0,\Omega} \leq h^2 |v|_{2,\Omega} \qquad \textit{and} \qquad |v - \mathcal{I}_h^1 v|_{1,\Omega} \leq h |v|_{2,\Omega}. \tag{1.10}$$

Proof. (1) Consider an interval $I_i \in \mathcal{T}_h$. Let $w \in H^1(I_i)$ be such that w vanishes at some point ξ in I_i. Then, owing to (1.8) we infer $\|w\|_{0,I_i} \leq h_i |w|_{1,I_i}$.
(2) Let $v \in H^2(\Omega)$, let $i \in \{0, \ldots, N\}$, and set $w_i = (v - \mathcal{I}_h^1 v)'_{|I_i}$. Note that $w_i \in H^1(I_i)$ and that w_i vanishes at some point ξ in I_i owing to the mean-value theorem. Applying the estimate derived in step 1 to w_i and using the fact that $(\mathcal{I}_h^1 v)''$ vanishes identically on I_i yields $|v - \mathcal{I}_h^1 v|_{1,I_i} \leq h_i |v|_{2,I_i}$. The second estimate in (1.10) is then obtained by summing over the mesh intervals. To prove the first estimate, observe that the result of step 1 can also be applied to $(v - \mathcal{I}_h^1 v)_{|I_i}$ yielding

$$\|v - \mathcal{I}_h^1 v\|_{0,I_i} \leq h_i |v - \mathcal{I}_h^1 v|_{1,I_i} \leq h_i^2 |v|_{2,I_i}.$$

Conclude by summing over the mesh intervals. □

Remark 1.6.

(i) The bound on the interpolation error involves second-order derivatives of v. This is reasonable since the larger the second derivative, the more the graph of v deviates from the piecewise linear interpolant.

(ii) If the function to be interpolated is in $H^1(\Omega)$ only, one can prove the following results:

$$\forall h,\ \|v-\mathcal{I}_h^1 v\|_{0,\Omega} \le h|v|_{1,\Omega} \qquad \text{and} \qquad \lim_{h\to 0} |v-\mathcal{I}_h^1 v|_{1,\Omega} = 0. \qquad \square$$

The proof of Proposition 1.5 shows that the operator $\mathcal{I}_h^1$ is endowed with *local* interpolation properties, i.e., the interpolation error is controlled elementwise before being controlled globally over Ω. This motivates the introduction of local interpolation operators. Let $I_i = [x_i, x_{i+1}] \in \mathcal{T}_h$ and let $\Sigma_i = \{\sigma_{i,0}, \sigma_{i,1}\}$ where $\sigma_{i,0}, \sigma_{i,1} \in \mathcal{L}(\mathbb{P}_1;\mathbb{R})$ are such that, for all $p \in \mathbb{P}_1$,

$$\sigma_{i,0}(p) = p(x_i) \qquad \text{and} \qquad \sigma_{i,1}(p) = p(x_{i+1}). \tag{1.11}$$

Note that Σ_i is a basis for $\mathcal{L}(\mathbb{P}_1;\mathbb{R})$. The triplet $\{I_i, \mathbb{P}_1, \Sigma_i\}$ is called a (one-dimensional) $\mathbb{P}_1$ *Lagrange finite element*, and the linear forms $\{\sigma_{i,0}, \sigma_{i,1}\}$ are the corresponding *local degrees of freedom*. The functions $\{\theta_{i,0}, \theta_{i,1}\}$ in the dual basis of Σ_i (i.e., $\sigma_{i,m}(\theta_{i,n}) = \delta_{mn}$ for $0 \le m, n \le 1$) are called the *local shape functions*. One readily verifies that

$$\theta_{i,0}(t) = 1 - \tfrac{t-x_i}{h_i} \qquad \text{and} \qquad \theta_{i,1}(t) = \tfrac{t-x_i}{h_i}. \tag{1.12}$$

Finally, introduce the family $\{\mathcal{I}_{I_i}^1\}_{I_i\in\mathcal{T}_h}$ of *local interpolation operators* such that, for $i \in \{0,\ldots,N\}$,

$$\mathcal{I}_{I_i}^1 : \mathcal{C}^0(I_i) \ni v \longmapsto \sum_{m=0}^{1} \sigma_{i,m}(v)\theta_{i,m}. \tag{1.13}$$

The proof of Propositions 1.4 and 1.5 can now be rewritten using the local interpolation operators $\mathcal{I}_{I_i}^1$. In particular, the key properties are, for $0 \le i \le N$ and $v \in H^2(I_i)$,

$$\|v-\mathcal{I}_{I_i}^1 v\|_{0,I_i} \le h_i^2 |v|_{2,I_i} \quad \text{and} \quad |v-\mathcal{I}_{I_i}^1 v|_{1,I_i} \le h_i |v|_{2,I_i}.$$

1.1.3 $\mathbb{P}_k$ Lagrange finite elements

The interpolation technique presented in §1.1.2 generalizes to higher-degree polynomials. Consider the mesh $\mathcal{T}_h = \{I_i\}_{0\le i\le N}$ introduced in §1.1.1. Let

$$P_h^k = \{v_h \in \mathcal{C}^0(\overline{\Omega});\, \forall i \in \{0,\ldots,N\},\, v_{h|I_i} \in \mathbb{P}_k\}. \tag{1.14}$$

To investigate the properties of the approximation space P_h^k and to construct an interpolation operator with codomain P_h^k, it is convenient to consider Lagrange polynomials. Recall the following:

Definition 1.7 (Lagrange polynomials). *Let $k \geq 1$ and let $\{s_0, \dots, s_k\}$ be $(k+1)$ distinct numbers. The Lagrange polynomials $\{\mathcal{L}_0^k, \dots, \mathcal{L}_k^k\}$ associated with the nodes $\{s_0, \dots, s_k\}$ are defined to be*

$$\mathcal{L}_m^k(t) = \frac{\prod_{l \neq m}(t - s_l)}{\prod_{l \neq m}(s_m - s_l)}, \qquad 0 \leq m \leq k. \tag{1.15}$$

The Lagrange polynomials satisfy the important property

$$\mathcal{L}_m^k(s_l) = \delta_{ml}, \quad 0 \leq m, l \leq k.$$

Figure 1.3 presents families of Lagrange polynomials with equi-distributed nodes in the reference interval $[0,1]$ for $k = 1$, 2, and 3.

For $i \in \{0, \dots, N\}$, introduce the *nodes* $\xi_{i,m} = x_i + \frac{m}{k} h_i$, $0 \leq m \leq k$, in the mesh interval I_i; see Figure 1.4. Let $\{\mathcal{L}_{i,0}^k, \dots, \mathcal{L}_{i,k}^k\}$ be the Lagrange polynomials associated with these nodes. For $j \in \{0, \dots, k(N+1)\}$ with $j = ki + m$ and $0 \leq m \leq k-1$, define the function φ_j elementwise as follows: For $1 \leq m \leq k-1$,

$$\varphi_{ki+m}(x) = \begin{cases} \mathcal{L}_{i,m}^k(x) & \text{if } x \in I_i, \\ 0 & \text{otherwise,} \end{cases}$$

and for $m = 0$,

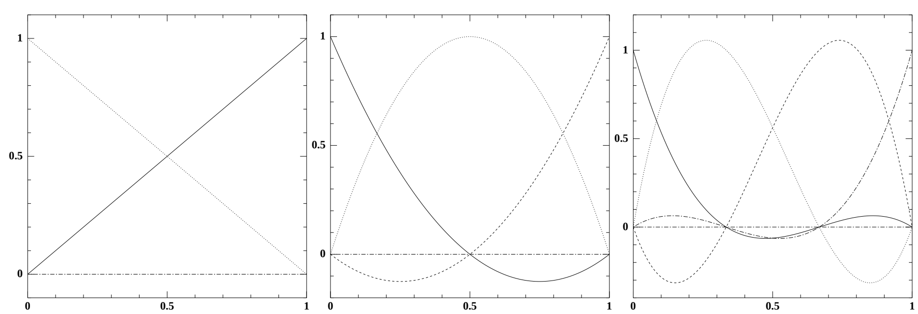


Fig. 1.3. Families of Lagrange polynomials with equi-distributed nodes in the reference interval $[0,1]$ and of degree $k = 1$ (left), 2 (center), and 3 (right).

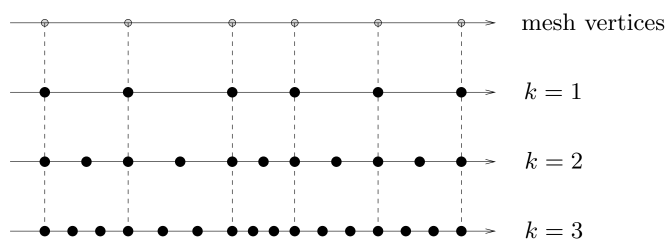


Fig. 1.4. Mesh vertices and nodes for $k = 1$, 2, and 3.

$$\varphi_{ki}(x) = \begin{cases} \mathcal{L}^k_{i-1,k}(x) & \text{if } x \in I_{i-1}, \\ \mathcal{L}^k_{i,0}(x) & \text{if } x \in I_i, \\ 0 & \text{otherwise,} \end{cases}$$

with obvious modifications if $i = 0$ or $N + 1$. The functions φ_j are illustrated in Figure 1.5 for $k = 2$. Note the difference between the support of the functions associated with mesh vertices (two adjacent intervals) and that of the functions associated with cell midpoints (one interval).

Lemma 1.8. $\varphi_j \in P^k_h$.

Proof. Let $j \in \{0, \ldots, k(N+1)\}$ with $j = ki + m$. If $1 \le m \le k-1$, $\varphi_j(x_i) = \varphi_j(x_{i+1}) = 0$; hence, $\varphi_j \in \mathcal{C}^0(\overline{\Omega})$. Moreover, the restrictions of φ_j to the mesh intervals are in $\mathbb{P}_k$ by construction. Therefore, $\varphi_j \in P^k_h$. Now, assume $m = 0$ (i.e., $j = ki$) and $0 < i < N + 1$. Clearly, φ_{ki} is continuous at x_i by construction and $\varphi_{ki}(x_{i-1}) = \varphi_{ki}(x_{i+1}) = 0$; hence, $\varphi_{ki} \in P^k_h$. The cases $i = 0$ and $i = N + 1$ are treated similarly. □

Introduce the set of nodes $\{a_j\}_{0 \le j \le k(N+1)}$ such that $a_j = \xi_{i,m}$ where $j = ik + m$. For $j \in \{0, \ldots, k(N+1)\}$, consider the linear form

$$\gamma_j : \mathcal{C}^0(\overline{\Omega}) \ni v \longmapsto \gamma_j(v) = v(a_j). \tag{1.16}$$

Proposition 1.9. $\{\varphi_0, \ldots, \varphi_{k(N+1)}\}$ *is a basis for* P^k_h*, and* $\{\gamma_0, \ldots, \gamma_{k(N+1)}\}$ *is a basis for* $\mathcal{L}(P^k_h; \mathbb{R})$.

Proof. Similar to that of Proposition 1.1 since $\gamma_j(\varphi_{j'}) = \delta_{jj'}$ for $0 \le j, j' \le k(N+1)$. □

The *global degrees of freedom* in P^k_h are chosen to be the $(k(N+1)+1)$ linear forms defined in (1.16); hence, the *global shape functions* in P^k_h are the functions $\{\varphi_0, \ldots, \varphi_{k(N+1)}\}$.

The main advantage of using high-degree polynomials is that smooth functions can be interpolated to high-order accuracy. Define the *interpolation operator* $\mathcal{I}^k_h$ to be

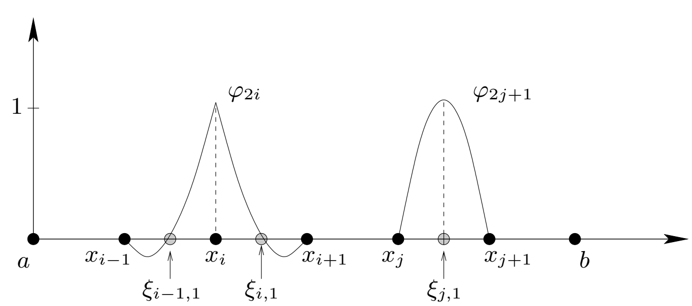


Fig. 1.5. Global shape functions in the approximation space P^2_h.

$$\mathcal{I}_h^k : \mathcal{C}^0(\overline{\Omega}) \ni v \longmapsto \sum_{j=0}^{k(N+1)} \gamma_j(v)\varphi_j \in P_h^k. \tag{1.17}$$

$\mathcal{I}_h^k v$ is called the *Lagrange interpolant* of v of degree k. Clearly, $\mathcal{I}_h^k$ is a linear operator, and $\mathcal{I}_h^k v$ is the unique function in P_h^k that takes the same value as v at all the mesh nodes. The approximation space P_h^k is the codomain of $\mathcal{I}_h^k$.

Lemma 1.10. $P_h^k \subset H^1(\Omega)$.

Proof. Similar to that of Lemma 1.3. □

To investigate the properties of $\mathcal{I}_h^k$, it is convenient to introduce a family of local interpolation operators. On $I_i = [x_i, x_{i+1}] \in \mathcal{T}_h$, choose the *local degrees of freedom* to be the $(k+1)$ linear forms $\{\sigma_{i,0}, \dots, \sigma_{i,k}\}$ defined as follows:

$$\sigma_{i,m} : \mathbb{P}_k \ni p \longmapsto \sigma_{i,m}(p) = p(\xi_{i,m}), \quad 0 \le m \le k. \tag{1.18}$$

The triplet $\{I_i, \mathbb{P}_k, \Sigma_i\}$ is called a (one-dimensional) $\mathbb{P}_k$ *Lagrange finite element*, and the points $\{\xi_{i,0}, \dots, \xi_{i,k}\}$ are called the *nodes* of the finite element. Clearly, the *local shape functions* $\{\theta_{i,0}, \dots, \theta_{i,k}\}$ are the $(k+1)$ Lagrange polynomials associated with the nodes $\{\xi_{i,0}, \dots, \xi_{i,k}\}$, i.e., $\theta_{i,m} = \mathcal{L}_{i,m}^k$ for $0 \le m \le k$. Finally, introduce the family $\{\mathcal{I}_{I_i}^k\}_{I_i \in \mathcal{T}_h}$ of *local interpolation operators* such that, for $i \in \{0, \dots, N\}$,

$$\mathcal{I}_{I_i}^k : \mathcal{C}^0(I_i) \ni v \longmapsto \sum_{m=0}^{k} \sigma_{i,m}(v)\theta_{i,m}, \tag{1.19}$$

i.e., for all $0 \le i \le N$ and $v \in \mathcal{C}^0(\overline{\Omega})$, $(\mathcal{I}_h^k v)_{|I_i} = \mathcal{I}_{I_i}^k(v_{|I_i})$.

Let us show that the family $\{\mathcal{I}_{I_i}^k\}_{I_i \in \mathcal{T}_h}$ can be generated from a single reference interpolation operator. Let $\widehat{K} = [0,1]$ be the *unit interval*, henceforth referred to as the *reference interval*. Set $\widehat{P} = \mathbb{P}_k$, and define the $(k+1)$ linear forms $\{\widehat{\sigma}_0, \dots, \widehat{\sigma}_k\}$ as follows:

$$\widehat{\sigma}_m : \mathbb{P}_k \ni \widehat{p} \longmapsto \widehat{\sigma}_m(\widehat{p}) = \widehat{p}(\widehat{\xi}_m), \quad 0 \le m \le k, \tag{1.20}$$

where $\widehat{\xi}_m = \frac{m}{k}$. Let $\{\widehat{\mathcal{L}}_0^k, \dots, \widehat{\mathcal{L}}_k^k\}$ be the Lagrange polynomials associated with the nodes $\{\widehat{\xi}_0, \dots, \widehat{\xi}_k\}$; see Figure 1.3. Set $\widehat{\theta}_m = \widehat{\mathcal{L}}_m^k$, $0 \le m \le k$, so that $\widehat{\sigma}_m(\widehat{\theta}_n) = \delta_{mn}$ for $0 \le m, n \le k$. Then, $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ is a $\mathbb{P}_k$ Lagrange finite element, and the corresponding interpolation operator is

$$\mathcal{I}_{\widehat{K}}^k : \mathcal{C}^0(\widehat{K}) \ni \widehat{v} \longmapsto \sum_{m=0}^{k} \widehat{\sigma}_m(\widehat{v})\widehat{\theta}_m.$$

$\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ is called the *reference finite element* and $\mathcal{I}_{\widehat{K}}^k$ the *reference interpolation operator*. For $i \in \{0, \dots, N\}$, consider the affine transformations

$$T_i : \widehat{K} \ni t \longmapsto x = x_i + th_i \in I_i. \tag{1.21}$$

Since $T_i(\widehat{K}) = I_i$, the mesh $\mathcal{T}_h$ can be constructed by applying the affine transformations T_i to the reference interval $\widehat{K}$. Moreover, owing to the fact that $T_i(\widehat{\xi}_m) = \xi_{i,m}$ for $0 \le m \le k$, it is clear that $\theta_{i,m} \circ T_i = \widehat{\theta}_m$ and $\sigma_{i,m}(v) = \widehat{\sigma}_m(v \circ T_i)$ for all $v \in \mathcal{C}^0(I_i)$. Hence, using

$$\begin{aligned}\mathcal{I}^k_{I_i}(v)(T_i(\widehat{x})) &= \sum_{m=0}^{k} \sigma_{i,m}(v)\theta_{i,m}(T_i(\widehat{x})) = \sum_{m=0}^{k} \sigma_{i,m}(v)\widehat{\theta}_m(\widehat{x}) = \\ &= \sum_{m=0}^{k} \widehat{\sigma}_m(v \circ T_i)\widehat{\theta}_m(\widehat{x}) = \mathcal{I}^k_{\widehat{K}}(v \circ T_i)(\widehat{x}),\end{aligned}$$

we infer

$$\forall v \in \mathcal{C}^0(I_i), \quad \mathcal{I}^k_{I_i}(v) \circ T_i = \mathcal{I}^k_{\widehat{K}}(v \circ T_i). \tag{1.22}$$

In other words, the family $\{\mathcal{I}^k_{I_i}\}_{I_i \in \mathcal{T}_h}$ is entirely generated by the transformations $\{T_i\}_{I_i \in \mathcal{T}_h}$ and the reference interpolation operator $\mathcal{I}^k_{\widehat{K}}$. The property (1.22) plays a key role when estimating the interpolation error; see the proof of Proposition 1.12 below.

Proposition 1.11. *$\mathcal{I}^k_h$ is a linear continuous mapping from $H^1(\Omega)$ to $H^1(\Omega)$, and $\|\mathcal{I}^k_h\|_{\mathcal{L}(H^1(\Omega);H^1(\Omega))}$ is uniformly bounded with respect to h.*

Proof. (1) To prove that $\mathcal{I}^k_h$ maps $H^1(\Omega)$ to $H^1(\Omega)$, use the argument of step 1 in the proof of Proposition 1.4.
(2) Let $v \in H^1(\Omega)$ and $I_i \in \mathcal{T}_h$. Since $\sum_{m=0}^{k} \theta'_{i,m} = 0$,

$$(\mathcal{I}^k_{I_i} v)' = \sum_{m=0}^{k} [v(\xi_{i,m}) - v(x_i)]\theta'_{i,m}.$$

Inequality (1.8) yields $|v(\xi_{i,m}) - v(x_i)| \le h_i^{\frac{1}{2}} |v|_{1,I_i}$ for $0 \le m \le k$. Furthermore, changing variables in the integral, it is clear that $|\theta_{i,m}|_{1,I_i} = h_i^{-\frac{1}{2}} |\widehat{\theta}_m|_{1,\widehat{K}}$. Set $c_k = \max_{0 \le m \le k} |\widehat{\theta}_m|_{1,\widehat{K}}$ and observe that this quantity is mesh-independent. A straightforward calculation yields

$$|\mathcal{I}^k_{I_i} v|_{1,I_i} \le (k+1)c_k |v|_{1,I_i},$$

showing that $|\mathcal{I}^k_h v|_{1,\Omega}$ is controlled by $|v|_{1,\Omega}$ uniformly with respect to h. In addition, since $\sum_{m=0}^{k} \theta_{i,m} = 1$,

$$\mathcal{I}^k_{I_i} v - v(x_i) = \sum_{m=0}^{k} [v(\xi_{i,m}) - v(x_i)]\theta_{i,m},$$

implying, for $x \in I_i$, $|\mathcal{I}_{I_i}^k v(x)| \leq \|v\|_{L^\infty(\Omega)} + (k+1) d_k h_i^{\frac{1}{2}} |v|_{1,I_i}$ with the mesh-independent constant $d_k = \max_{0 \leq m \leq k} \|\widehat{\theta}_m\|_{L^\infty(\widehat{K})}$. Then, using(1.9) yields $\|\mathcal{I}_h^k v\|_{L^\infty(\Omega)}$ is controlled by $\|v\|_{1,\Omega}$ uniformly with respect to h. To conclude, use the fact that $\|\mathcal{I}_h^k v\|_{0,\Omega} \leq |b-a|^{\frac{1}{2}} \|\mathcal{I}_h^k v\|_{L^\infty(\Omega)}$. $\square$

we

Proposition 1.12. *Let $0 \leq l \leq k$. Then, there exists c such that, for all h and $v \in H^{l+1}(\Omega)$,*

$$\|v - \mathcal{I}_h^k v\|_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} \leq c\, h^{l+1} |v|_{l+1,\Omega}, \tag{1.23}$$

and for $l \geq 1$,

$$\sum_{m=2}^{l+1} h^m \left(\sum_{i=0}^{N} |v - \mathcal{I}_h^k v|_{m,I_i}^2 \right)^{\frac{1}{2}} \leq c\, h^{l+1} |v|_{l+1,\Omega}. \tag{1.24}$$

Proof. Let $0 \leq l \leq k$ and $0 \leq m \leq l+1$. Let $v \in H^{l+1}(\Omega)$.
(1) Consider a mesh interval I_i. Set $\widehat{v} = v \circ T_i$. Then, use (1.22) and change variables in the integral to obtain

$$|v - \mathcal{I}_{I_i}^k v|_{m,I_i} = h_i^{-m+\frac{1}{2}} |\widehat{v} - \mathcal{I}_{\widehat{K}}^k \widehat{v}|_{m,\widehat{K}}.$$

Similarly, $|\widehat{v}|_{l+1,\widehat{K}} = h_i^{l+\frac{1}{2}} |v|_{l+1,I_i}$.
(2) Consider the linear mapping

$$\mathcal{F} : H^{l+1}(\widehat{K}) \ni \widehat{v} \longmapsto \widehat{v} - \mathcal{I}_{\widehat{K}}^k \widehat{v} \in H^m(\widehat{K}).$$

Note that $\mathcal{I}_{\widehat{K}}^k \widehat{v}$ is meaningful since in one dimension, $\widehat{v} \in H^{l+1}(\widehat{K})$ with $l \geq 0$ implies $\widehat{v} \in \mathcal{C}^0(\widehat{K})$. Moreover, $\mathcal{F}$ is continuous from $H^{l+1}(\widehat{K})$ to $H^m(\widehat{K})$. Indeed, one can easily adapt the proof of Proposition 1.11 to prove that $\mathcal{I}_{\widehat{K}}^k$ is continuous from $H^1(\widehat{K})$ to $H^s(\widehat{K})$ for all $s \geq 1$. Furthermore, it is clear that $\mathbb{P}_k$ is invariant under $\mathcal{F}$ since, for all $\widehat{p} \in \mathbb{P}_k$ with $\widehat{p} = \sum_{n=0}^k \alpha_n \widehat{\theta}_n$,

$$\mathcal{I}_{\widehat{K}} \widehat{p} = \sum_{m,n=0}^{k} \alpha_n \widehat{\sigma}_m(\widehat{\theta}_n) \widehat{\theta}_m = \sum_{m,n=0}^{k} \alpha_n \delta_{mn} \widehat{\theta}_m = \sum_{n=0}^{k} \alpha_n \widehat{\theta}_n = \widehat{p}.$$

(3) Since $l \leq k$, $\mathbb{P}_l$ is invariant under $\mathcal{F}$. As a result,

$$\begin{aligned} |\widehat{v} - \mathcal{I}_{\widehat{K}}^k \widehat{v}|_{m,\widehat{K}} &= |\mathcal{F}(\widehat{v})|_{m,\widehat{K}} = \inf_{\widehat{p} \in \mathbb{P}_l} |\mathcal{F}(\widehat{v} + \widehat{p})|_{m,\widehat{K}} \\ &\leq \|\mathcal{F}\|_{\mathcal{L}(H^{l+1}(\widehat{K}); H^m(\widehat{K}))} \inf_{\widehat{p} \in \mathbb{P}_l} \|\widehat{v} + \widehat{p}\|_{l+1,\widehat{K}} \\ &\leq c \inf_{\widehat{p} \in \mathbb{P}_l} \|\widehat{v} + \widehat{p}\|_{l+1,\widehat{K}} \leq c\, |\widehat{v}|_{l+1,\widehat{K}}, \end{aligned}$$

the last estimate resulting from the Deny–Lions Lemma; see Lemma B.67. The identities derived in step 1 yield

$$|v - \mathcal{I}_{I_i}^k v|_{m,I_i} = h_i^{-m+\frac{1}{2}} |\widehat{v} - \mathcal{I}_{\widehat{K}}^k \widehat{v}|_{m,\widehat{K}}$$
$$\leq c\, h_i^{-m+\frac{1}{2}} |\widehat{v}|_{l+1,\widehat{K}} \leq c\, h_i^{l+1-m} |v|_{l+1,I_i}.$$

(4) To derive the estimates (1.23) and (1.24), sum over the mesh intervals. When $m = 0$ or 1, global norms over Ω can be used since $P_h^k \subset H^1(\Omega)$ owing to Lemma 1.10. □

Remark 1.13.

(i) The proof of Proposition 1.12 shows that the interpolation properties of $\mathcal{I}_h^k$ are local.

(ii) If the function to be interpolated is smooth enough, say $v \in H^{k+1}(\Omega)$, the interpolation error is of optimal order. In particular, (1.23) yields

$$\forall h,\, \forall v \in H^{k+1}(\Omega), \quad \|v - \mathcal{I}_h^k v\|_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} \leq c\, h^{k+1} |v|_{k+1,\Omega}.$$

However, one should bear in mind that the order of the interpolation error may not be optimal if the function to be interpolated is not smooth. For instance, if $v \in H^s(\Omega)$ and $v \notin H^{s+1}(\Omega)$ with $s \geq 2$, considering polynomials of degree larger than $s - 1$ does not improve the interpolation error.

(iii) If the function to be interpolated is in $H^1(\Omega)$ only, one can still prove $\lim_{h\to 0} |v - \mathcal{I}_h^k v|_{1,\Omega} = 0$. To this end, use the density of $H^2(\Omega)$ in $H^1(\Omega)$ and (1.23); details are left as an exercise. □

1.1.4 Interpolation by discontinuous functions

Let

$$P_{\mathrm{d},h}^k = \{v_h \in L^1(\Omega); \forall i \in \{0, \dots, N\},\ v_{h|I_i} \in \mathbb{P}_k\}.$$

Since the restriction of a function $v_h \in P_{\mathrm{d},h}^k$ to an interval I_i can be chosen independently of its restriction to the other intervals, $P_{\mathrm{d},h}^k$ is a vector space of dimension $(k + 1) \times (N + 1)$. However, instead of taking the Lagrange polynomials as local shape functions, it is often more convenient to consider the Legendre polynomials or modifications thereof based on the concept of hierarchical bases; see §1.1.5. Let $\widehat{K} = [0, 1]$ be the reference interval.

Definition 1.14 (Legendre polynomials). *The Legendre polynomials on the reference interval* $[0, 1]$ *are defined to be* $\widehat{\mathcal{E}}_k(t) = \frac{1}{k!} \frac{d^k}{dt^k}(t^2 - t)^k$ *for* $k \geq 0$.

The Legendre polynomial $\widehat{\mathcal{E}}_k$ is of degree k, $\widehat{\mathcal{E}}_k(0) = (-1)^k$, $\widehat{\mathcal{E}}_k(1) = 1$, and its k roots are in $\widehat{K}$. The roots of the Legendre polynomials are called *Gauß–Legendre points* and play an important role in the design of quadratures; see §8.1. The first four Legendre polynomials are (see Figure 1.6)

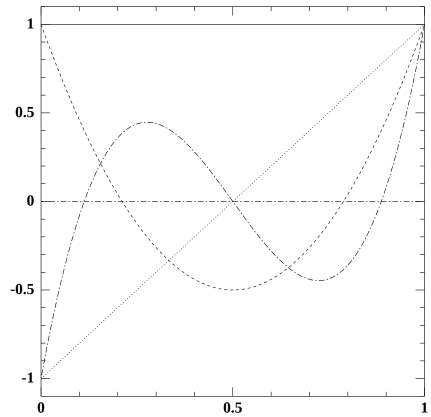


Fig. 1.6. Legendre polynomials of degree at most 3 on the reference interval $[0,1]$.

$$\widehat{\mathcal{E}}_0(t)=1, \qquad \widehat{\mathcal{E}}_1(t)=2t-1,$$
$$\widehat{\mathcal{E}}_2(t)=6t^2-6t+1, \qquad \widehat{\mathcal{E}}_3(t)=20t^3-30t^2+12t-1.$$

In the literature, the Legendre polynomials are sometimes defined using the reference interval $[-1,+1]$. Up to rescaling, both definitions are equivalent. In the context of finite elements, an important property of Legendre polynomials is that

$$\int_0^1 \widehat{\mathcal{E}}_m(t)\widehat{\mathcal{E}}_n(t)\,\mathrm{d}t = \tfrac{1}{2m+1}\delta_{mn}. \tag{1.25}$$

Introduce the functions $\{\varphi_{i,m}\}_{0\le i\le N, 0\le m\le k}$ such that $\varphi_{i,m|I_j}=\delta_{ij}\ \widehat{\mathcal{E}}_m\circ T_i^{-1}$ where the geometric transformation T_i is defined in (1.21). Clearly, $\{\varphi_{i,m}\}_{0\le i\le N, 0\le m\le k}$ is a basis for $P^k_{\mathrm{d},h}$. The corresponding degrees of freedom are the linear forms $\gamma_{i,m}$, $0\le i\le N$ and $0\le m\le k$, such that

$$\gamma_{i,m}: L^1(\Omega)\ni v \longmapsto \gamma_{i,m}(v)=\tfrac{2m+1}{h_i}\int_{I_i} v(x)\,\widehat{\mathcal{E}}_m\circ T_i^{-1}(x)\,\mathrm{d}x,$$

since, for $0\le i,i'\le N$ and $0\le m,m'\le k$,

$$\begin{aligned}\gamma_{i,m}(\varphi_{i',m'}) &= \tfrac{2m+1}{h_i}\int_{I_i}\varphi_{i',m'}(x)\,\widehat{\mathcal{E}}_m\circ T_i^{-1}(x)\,\mathrm{d}x\\ &= (2m+1)\delta_{ii'}\delta_{mm'}\int_{\widehat{K}}\widehat{\mathcal{E}}_m(t)^2\,\mathrm{d}t=\delta_{ii'}\delta_{mm'}.\end{aligned}$$

Define the *interpolation operator* $\mathcal{I}^k_{\mathrm{d},h}$ by

$$\mathcal{I}^k_{\mathrm{d},h}: L^1(\Omega)\ni v \longmapsto \sum_{i=0}^{N}\sum_{m=0}^{k}\gamma_{i,m}(v)\varphi_{i,m}\in P^k_{\mathrm{d},h}. \tag{1.26}$$

For instance, $\mathcal{I}^0_{\mathrm{d},h}v$ is the unique piecewise constant function that takes the same mean value as v over the mesh intervals.

Let $I_i = [x_i, x_{i+1}] \in \mathcal{T}_h$ and choose for the local degrees of freedom in $\mathbb{P}_k$ the set $\Sigma_i = \{\gamma_{i,m}\}_{0\le m\le k}$. The triplet $\{I_i, \mathbb{P}_k, \Sigma_i\}$ is often called a *modal finite element*; see §1.1.5 for further insight. The local shape functions are $\theta_{i,m} = \widehat{\mathcal{E}}_m \circ T_i^{-1}$. Introduce the family $\{\mathcal{I}^k_{\mathrm{d},I_i}\}_{I_i\in\mathcal{T}_h}$ of *local interpolation operators* such that, for $0 \le i \le N$,

$$\mathcal{I}^k_{\mathrm{d},I_i} : L^1(I_i) \ni v \longmapsto \sum_{m=0}^{k} \sigma_{i,m}(v)\theta_{i,m}. \tag{1.27}$$

Then, it is clear that, for all $v \in L^1(\Omega)$, $(\mathcal{I}^k_{\mathrm{d},h}v)_{|I_i} = \mathcal{I}^k_{\mathrm{d},I_i}(v_{|I_i})$. Using the family $\{\mathcal{I}^k_{\mathrm{d},I_i}\}_{I_i\in\mathcal{T}_h}$, one easily verifies the following results:

Proposition 1.15. *$\mathcal{I}^k_{\mathrm{d},h}$ is a linear continuous mapping from $L^1(\Omega)$ to $L^1(\Omega)$, and $\|\mathcal{I}^k_{\mathrm{d},h}\|_{\mathcal{L}(L^1(\Omega);L^1(\Omega))}$ is uniformly bounded with respect to h.*

Proposition 1.16. *Let $k \ge 0$ and let $0 \le l \le k$. Then, there exists c such that, for all h and $v \in H^{l+1}(\Omega)$,*

$$\|v - \mathcal{I}^k_{\mathrm{d},h}v\|_{0,\Omega} + \sum_{m=1}^{l+1} h^m \left(\sum_{i=0}^{N} |v - \mathcal{I}^k_{\mathrm{d},h}v|^2_{m,I_i}\right)^{\frac{1}{2}} \le c\, h^{l+1}|v|_{l+1,\Omega}.$$

Proof. Use steps 1, 2, and 3 in the proof of Proposition 1.12. □

Example 1.17. Taking $k = l = 0$ in Proposition 1.16 yields, for all h and $v \in H^1(\Omega)$, $\|v - \mathcal{I}^0_{\mathrm{d},h}v\|_{0,\Omega} \le c\, h|v|_{1,\Omega}$. □

1.1.5 Hierarchical polynomial bases

Although the emphasis in this book is set on h-type finite element methods for which convergence is achieved by refining the mesh, it is also possible to consider p-type finite element methods for which convergence is achieved by increasing the polynomial degree of the interpolation in every element. The hp-type finite element method is a combination of these two strategies. The idea that the p version of the finite element method can be as efficient as the h version is rooted in a series of papers by Babuška et al. [BaS81, BaD81].

When working with high-degree polynomials, it is important to select carefully the polynomial basis. The material presented herein is set at an introductory level; see, e.g., [KaS99^b, pp. 31–59]. The following definition plays an important role in the construction of polynomial bases:

Definition 1.18 (Hierarchical modal basis). *A family $\{\mathcal{B}_k\}_{k\ge0}$, where $\mathcal{B}_k$ is a set of polynomials, is said to be a* hierarchical modal basis *if, for all $k \ge 0$:*

(i) *$\mathcal{B}_k$ is a basis for $\mathbb{P}_k$.*

(ii) $\mathcal{B}_k \subset \mathcal{B}_{k+1}$.

Example 1.19. The simplest example of hierarchical modal basis is $\mathcal{B}_k = \{1, x, \dots, x^k\}$. □

So far, the local shape functions $\{\widehat{\theta}_0, \dots, \widehat{\theta}_k\}$ we have used are the Lagrange polynomials $\{\widehat{\mathcal{L}}_0^k, \dots, \widehat{\mathcal{L}}_k^k\}$ or the Legendre polynomials $\{\widehat{\mathcal{E}}_0, \dots, \widehat{\mathcal{E}}_k\}$. Clearly, the Legendre polynomial basis is a *hierarchical modal basis*. This is not the case for the Lagrange polynomial basis, which instead has the remarkable property that $\widehat{\mathcal{L}}_l^k(\widehat{\xi}_{l'}) = \delta_{ll'}$ at the associated nodes $\{\widehat{\xi}_0, \dots, \widehat{\xi}_k\}$. Because of this property, the Lagrange polynomial basis is said to be a *nodal basis*.

A first important criterion to select a high-degree polynomial basis is that the basis is orthogonal or nearly orthogonal with respect to an appropriate inner product. Let $\widehat{K} = [0, 1]$ be the reference interval and define the matrix $\mathcal{M}_{\widehat{K}}$ of order $k+1$ with entries

$$\forall m, n \in \{0, \dots, k\}, \quad \mathcal{M}_{\widehat{K}, mn} = \int_{\widehat{K}} \widehat{\theta}_m(t) \widehat{\theta}_n(t) \, \mathrm{d}t. \tag{1.28}$$

The matrix $\mathcal{M}_{\widehat{K}}$ is symmetric positive definite and is called the *elemental mass matrix*. The high-degree polynomial basis can be constructed so that $M_{\widehat{K}}$ is diagonal or "almost" diagonal. Define the condition number of $\mathcal{M}_{\widehat{K}}$ to be the ratio between its largest and smallest eigenvalue; see §9.1. Instead of diagonality, an alternative criterion to select a polynomial basis can be that the condition number of $\mathcal{M}_{\widehat{K}}$ does not increase "too much" as k grows; see Remark 1.20(i).

A second important criterion is that interface conditions between adjacent mesh elements can be imposed easily. For instance, imposing continuity at the interfaces ensures that the codomain of the global interpolation operator is in $H^1(\Omega)$; see, e.g., Lemmas 1.3 and 1.10.

Remark 1.20.

(i) The conditioning of the elemental mass matrix has important consequences on computational efficiency. For instance, in time-dependent problems discretized with explicit time-marching algorithms, this matrix has to be inverted at each time step; see, e.g.,(6.27). Furthermore, for time-dependent advection problems, explicit time step restrictions are less severe when the elemental mass matrix is well-conditioned; see [KaS99^b, p. 187] and also Exercises 6.7 and 6.9.

(ii) Instead of the elemental mass matrix, one can also consider the *elemental stiffness matrix* $\mathcal{A}_{\widehat{K}}$ defined by

$$\forall m, n \in \{0, \dots, k\}, \quad \mathcal{A}_{\widehat{K}, mn} = \int_{\widehat{K}} \tfrac{d}{dt} \widehat{\theta}_m(t) \tfrac{d}{dt} \widehat{\theta}_n(t) \, \mathrm{d}t.$$

This matrix, which is symmetric and positive, arises when approximating the Laplace equation; see §3.1. The high-degree polynomial basis can then be constructed so that $\mathcal{A}_{\widehat{K}}$ remains relatively well-conditioned. □

The Legendre polynomial basis satisfies the first criterion above. Owing to (1.25), the mass matrix is diagonal and its condition number is $(2k+1)$. However, Legendre polynomials do not vanish at the boundary of $\widehat{K}$, making it cumbersome to enforce $\mathcal{C}^0$-continuity between adjacent mesh intervals. On the other hand, the Lagrange polynomial basis satisfies the $\mathcal{C}^0$-continuity criterion provided the nodal points contain the interval endpoints, but the mass matrix is dense and its condition number explodes exponentially with k; see [OlD95] for a proof and [KaS99^b, p. 44] for an illustration. We now discuss appropriate modifications of the above bases designed to better fulfill the above criteria.

Modal ($\mathcal{C}^0$-continuous) basis. We first define the Jacobi polynomials.

Definition 1.21 (Jacobi polynomials). *Let $\alpha > -1$ and $\beta > -1$. The Jacobi polynomials $\{\mathcal{J}_k^{\alpha,\beta}\}_{k\geq 0}$ are defined by*

$$\mathcal{J}_k^{\alpha,\beta}(t) = \tfrac{(-1)^k}{k!} 2^{-\alpha-\beta}(1-t)^{-\alpha} t^{-\beta} \tfrac{d^k}{dt^k}\left((1-t)^{\alpha+k} t^{\beta+k}\right). \tag{1.29}$$

The Jacobi polynomials satisfy the important orthogonality property

$$\int_{\widehat{K}} (1-t)^\alpha t^\beta \mathcal{J}_m^{\alpha,\beta}(t) \mathcal{J}_n^{\alpha,\beta}(t)\,\mathrm{d}t = c_{m,\alpha,\beta}\delta_{mn}, \tag{1.30}$$

with constant $c_{m,\alpha,\beta} = \frac{1}{2m+\alpha+\beta+1}\frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!\Gamma(m+\alpha+\beta+1)}$. The first Jacobi polynomials for $\alpha = \beta = 1$ are $\mathcal{J}_0^{1,1}(t) = 1$, $\mathcal{J}_1^{1,1}(t) = 4t-2$, and $\mathcal{J}_2^{1,1}(t) = 15t^2 - 15t + 3$. Note that the Legendre polynomials introduced in Definition 1.14 are Jacobi polynomials with parameters $\alpha = \beta = 0$. For more details on Jacobi polynomials, see [AbS72, Chap. 22] and [KaS99^b, p. 350].

The modal ($\mathcal{C}^0$-continuous) basis is the set of functions $\{\widehat{\theta}_0, \ldots, \widehat{\theta}_k\}$ such that

$$\widehat{\theta}_l(t) = \begin{cases} 1-t & \text{if } l = 0, \\ (1-t)t\,\mathcal{J}_{l-1}^{1,1}(t) & \text{if } 0 < l < k, \\ t & \text{if } l = k. \end{cases} \tag{1.31}$$

This basis possesses several attractive features:

(i) It is a hierarchical modal basis according to Definition 1.18.
(ii) $\mathcal{C}^0$-continuity at element endpoints can be easily enforced since only the first and last basis functions do not vanish at the endpoints.
(iii) Owing to the use of Jacobi polynomials with parameters $\alpha = \beta = 1$, the elemental mass matrix $\mathcal{M}_{\widehat{K}}$ is such that $\mathcal{M}_{\widehat{K},mn} = 0$ for $|m-n| > 2$ and $0 \leq m, n \leq k$, unless $m = k$ and $n \leq 2$ or $n = k$ and $m \leq 2$. Furthermore, this matrix remains relatively well-conditioned. A precise result in arbitrary dimension d using tensor products of modal hierarchical bases is that the condition number of the elemental mass matrix (resp., stiffness matrix) is equivalent to 4^{kd} (resp., $4^{k(d-1)}$) uniformly in k; see [HuG98].

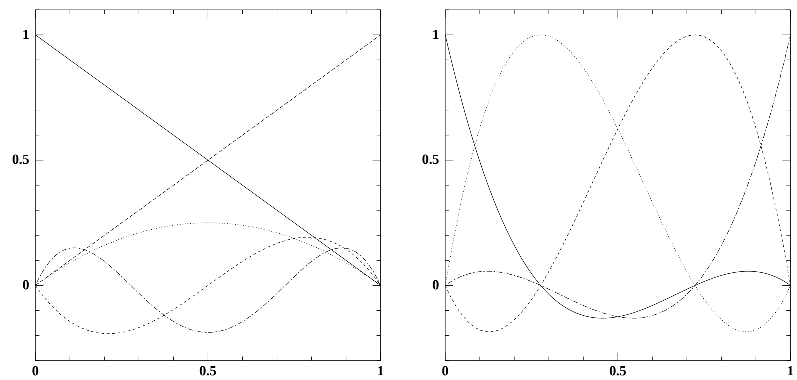


Fig. 1.7. Left: Modal ($\mathcal{C}^0$-continuous) basis functions of degree at most 4 on the reference interval $[0, 1]$. Right: Nodal ($\mathcal{C}^0$-continuous) basis functions of degree at most 3 on the same interval.

The modal ($\mathcal{C}^0$-continuous) basis functions are shown in the left panel of Figure 1.7 for $k = 5$.

Remark 1.22. Note that in the present case the degrees of freedom have no evident definition. It is more natural to define directly the local shape functions without resorting to the notion of degrees of freedom. □

Nodal ($\mathcal{C}^0$-continuous) basis. Nodal basis functions are interesting in the context of quadratures; see §8.1 for an introduction to these techniques. The principle of quadratures is to approximate the integral of a function over $\widehat{K}$ by a linear combination of the values it takes at $(k + 1)$ points in $\widehat{K}$, say $\{\widehat{\xi}_0, \dots, \widehat{\xi}_k\}$, in the form

$$\int_{\widehat{K}} f(t)\,\mathrm{d}t \simeq \sum_{l=0}^{k} \widehat{\omega}_l f(\widehat{\xi}_l). \tag{1.32}$$

The points $\{\widehat{\xi}_0, \dots, \widehat{\xi}_k\}$ are called the *quadrature nodes* and the numbers $\{\widehat{\omega}_0, \dots, \widehat{\omega}_k\}$ the *quadrature weights*. For $k \geq 2$, the *Gauß–Lobatto* quadrature nodes are defined to be the two endpoints of $\widehat{K}$ and the $(k - 1)$ roots of $\widehat{\mathcal{E}}_k'$. The resulting quadrature rule is exact for polynomials up to degree $2k-1$.

Define the local degrees of freedom $\{\widehat{\sigma}_0, \dots, \widehat{\sigma}_k\}$ such that, for $0 \leq i \leq k$,

$$\widehat{\sigma}_i : \mathbb{P}_k \ni \widehat{p} \longmapsto \widehat{\sigma}_i(\widehat{p}) = p(\widehat{\xi}_i) \in \mathbb{R}.$$

Then, the local shape functions $\{\widehat{\theta}_0, \dots, \widehat{\theta}_k\}$ are the Lagrange polynomials associated with the nodes $\{\widehat{\xi}_0, \dots, \widehat{\xi}_k\}$. Using standard induction relations on the Legendre polynomials, it is possible to show that the local shape functions $\{\widehat{\theta}_0, \dots, \widehat{\theta}_k\}$ are given by

$$\forall m \in \{0\dots,k\}, \quad \widehat{\theta}_m(t) = \frac{(t-1)t\widehat{\mathcal{E}}_k'(t)}{k(k+1)\widehat{\mathcal{E}}_k(\widehat{\xi}_m)(t-\widehat{\xi}_m)}. \tag{1.33}$$

These functions are shown in the right panel of Figure 1.7 for $k = 4$. Although these nodal basis functions are not hierarchical, they present attractive features in the context of *spectral element methods*; see [KaS99^b, p. 51] and [Pat84] for more details. If the quadrature (1.32) is used to evaluate $\mathcal{M}_{\widehat{K}}$ in (1.28), the elemental mass matrix becomes diagonal, and each diagonal entry is equal to the row-wise sum of the entries of the exact elemental matrix. Summing row-wise the entries of the mass matrix and using the result as diagonal entries is often referred to as *lumping*.

1.2 Finite Elements: Definitions and Examples

The purpose of this section is to give a general definition of finite elements and local interpolation operators. Numerous two- and three-dimensional examples are listed.

1.2.1 Main definitions

Following Ciarlet, a finite element is defined as a triplet $\{K, P, \Sigma\}$; see, e.g., [Cia91, p. 93].

Definition 1.23. *A finite element consists of a triplet $\{K, P, \Sigma\}$ where:*

(i) *K is a compact, connected, Lipschitz subset of $\mathbb{R}^d$ with non-empty interior.*
(ii) *P is a vector space of functions $p : K \to \mathbb{R}^m$ for some positive integer m (typically $m = 1$ or d).*
(iii) *Σ is a set of n_{sh} linear forms $\{\sigma_1, \dots, \sigma_{n_{\mathrm{sh}}}\}$ acting on the elements of P, and such that the linear mapping*

$$P \ni p \longmapsto \big(\sigma_1(p), \dots, \sigma_{n_{\mathrm{sh}}}(p)\big) \in \mathbb{R}^{n_{\mathrm{sh}}}, \tag{1.34}$$

is bijective, i.e., Σ is a basis for $\mathcal{L}(P;\mathbb{R})$. The linear forms $\{\sigma_1, \dots, \sigma_{n_{\mathrm{sh}}}\}$ are called the local degrees of freedom.

Proposition 1.24. *There exists a basis $\{\theta_1, \dots, \theta_{n_{\mathrm{sh}}}\}$ in P such that*

$$\sigma_i(\theta_j) = \delta_{ij}, \qquad 1 \le i, j \le n_{\mathrm{sh}}.$$

Proof. Direct consequence of the bijectivity of the mapping (1.34). □

Definition 1.25. $\{\theta_1, \dots, \theta_{n_{\mathrm{sh}}}\}$ *are called the* local shape functions.

Remark 1.26. Condition (iii) in Definition 1.23 amounts to proving that

$$\forall(\alpha_1,\dots,\alpha_{n_{\mathrm{sh}}})\in\mathbb{R}^{n_{\mathrm{sh}}},\ \exists! p\in P,\quad \sigma_i(p)=\alpha_i\ \text{ for } 1\le i\le n_{\mathrm{sh}},$$

which, in turn, is equivalent to

$$\begin{cases} \dim P = \operatorname{card}\Sigma = n_{\mathrm{sh}}, \\ \forall p\in P,\ \ (\sigma_i(p)=0,\ 1\le i\le n_{\mathrm{sh}}) \implies (p=0). \end{cases}$$

This property is usually referred to as *unisolvence*. In the literature, the bijectivity of the mapping (1.34) is sometimes not included in the definition and, if this property holds, the finite element is said to be *unisolvent*. □

Definition 1.27 (Lagrange finite element). *Let $\{K,P,\Sigma\}$ be a finite element. If there is a set of points $\{a_1,\dots,a_{n_{\mathrm{sh}}}\}$ in K such that, for all $p\in P$, $\sigma_i(p)=p(a_i)$, $1\le i\le n_{\mathrm{sh}}$, $\{K,P,\Sigma\}$ is called a* Lagrange finite element. *The points $\{a_1,\dots,a_{n_{\mathrm{sh}}}\}$ are called the* nodes *of the finite element, and the local shape functions $\{\theta_1,\dots,\theta_{n_{\mathrm{sh}}}\}$ (which are such that $\theta_i(a_j)=\delta_{ij}$ for $1\le i,j\le n_{\mathrm{sh}}$) are called the* nodal basis *of P.*

Example 1.28. See §1.1.2 and §1.1.3 for one-dimensional examples of Lagrange finite elements. □

Remark 1.29. In the literature, Lagrange finite elements as defined above are also called *nodal finite elements*. □

1.2.2 Local interpolation operator

Let $\{K,P,\Sigma\}$ be a finite element. Assume that there exists a normed vector space $V(K)$ of functions $v:K\to\mathbb{R}^m$, such that:

(i) $P\subset V(K)$.
(ii) The linear forms $\{\sigma_1,\dots,\sigma_{n_{\mathrm{sh}}}\}$ can be extended to $V(K)'$.

Then, the *local interpolation operator* $\mathcal{I}_K$ can be defined as follows:

$$\mathcal{I}_K : V(K)\ni v \longmapsto \sum_{i=1}^{n_{\mathrm{sh}}}\sigma_i(v)\theta_i\in P. \tag{1.35}$$

$V(K)$ is the domain of $\mathcal{I}_K$ and P is its codomain. Note that the term "interpolation" is used in a broad sense since $\mathcal{I}_K v$ is not necessarily defined by matching point values of v.

Proposition 1.30. *P is invariant under $\mathcal{I}_K$, i.e., $\forall p\in P$, $\mathcal{I}_K p=p$.*

Proof. Letting $p=\sum_{j=1}^{n_{\mathrm{sh}}}\alpha_j\theta_j$ yields $\mathcal{I}_K p=\sum_{i,j=1}^{n_{\mathrm{sh}}}\alpha_j\sigma_i(\theta_j)\theta_i=p$. □

Example 1.31.

(i) For Lagrange finite elements, one may choose $V(K) = [\mathcal{C}^0(K)]^m$ or $V(K) = [H^s(K)]^m$ with $s > \frac{d}{2}$. The *local Lagrange interpolation operator* is defined as follows:

$$\mathcal{I}_K : V(K) \ni v \longmapsto \mathcal{I}_K v = \sum_{i=1}^{n_{\mathrm{sh}}} v(a_i)\theta_i, \tag{1.36}$$

i.e., the Lagrange interpolant is constructed by matching the point values at the Lagrange nodes.

(ii) For the modal finite elements discussed in §1.1.4, an admissible choice is $V(K) = L^1(K)$. □

Remark 1.32. It may seem more appropriate to define a finite element as a quadruplet $\{K, P, \Sigma, V(K)\}$, where the triplet $\{K, P, \Sigma\}$ complies with Definition 1.23 and $V(K)$ satisfies properties (i)–(ii). However, for the sake of simplicity, we hereafter employ the well-established triplet-based definition, and always implicitly assume that there exists a normed vector space $V(K)$ satisfying properties (i)–(ii). In many textbooks, $V(K)$ is implicitly assumed to be of the form $\mathcal{C}^s(K)$ for some integer $s \geq 0$; see, e.g., [Cia91, p. 96] or [BrS94, p. 79]. □

1.2.3 Simplicial Lagrange finite elements

Simplices and barycentric coordinates. Let $\{a_0, \dots, a_d\}$ be a family a points in $\mathbb{R}^d$, $d \geq 1$. Assume that the vectors $\{a_1 - a_0, \dots, a_d - a_0\}$ are linearly independent. Then, the convex hull of $\{a_0, \dots, a_d\}$ is called a *simplex*, and the points $\{a_0, \dots, a_d\}$ are called the *vertices* of the simplex. The *unit simplex* of $\mathbb{R}^d$ is the set

$$\left\{ x \in \mathbb{R}^d;\ x_i \geq 0,\ 1 \leq i \leq d,\ \text{ and } \sum_{i=1}^{d} x_i \leq 1 \right\}.$$

A simplex can be equivalently defined to be the image of the unit simplex by a bijective affine transformation. For $0 \leq i \leq d$, define F_i to be the face of K opposite to a_i, and define n_i to be the outward normal to F_i. Note that in dimension 2 a face is also called an edge, but this distinction will not be made unless necessary.

Given a simplex K in $\mathbb{R}^d$, it is often convenient to consider the associated *barycentric coordinates* $\{\lambda_0, \dots, \lambda_d\}$ defined as follows: For $0 \leq i \leq d$,

$$\lambda_i : \mathbb{R}^d \ni x \longmapsto \lambda_i(x) = 1 - \frac{(x - a_i) \cdot n_i}{(a_j - a_i) \cdot n_i} \in \mathbb{R}, \tag{1.37}$$

where a_j is an arbitrary vertex in F_i (the definition of λ_i is clearly independent of the choice of the vertex in F_i). The barycentric coordinate λ_i is an affine function; it is equal to 1 at a_i and vanishes at F_i. Furthermore, its level-sets are hyperplanes parallel to F_i. Note that the barycenter G of K has barycentric coordinates $(\frac{1}{d+1}, \dots, \frac{1}{d+1})$. The barycentric coordinates satisfy the following properties: For all $x \in K$, $0 \le \lambda_i(x) \le 1$, and for all $x \in \mathbb{R}^d$,

$$\sum_{i=1}^{d+1} \lambda_i(x) = 1 \qquad \text{and} \qquad \sum_{i=1}^{d+1} \lambda_i(x)(x - a_i) = 0.$$

See Exercise 1.4 for further properties in dimension 2 and 3.

Example 1.33. In the unit simplex, $\lambda_0 = 1 - x_1 - x_2$, $\lambda_1 = x_1$, and $\lambda_2 = x_2$ in dimension 2, and $\lambda_0 = 1 - x_1 - x_2 - x_3$, $\lambda_1 = x_1$, $\lambda_2 = x_2$, $\lambda_3 = x_3$ in dimension 3. □

The polynomial space $\mathbb{P}_k$. Let $x = (x_1, \dots, x_d)$ and let $\mathbb{P}_k$ be the space of polynomials in the variables $x_1, \dots, x_d$, with real coefficients and of global degree at most k,

$$\mathbb{P}_k = \left\{ p(x) = \sum_{\substack{0 \le i_1, \dots, i_d \le k \\ i_1 + \dots + i_d \le k}} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d}; \quad \alpha_{i_1 \dots i_d} \in \mathbb{R} \right\}.$$

One readily verifies that $\mathbb{P}_k$ is a vector space of dimension

$$\dim \mathbb{P}_k = \binom{d+k}{k} = \begin{cases} k+1 & \text{if } d = 1, \\ \frac{1}{2}(k+1)(k+2) & \text{if } d = 2, \\ \frac{1}{6}(k+1)(k+2)(k+3) & \text{if } d = 3. \end{cases}$$

Proposition 1.34. *Let K be a simplex in $\mathbb{R}^d$. Let $k \ge 1$, let $P = \mathbb{P}_k$, and let $n_{\mathrm{sh}} = \dim \mathbb{P}_k$. Consider the set of nodes $\{a_i\}_{1 \le i \le n_{\mathrm{sh}}}$ with barycentric coordinates*

$$\left(\tfrac{i_0}{k}, \dots, \tfrac{i_d}{k}\right), \quad 0 \le i_0, \dots, i_d \le k, \;\; i_0 + \dots + i_d = k.$$

Let $\Sigma = \{\sigma_1, \dots, \sigma_{n_{\mathrm{sh}}}\}$ be the linear forms such that $\sigma_i(p) = p(a_i)$, $1 \le i \le n_{\mathrm{sh}}$. Then, $\{K, P, \Sigma\}$ is a Lagrange finite element.

Proof. See Exercise 1.3. □

Table 1.1 presents examples for $k = 1$, 2, and 3 in dimension 2 and 3. For $k = 1$, the $(d + 1)$ local shape functions are the barycentric coordinates

$$\theta_i = \lambda_i, \quad 0 \le i \le d.$$

For $k = 2$, the local shape functions are

$$\begin{cases} \lambda_i(2\lambda_i - 1), & 0 \le i \le d, \\ 4\lambda_i\lambda_j, & 0 \le i < j \le d, \end{cases}$$

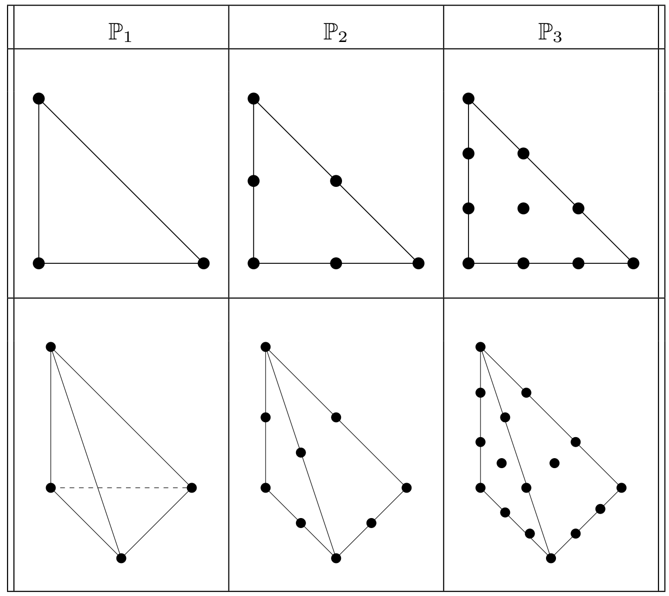


Table 1.1. Two- and three-dimensional $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ Lagrange finite elements; in three dimensions, only visible degrees of freedom are shown.

and for $k = 3$,

$$\begin{cases} \frac{1}{2}\lambda_i(3\lambda_i - 1)(3\lambda_i - 2), & 0 \le i \le d, \\ \frac{9}{2}\lambda_i(3\lambda_i - 1)\lambda_j, & 0 \le i, j \le d,\ i \ne j, \\ 27\lambda_i\lambda_j\lambda_k, & 0 \le i < j < k \le d. \end{cases}$$

1.2.4 Tensor product Lagrange finite elements

Cuboids. Given a set of d intervals $\{[c_i, d_i]\}_{1 \le i \le d}$, all with non-zero measure, the set $K = \prod_{i=1}^{d}[c_i, d_i]$ is called a *cuboid*. For $x \in K$, there exists a unique vector $(t_1, \dots, t_d) \in [0,1]^d$ such that, for all $1 \le i \le d$, $x_i = c_i + t_i(d_i - c_i)$. The vector $(t_1, \dots, t_d)$ is called the local coordinate vector of x in K.

The polynomial space $\mathbb{Q}_k$. Let $\mathbb{Q}_k$ be the polynomial space in the variables $x_1, \dots, x_d$, with real coefficients and of degree at most k in each variable. In dimension 1, $\mathbb{Q}_k = \mathbb{P}_k$; in dimension $d \ge 2$,

$$\mathbb{Q}_k = \left\{ q(x) = \sum_{0 \le i_1, \dots, i_d \le k} \alpha_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d}; \quad \alpha_{i_1 \dots i_d} \in \mathbb{R} \right\}.$$

One readily verifies that $\mathbb{Q}_k$ is a vector space of dimension

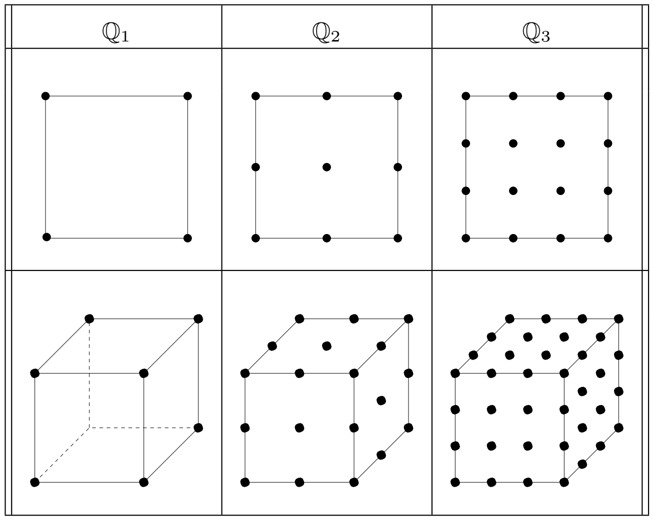


Table 1.2. Two- and three-dimensional $\mathbb{Q}_1$, $\mathbb{Q}_2$, and $\mathbb{Q}_3$ Lagrange finite elements; in three dimensions, only visible degrees of freedom are shown.

$$\dim \mathbb{Q}_k = (k+1)^d = \begin{cases} (k+1)^2 & \text{if } d = 2, \\ (k+1)^3 & \text{if } d = 3. \end{cases}$$

Note the inclusions $\mathbb{P}_k \subset \mathbb{Q}_k \subset \mathbb{P}_{kd}$.

Proposition 1.35. *Let K be a cuboid in $\mathbb{R}^d$. Let $k \geq 1$, let $P = \mathbb{Q}_k$, and let $n_{\mathrm{sh}} = \dim \mathbb{Q}_k$. Consider the set of nodes $\{a_i\}_{1 \leq i \leq n_{\mathrm{sh}}}$ with local coordinates*

$$\left(\tfrac{i_1}{k}, \ldots, \tfrac{i_d}{k}\right), \quad 0 \leq i_1, \ldots, i_d \leq k.$$

Let $\Sigma = \{\sigma_1, \ldots, \sigma_{n_{\mathrm{sh}}}\}$ be the linear forms such that $\sigma_i(p) = p(a_i)$, $1 \leq i \leq n_{\mathrm{sh}}$. Then, $\{K, P, \Sigma\}$ is a Lagrange finite element.

Table 1.2 presents examples for $k = 1$, 2, and 3 in dimension 2 and 3. For $1 \leq i \leq d$, set $\xi_{i,l} = c_i + \frac{l}{k}(d_i - c_i)$, $0 \leq l \leq k$, and let $\{\mathcal{L}^k_{i,0}, \ldots, \mathcal{L}^k_{i,k}\}$ be the Lagrange polynomials in the variable x_i associated with the nodes $\{\xi_{i,0}, \ldots, \xi_{i,k}\}$; see Definition 1.7. Then, the local shape functions are

$$\theta_{i_1 \ldots i_d}(x) = \mathcal{L}^k_{1,i_1}(x_1) \ldots \mathcal{L}^k_{d,i_d}(x_d), \quad 0 \leq i_1, \ldots, i_d \leq k.$$

1.2.5 Prismatic Lagrange finite elements

Prisms. For $x \in \mathbb{R}^d$, set $x' = (x_1, \ldots, x_{d-1})$. Let K' be a simplex in $\mathbb{R}^{d-1}$ and let $[a, b]$ be an interval with non-zero measure. Then, the set $K = \{x \in \mathbb{R}^d;\, x' \in K';\, x_d \in [a, b]\}$ is called a *prism*. Let $(\lambda_0, \ldots, \lambda_{d-1})$ be the barycentric coordinates of x' in K' and let $t \in [0, 1]$ be such that $x_d = a + t(b - a)$. Then, the prismatic coordinates of $x \in K$ are defined to be $(\lambda_0, \ldots, \lambda_{d-1}; t)$.

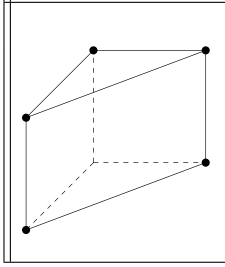
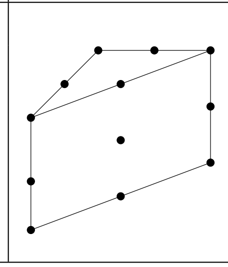
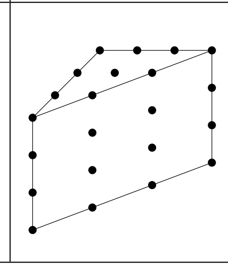
$\mathbb{PR}_1$	$\mathbb{PR}_2$	$\mathbb{PR}_3$
		

Table 1.3. Prismatic Lagrange finite elements of degree 1, 2, and 3; only visible degrees of freedom are shown.

Prismatic polynomials. Let $\mathbb{P}_k[x']$ (resp., $\mathbb{P}_k[x_d]$) be the set of polynomials with real coefficients in the variable x' (resp., x_d) of global degree at most k. Set

$$\mathbb{PR}_k = \{p(x) = p_1(x')\, p_2(x_d);\, p_1 \in \mathbb{P}_k[x'],\, p_2 \in \mathbb{P}_k[x_d]\}.$$

Clearly, $\mathbb{P}_k \subset \mathbb{PR}_k$ and $\dim \mathbb{PR}_k = \frac{1}{2}(k+1)^2(k+2)$ in dimension 3.

Proposition 1.36. *Let K be a prism in $\mathbb{R}^d$. Let $k \geq 1$, let $P = \mathbb{PR}_k$, and let $n_{\text{sh}} = \dim \mathbb{PR}_k$. Consider the set of nodes $\{a_i\}_{1\leq i\leq n_{\text{sh}}}$ with prismatic coordinates*

$$\left(\tfrac{i_0}{k}, \dots, \tfrac{i_{d-1}}{k}; \tfrac{i_d}{k}\right), \quad 0 \leq i_0, \dots, i_{d-1}, i_d \leq k, \;\; i_0 + \ldots + i_{d-1} = k.$$

Let $\varSigma = \{\sigma_1, \dots, \sigma_{n_{\text{sh}}}\}$ be the linear forms such that $\sigma_i(p) = p(a_i)$, $1 \leq i \leq n_{\text{sh}}$. Then, $\{K, P, \varSigma\}$ is a Lagrange finite element.

Table 1.3 presents examples for $k = 1$, 2, and 3. The local shape functions can be expressed in tensor product form using the local shape functions on the simplex K' and the Lagrange polynomials in x_d.

1.2.6 The Crouzeix–Raviart finite element

Let K be a simplex in $\mathbb{R}^d$, set $P = \mathbb{P}_1$, and take for the local degrees of freedom the mean-value over the $(d+1)$ faces of K, i.e., for $0 \leq i \leq d$,

$$\sigma_i(p) = \frac{1}{\text{meas}(F_i)} \int_{F_i} p.$$

Proposition 1.37. *Let $\varSigma = \{\sigma_i\}_{0\leq i\leq d}$. Then, $\{K, \mathbb{P}_1, \varSigma\}$ is a finite element.*

Using the barycentric coordinates $\{\lambda_0, \dots, \lambda_d\}$ defined in (1.37), the local shape functions are

$$\theta_i(x) = d\Big(\frac{1}{d} - \lambda_i(x)\Big), \quad 0 \leq i \leq d. \tag{1.38}$$

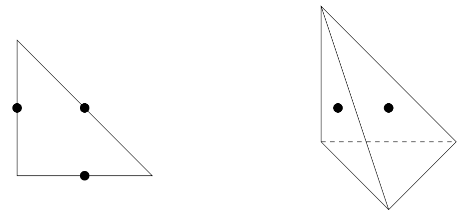


Fig. 1.8. Crouzeix–Raviart finite element in two (left) and three (right) dimensions; in three dimensions, only visible degrees of freedom are shown.

Indeed, $\theta_i \in \mathbb{P}_1$ and $\sigma_j(\theta_i) = \delta_{ij}$ for $0 \leq i, j \leq d$. Note that $\theta_{i|F_i} = 1$ and $\theta_i(a_i) = 1 - d$.

A conventional representation of the Crouzeix–Raviart finite element is shown in Figure 1.8. The dot means that the mean-value is taken over the corresponding face. This finite element has been introduced by Crouzeix and Raviart; see [CrR73] and also [BrF91^b, pp. 107–109].

An admissible choice for the domain of the local interpolation operator is $V(K) = W^{1,1}(K)$. Indeed, owing to the Trace Theorem B.52 applied with $p = 1$, the trace of a function in $W^{1,1}(K)$ is in $L^1(\partial K)$. The *local Crouzeix–Raviart interpolation operator* is then defined as follows:

$$\mathcal{I}_K^{\mathrm{CR}} : V(K) \ni v \longmapsto \mathcal{I}_K^{\mathrm{CR}} v = \sum_{i=0}^{d} \left(\frac{1}{\operatorname{meas} F_i} \int_{F_i} v \right) \theta_i \in \mathbb{P}_1. \tag{1.39}$$

Remark 1.38.

(i) Since a polynomial in P is linear, its mean-value over a face is equal to the value it takes at the barycenter. Therefore, another possible choice for the degrees of freedom is to take the value at the face barycenters. The resulting finite element is a Lagrange finite element according to Definition 1.27. The only difference with the Crouzeix–Raviart finite element is that it is no longer possible to take $W^{1,1}(K)$ for the domain of the local interpolation operator; an admissible choice is, for instance, $V(K) = \mathcal{C}^0(K)$.

(ii) Another choice for the local degrees of freedom is $\sigma_i(p) = \int_{F_i} p$ for $0 \leq i \leq d$; then, the local shape functions are $\theta_i = \frac{d}{\operatorname{meas}(F_i)} \left(\frac{1}{d} - \lambda_i\right)$. □

1.2.7 The Raviart–Thomas finite element

Let K be a simplex in $\mathbb{R}^d$. Consider the vector space of $\mathbb{R}^d$-valued polynomials

$$\mathbb{RT}_0 = [\mathbb{P}_0]^d \oplus x\, \mathbb{P}_0. \tag{1.40}$$

Clearly, the dimension of $\mathbb{RT}_0$ is $d + 1$. For $p \in \mathbb{RT}_0$, the local degrees of freedom are chosen to be the value of the flux of the normal component of p across the faces of K, i.e., for $0 \leq i \leq d$,

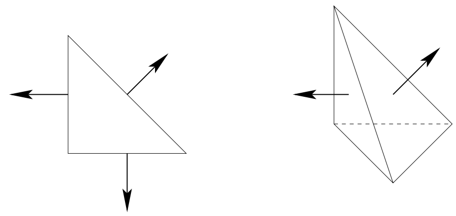


Fig. 1.9. Raviart–Thomas finite element in two (left) and three (right) dimensions; in three dimensions, only visible degrees of freedom are shown.

$$\sigma_i(p) = \int_{F_i} p\cdot n_i.$$

Proposition 1.39. *Let $\Sigma = \{\sigma_i\}_{0\le i\le d}$. Then, $\{K, \mathbb{RT}_0, \Sigma\}$ is a finite element.*

The local shape functions are

$$\theta_i(x) = \frac{1}{d\,\mathrm{meas}(K)}(x - a_i), \quad 0 \le i \le d. \tag{1.41}$$

Indeed, $\theta_i \in \mathbb{RT}_0$ and $\sigma_j(\theta_i) = \delta_{ij}$ for $0 \le i, j \le d$. Note that the normal component of a local shape function is constant on the face with which it is associated and is zero on the other faces.

A conventional representation of the degrees of freedom of the Raviart–Thomas finite element is shown in Figure 1.9. An arrow means that the flux of the normal component is taken over the corresponding face. This finite element has been introduced by Raviart and Thomas and is often referred to as the $\mathbb{RT}_0$ finite element [RaT77]. It is used, for instance, in applications related to fluid mechanics where the functions to be interpolated are velocities.

The domain of the local interpolation operator can be taken to be $V^{\mathrm{div}}(K) = \{v \in [L^p(K)]^d; \nabla\cdot v \in L^s(K)\}$ for $p > 2$, $s \ge q$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$. Note that $V^{\mathrm{div}}(K) = W^{1,t}(K)$ with $t > \frac{2d}{d+2}$ is also an admissible choice. Indeed, one can show that for $v \in V^{\mathrm{div}}(K)$ and for a face F_i of K, the quantity $\int_{F_i} v\cdot n_i$ is meaningful. The *local Raviart–Thomas interpolation operator* is then defined as follows:

$$\mathcal{I}_K^{\mathrm{RT}} : V^{\mathrm{div}}(K) \ni v \longmapsto \mathcal{I}_K^{\mathrm{RT}} v = \sum_{i=0}^{d} \left(\int_{F_i} v\cdot n_i \right) \theta_i \in \mathbb{RT}_0. \tag{1.42}$$

Remark 1.40.

(i) See Exercise 1.5 for the proofs of the above results and for an alternative expression of the local shape functions in terms of barycentric coordinates. Further results can be found in [BrF91^b, p. 113] and [QuV97, p. 82].

(ii) In the spirit of Remark 1.38, the Raviart–Thomas finite element can be defined as a Lagrange finite element. Another choice for the degrees of freedom is $\sigma_i(p) = \frac{1}{\mathrm{meas}(F_i)} \int_{F_i} p{\cdot}n_i$; then, the local shape functions are $\theta_i = \frac{\mathrm{meas}\,(F_i)}{d\,\mathrm{meas}(K)}(x - a_i)$. □

Lemma 1.41. *Let $\mathcal{I}_K^{\mathrm{RT}}$ be defined in (1.42). Let π_K^0 be the orthogonal projection from $L^2(K)$ to $\mathbb{P}_0$. The following diagram commutes:*

$$\begin{tikzcd} V^{\mathrm{div}}(K) \arrow[r, "\nabla\cdot"] \arrow[d, "\mathcal{I}_K^{\mathrm{RT}}"] & L^2(K) \arrow[d, "\pi_K^0"] \\ \mathbb{RT}_0 \arrow[r, "\nabla\cdot"] & \mathbb{P}_0 \end{tikzcd}$$

Proof. Left as an exercise. □

1.2.8 The Nédélec (or edge) finite element

Let K be a simplex in $\mathbb{R}^d$, $d = 2$ or 3. Define the polynomial space of dimension $\frac{1}{2}d(d+1)$,

$$\mathbb{N}_0 = [\mathbb{P}_0]^d \oplus \mathcal{R}_1, \qquad \mathcal{R}_1 = \{p \in [\mathbb{P}_1]^d;\ x{\cdot}p = 0\}. \tag{1.43}$$

Introducing the mapping $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{R}(x_1, x_2) = (x_2, -x_1)$, the following equivalent definition of $\mathbb{N}_0$ holds in dimension 2:

$$\mathbb{N}_0 = [\mathbb{P}_0]^2 \oplus (\mathcal{R}(x)\mathbb{P}_0). \tag{1.44}$$

In dimension 3, the following equivalent definition of $\mathbb{N}_0$ holds:

$$\mathbb{N}_0 = [\mathbb{P}_0]^3 \oplus (x \times [\mathbb{P}_0]^3). \tag{1.45}$$

For $p \in \mathbb{N}_0$, the local degrees of freedom are chosen to be the integral of the tangential component of p along the three (resp., six) edges of K in two (resp., three) dimensions. Set $n_{\mathrm{e}} = 3$ if $d = 2$ and $n_{\mathrm{e}} = 6$ if $d = 3$. Denote by $\{e_i\}_{1 \le i \le n_{\mathrm{e}}}$ the set of edges of K and, for each edge e_i, let t_i be one of the two unit vectors parallel to e_i. For $1 \le i \le n_{\mathrm{e}}$, the local degrees of freedom are

$$\sigma_i(p) = \int_{e_i} p{\cdot}t_i.$$

Proposition 1.42. *Let $\Sigma = \{\sigma_i\}_{1 \le i \le n_{\mathrm{e}}}$. Then, $\{K, \mathbb{N}_0, \Sigma\}$ is a finite element.*

In two dimensions, the local shape function associated with the edge e_i, $1 \le i \le 3$, is

$$\theta_i(x) = \frac{\mathcal{R}(x - a_i)}{t_i{\cdot}\big[\mathcal{R}(\frac{a_{i_1} + a_{i_2}}{2} - a_i)\big]\,\mathrm{meas}(e_i)}, \quad i_1, i_2 \ne i. \tag{1.46}$$

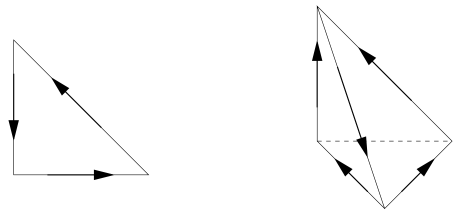


Fig. 1.10. Edge finite element in dimension 2 (left) and 3 (right); in three dimensions, only visible degrees of freedom are shown.

In three dimensions, define the mapping $j : \{1, \dots, 6\} \to \{1, \dots, 6\}$ such that $j(i)$ is the index of the edge opposite to e_i, i.e., e_i does not intersect $e_{j(i)}$. Note that $j = j^{-1}$. Let m_i be the midpoint of e_i. Then, the local shape function associated with the edge e_i, $1 \le i \le 6$, is

$$\theta_i(x) = \frac{(x - m_{j(i)}) \times t_{j(i)}}{t_i\cdot\big[(m_i - m_{j(i)}) \times t_{j(i)}\big] \operatorname{meas}(e_i)}. \tag{1.47}$$

In both cases, the tangential component of a local shape function is constant along the edge with which it is associated and vanishes along the other edges.

A conventional representation of the edge finite element is shown in Figure 1.10. An arrow means that the integral of the component parallel to this direction is taken over the corresponding edge. This finite element has been introduced by Nédélec [Néd80, Néd86]; see also [Whi57]. It is used, for instance, in electromagnetism and in magneto-hydrodynamics; see [Bos93, Chap. 3].

In two dimensions, the domain of the local interpolation operator can be taken to be $V^{\text{curl}}(K) = \{v = (v_1, v_2) \in [L^p(K)]^2;\, \partial_2 v_1 - \partial_1 v_2 \in L^p(K)\}$ for $p > 2$. Indeed, one can show that for $v \in V^{\text{curl}}(K)$ and for an edge e_i of K, the quantity $\int_{e_i} v\cdot t_i$ is meaningful. In three dimensions, a suitable choice is $V^{\text{curl}}(K) = \{v \in [H^s(K)]^3;\, \nabla\times v \in [L^p(K)]^3\}$ for $s > \frac{1}{2}$ and $p > 2$; see, e.g., [AmB98]. The *local Nédélec interpolation operator* is then defined as follows:

$$\mathcal{I}_K^{\mathrm{N}} : V^{\text{curl}}(K) \ni v \longmapsto \mathcal{I}_K^{\mathrm{N}} v = \sum_{i=1}^{n_{\mathrm{e}}} \left(\int_{e_i} v\cdot t_i \right) \theta_i \in \mathbb{N}_0. \tag{1.48}$$

Remark 1.43.

(i) See Exercise 1.6 for the proofs of the above results and for an alternative expression of the local shape functions in terms of barycentric coordinates.

(ii) In the spirit of Remark 1.38, the Nédélec finite element can be defined as a Lagrange finite element. Another choice for the degrees of freedom is $\sigma_i(p) = \frac{1}{\operatorname{meas}(e_i)} \int_{e_i} p\cdot t_i$ for $1 \le i \le n_{\mathrm{e}}$; the local shape functions are then readily derived from (1.46) and (1.47). □

Lemma 1.44. *Assume $d = 3$. Let $\mathcal{I}_K^{\mathrm{RT}}$ and $\mathcal{I}_K^{\mathrm{N}}$ be defined in* (1.42) *and* (1.48), *respectively. The following diagram commutes:*

$$
\begin{tikzcd}
V^{\mathrm{curl}}(K) \arrow[r, "\nabla\times"] \arrow[d, "\mathcal{I}_K^{\mathrm{N}}"] & V^{\mathrm{div}}(K) \arrow[d, "\mathcal{I}_K^{\mathrm{RT}}"] \\
\mathbb{N}_0 \arrow[r, "\nabla\times"] & \mathbb{RT}_0
\end{tikzcd}
$$

Proof. Let $v \in V^{\mathrm{curl}}(K)$. It is clear that $\nabla\times\mathcal{I}_K^{\mathrm{N}} v \in [\mathbb{P}_0]^3 \subset \mathbb{RT}_0$. Let F be a face of K and n_F be the corresponding outward normal. Then,

$$
\begin{aligned}
\int_F (\nabla\times(\mathcal{I}_K^{\mathrm{N}} v))\cdot n_F &= \sum_{e\subset\partial F} \int_e \mathcal{I}_K^{\mathrm{N}} v\cdot t_e = \sum_{e\subset\partial F} \int_e v\cdot t_e \\
&= \int_F (\nabla\times v)\cdot n_F = \int_F (\mathcal{I}_K^{\mathrm{RT}}(\nabla\times v))\cdot n_F,
\end{aligned}
$$

where t_e is a unit vector parallel to the edge e so that the edge integrals are taken anti-clockwise along ∂F. The above equality implies $\mathcal{I}_K^{\mathrm{RT}}(\nabla\times v) = \nabla\times(\mathcal{I}_K^{\mathrm{N}} v)$, since these two functions are in $\mathbb{RT}_0$ and their fluxes across the faces of K are identical. $\square$

Lemma 1.45. *Assume $d = 2$ or 3. Let $\mathcal{I}_K^1$ be the interpolation operator associated with the $\mathbb{P}_1$ Lagrange finite element and let $V^1(K) = H^s(K)$ be its domain, $s > \frac{d}{2}$. The following diagram commutes:*

$$
\begin{tikzcd}
V^1(K) \arrow[r, "\nabla"] \arrow[d, "\mathcal{I}_K^1"] & V^{\mathrm{curl}}(K) \arrow[d, "\mathcal{I}_K^{\mathrm{N}}"] \\
\mathbb{P}_1 \arrow[r, "\nabla"] & \mathbb{N}_0
\end{tikzcd}
$$

Proof. Le $v \in V^1(K)$. Let e be an edge of K and denote by a_1, a_2 the two vertices of e. Set $t = \frac{a_2 - a_1}{\|a_2 - a_1\|_d}$ to obtain

$$
\begin{aligned}
\int_e \nabla(\mathcal{I}_K^1 v)\cdot t &= \mathcal{I}_K^1 v(a_2) - \mathcal{I}_K^1 v(a_1) = v(a_2) - v(a_1) \\
&= \int_e (\nabla v)\cdot t = \int_e \mathcal{I}_K^{\mathrm{N}}(\nabla v)\cdot t.
\end{aligned}
$$

Conclude using the fact that both $\mathcal{I}_K^{\mathrm{N}}(\nabla v)$ and $\nabla(\mathcal{I}_K^1 v)$ belong to $\mathbb{N}_0$. $\square$

1.2.9 High-order finite elements

As in the one-dimensional case, basis functions must be selected carefully when working with high-degree polynomials.

Nodal finite elements. When K is a simplex in $\mathbb{R}^d$ and $P = \mathbb{P}_k$ with k large, it is possible to define sets of quadrature points with near optimal interpolation properties: the so-called *Fekete points*; see [ChB95, TaW00]. Then, these points can be used as *Lagrange nodes* to define nodal bases. Finite element methods using the Fekete points as Lagrange nodes when k is large are often referred to as *spectral element methods*; see, e.g., [KaS99^b].

When K is a cuboid in $\mathbb{R}^d$ and $P = \mathbb{Q}_k$, one can use the tensor product of Gauß–Lobatto nodes instead of equi-distributing the Lagrange nodes in each space direction. Then, the local shape functions are

$$\theta_{i_1,\dots,i_d}(x_1,\dots,x_d) = \theta_{i_1}(x_1)\dots\theta_{i_d}(x_d), \quad 0 \le i_1,\dots,i_d \le k, \tag{1.49}$$

where the functions $\{\theta_i\}_{0\le i\le k}$ are the images by suitable mappings of the *nodal ($\mathcal{C}^0$-continuous) basis functions* defined in (1.33). An interesting property of the Gauß–Lobatto points is that they are the Fekete points for the d-dimensional cuboid, i.e., these points have near optimal interpolation properties; see [BoT01].

Modal finite elements. When K is a cuboid, hierarchical modal bases can be constructed using tensor products of one-dimensional hierarchical modal bases. For instance, we can consider the basis functions defined in (1.49), where the functions $\{\theta_i\}_{0\le i\le k}$ are now the images by suitable mappings of the *modal ($\mathcal{C}^0$-continuous) basis functions* defined in (1.31).

When K is a simplex or a prism, the construction of hierarchical bases is more technical. The idea is to introduce a nonlinear transformation mapping K to a square or a cube and to use tensor products of one-dimensional bases. See [KaS99^b, pp. 70–94] for a detailed presentation.

1.3 Meshes: Basic Concepts

This section presents the general principles governing the construction of a mesh. Implementation aspects are investigated in Chapter 7.

1.3.1 Domains and meshes

Throughout this book, we shall use the following:

Definition 1.46 (Domain). *In dimension* 1*, a domain is an open, bounded interval. In dimension* $d \ge 2$*, a domain is an open, bounded, connected set in* $\mathbb{R}^d$ *such that its boundary* $\partial\Omega$ *satisfies the following property: There are* $\alpha > 0$*,* $\beta > 0$*, a finite number* R *of local coordinate systems* $x^r = (x^{r\prime}, x^r_d)$*,* $1 \le r \le R$*, where* $x^{r\prime} \in \mathbb{R}^{d-1}$ *and* $x^r_d \in \mathbb{R}$*, and* R *local maps* ϕ^r *that are* Lipschitz *on their definition domain* $\{x^{r\prime} \in \mathbb{R}^{d-1};\ |x^{r\prime}| < \alpha\}$ *and such that*

$$\begin{aligned}
&\partial\Omega = \bigcup_{r=1}^{R} \{(x^{r\prime}, x_d^r);\, x_d^r = \phi^r(x^{r\prime});\, |x^{r\prime}| < \alpha\},\\
&\{(x^{r\prime}, x_d^r);\, \phi^r(x^{r\prime}) < x_d^r < \phi^r(x^{r\prime}) + \beta;\, |x^{r\prime}| < \alpha\} \subset \Omega, \qquad \forall r,\\
&\{(x^{r\prime}, x_d^r);\, \phi^r(x^{r\prime}) - \beta < x_d^r < \phi^r(x^{r\prime});\, |x^{r\prime}| < \alpha\} \subset \mathbb{R}^d \backslash \overline{\Omega}, \qquad \forall r,
\end{aligned}$$

where $|x^{r\prime}| \leq \alpha$ *means that* $|x_i^{r\prime}| \leq \alpha$ *for all* $1 \leq i \leq d-1$. *For* $m \geq 1$, Ω *is said to be of* class $\mathcal{C}^m$ *(resp.,* piecewise of class $\mathcal{C}^m$*) if all the local maps* Φ^r *are of class* $\mathcal{C}^m$ *(resp., piecewise of class* $\mathcal{C}^m$*).*

Definition 1.47 (Polygon, polyhedron). *In dimension* 2, *a* polygon *is a domain whose boundary is a finite union of segments. In dimension* 3, *a* polyhedron *is a domain whose boundary is a finite union of polygons. When the distinction is not relevant, the term polyhedron is also employed for polygons.*

Remark 1.48.

(i) Definition 1.46 implies that a domain is necessarily located on one side of its boundary $\partial\Omega$, i.e., it excludes sets with slits. This assumption can be weakened, but this involves technical complexities that go beyond the scope of this book; see, e.g., [CoD02].

(ii) For a domain Ω in $\mathbb{R}^d$ with $d \geq 2$, the outward normal, say n, is defined for a.e. $x \in \partial\Omega$. For a domain of class $\mathcal{C}^m$, $m \geq 1$, n is defined for all $x \in \partial\Omega$ and is a function of class $\mathcal{C}^{m-1}$.

(iii) Definition 1.47 can be extended to arbitrary dimension d by induction: a polyhedron in $\mathbb{R}^d$ is a domain whose boundary is a finite union of polyhedra in $\mathbb{R}^{d-1}$. □

Definition 1.49 (Mesh). *Let* Ω *be a domain in* $\mathbb{R}^d$. *A mesh is a union of a finite number* N_{el} *of compact, connected, Lipschitz sets* K_m *with non-empty interior such that* $\{K_m\}_{1 \leq m \leq N_{\mathrm{el}}}$ *forms a partition of* Ω, *i.e.,*

$$\overline{\Omega} = \bigcup_{m=1}^{N_{\mathrm{el}}} K_m \qquad \text{and} \qquad \overset{\circ}{K}_m \cap \overset{\circ}{K}_n = \emptyset \qquad \text{for } \ m \neq n. \tag{1.50}$$

The subsets K_m *are called* mesh cells *or* mesh elements *(or simply elements when there is no ambiguity).*

Figure 1.11 presents an example of a mesh of the unit square in $\mathbb{R}^2$ involving triangles and quadrangles. In the sequel, a mesh $\{K_m\}_{1 \leq m \leq N_{\mathrm{el}}}$ is denoted by $\mathcal{T}_h$. The subscript h refers to the level of refinement of the mesh. Setting

$$\forall K \in \mathcal{T}_h, \quad h_K = \mathrm{diam}(K) = \max_{x_1, x_2 \in K} \|x_1 - x_2\|_d,$$

where $\|\cdot\|_d$ is the Euclidean norm in $\mathbb{R}^d$, the parameter h is defined by

$$h = \max_{K \in \mathcal{T}_h} h_K.$$

A sequence of successively refined meshes is denoted by $\{\mathcal{T}_h\}_{h>0}$.

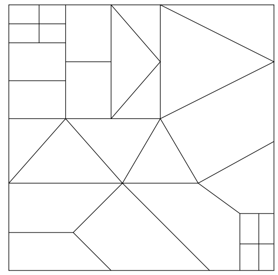


Fig. 1.11. Example of a mesh of the unit square in $\mathbb{R}^2$.

1.3.2 Mesh generation

In practice, a mesh is generated from a *reference cell*, say $\widehat{K}$, and a set of geometric transformations mapping $\widehat{K}$ to the actual mesh cells. We shall henceforth assume that the geometric transformations are $\mathcal{C}^1$-diffeomorphisms. For $K \in \mathcal{T}_h$, denote by $T_K : \widehat{K} \to K$ the corresponding transformation. Usually T_K is specified using a Lagrange finite element $\{\widehat{K}, \widehat{P}_{\text{geo}}, \widehat{\Sigma}_{\text{geo}}\}$. Let $n_{\text{geo}} = \text{card}(\widehat{\Sigma}_{\text{geo}})$, let $\{\widehat{g}_1, \dots, \widehat{g}_{n_{\text{geo}}}\}$ be the nodes of $\widehat{K}$ associated with $\widehat{\Sigma}_{\text{geo}}$, and let $\{\widehat{\psi}_1, \dots, \widehat{\psi}_{n_{\text{geo}}}\}$ be the local shape functions.

Definition 1.50. $\{\widehat{K}, \widehat{P}_{\text{geo}}, \widehat{\Sigma}_{\text{geo}}\}$ *is called the* geometric (reference) finite element*,* $\{\widehat{g}_1, \dots, \widehat{g}_{n_{\text{geo}}}\}$ *the* geometric (reference) nodes*, and* $\{\widehat{\psi}_1, \dots, \widehat{\psi}_{n_{\text{geo}}}\}$ *the* geometric (reference) shape functions.

For the sake of simplicity, assume that all the mesh cells are generated using the *same* geometric reference finite element. This assumption can be easily lifted. When $\widehat{K}$ is a simplex, $\mathcal{T}_h$ is called a *simplicial mesh*.

A mesh generator usually provides a list of n_{geo}-uplets

$$\{g_1^m, \dots, g_{n_{\text{geo}}}^m\}_{1 \le m \le N_{\text{el}}},$$

where $g_i^m \in \mathbb{R}^d$ and N_{el} is the number of mesh elements. The points $\{g_1^m, \dots, g_{n_{\text{geo}}}^m\}$ are called the *geometric nodes* of the m-th element. For $1 \le m \le N_{\text{el}}$, define the *geometric transformation*

$$T_m : \widehat{K} \ni \widehat{x} \longmapsto T_m(\widehat{x}) = \sum_{i=1}^{n_{\text{geo}}} g_i^m \widehat{\psi}_i(\widehat{x}) \in \mathbb{R}^d, \tag{1.51}$$

so that $T_m(\widehat{g}_i) = g_i^m$ for $1 \le i \le n_{\text{geo}}$, and set $K_m = T_m(\widehat{K})$.

Remark 1.51. The hypothesis that the geometric transformation T_m is a $\mathcal{C}^1$-diffeomorphism requires that the numbering of the nodes $\{g_1^m, \dots, g_{n_{\text{geo}}}^m\}$ and that employed in the reference element are compatible; see Figure 1.12. An usual convention is to impose the additional requirement that the numbering

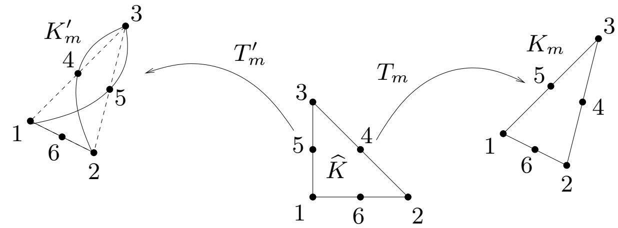


Fig. 1.12. The numbering of the nodes of K_m (right) is compatible with that of $\widehat{K}$; the numbering in K'_m (left) is not.

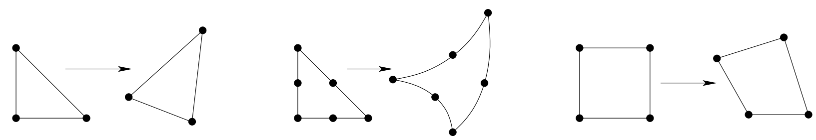


Fig. 1.13. Examples of geometric transformations: $\mathbb{P}_1$ transformation of a triangle (left); $\mathbb{P}_2$ transformation of a triangle (center); $\mathbb{Q}_1$ transformation of a quadrangle (right).

is such that the Jacobian determinant of the transformation T_m is positive. For instance, in two dimensions, there are three compatible ways of numbering the nodes of a triangle, and there are four compatible ways of numbering the nodes of a square. In three dimensions, there are 3×4 compatible ways of numbering the nodes of a tetrahedron, and there are 4×6 compatible ways of numbering the nodes of a cube. □

Example 1.52. Figure 1.13 presents three examples in dimension 2:

(i) A transformation based on the Lagrange finite element $\mathbb{P}_1$ maps the unit simplex to a non-degenerate triangle.

(ii) A transformation based on the Lagrange finite element $\mathbb{P}_2$ maps the unit simplex to a curved triangle.

(iii) A transformation based on the Lagrange finite element $\mathbb{Q}_1$ maps the unit square to a non-degenerate quadrangle. □

Definition 1.53 (Affine meshes). *When the transformations $\{T_m\}_{1\le m\le N_{\rm el}}$ are affine, the mesh is said to be* affine. *In dimension* 2*, when the reference cell $\widehat{K}$ is a simplex, an* affine mesh *is also called a* triangulation. *This terminology is used henceforth in any dimension for an affine, simplicial mesh.*

Examples of affine meshes include the following:

(i) When the geometric reference finite element is the Lagrange finite element $\mathbb{P}_1$, all the mesh elements are triangles in dimension 2 and tetrahedra in dimension 3.
(ii) When the geometric reference finite element is the Lagrange finite element $\mathbb{Q}_1$, all the mesh elements are parallelograms in dimension 2 and parallelepipeds in dimension 3.

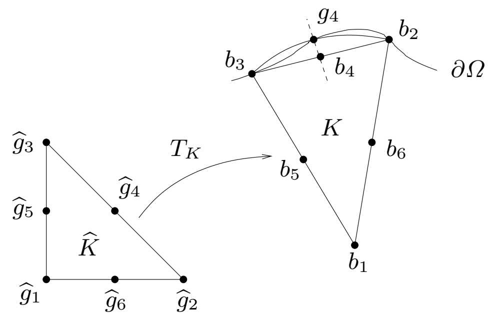


Fig. 1.14. Geometric construction of a curved triangle.

Domains with curved boundary. For domains with curved boundary, the use of affine meshes generates an interpolation error in the neighborhood of the boundary. Hence, when high-order accuracy is required, it is necessary to generate the mesh with geometric transformations of degree k_{geo} larger than one; the mesh then contains curved elements.

A relatively straightforward way to proceed is the following:

(i) Construct an affine mesh $\widetilde{\mathcal{T}}_h$ so that all the vertices of the resulting polyhedron lie on the curved boundary $\partial\Omega$.
(ii) For each element $\widetilde{K} \in \widetilde{\mathcal{T}}_h$ having a non-empty intersection with $\partial\Omega$, design a polynomial transformation (of degree larger than 1) that approximates the boundary more accurately than the first-order interpolation. The resulting element K replaces $\widetilde{K}$ in the mesh.

Example 1.54. The following example illustrates a simple technique relying on $\mathbb{P}_2$ Lagrange finite elements to approximate a curved boundary in $\mathbb{R}^2$ (see Figure 1.14):

(i) Let $\widetilde{K}$ be an element having an edge whose vertices lie on $\partial\Omega$. Let $\{b_1, \dots, b_{n_{\text{geo}}}\}$ be the geometric nodes of $\widetilde{K}$ ($n_{\text{geo}} = 6$).
(ii) For each b_i, $1 \le i \le n_{\text{geo}}$, construct a new node g_i as follows:
- If b_i is located on an edge whose vertices lie on $\partial\Omega$, g_i is defined as the intersection with $\partial\Omega$ of the line normal to the corresponding edge and passing through b_i.
- Otherwise, set $g_i = b_i$.

(iii) Replace $\{b_1, \dots, b_{n_{\text{geo}}}\}$ by $\{g_1, \dots, g_{n_{\text{geo}}}\}$ in the list of n_{geo}-uplets provided by the mesh generator. In other words, replace $\widetilde{K}$ by the curved triangle $K = T_K(\widehat{K})$ where

$$\forall \widehat{x} \in \widehat{K}, \quad T_K(\widehat{x}) = \sum_{i=1}^{n_{\text{geo}}} g_i \widehat{\psi}_i(\widehat{x}),$$

and $\{\widehat{\psi}_1, \dots, \widehat{\psi}_{n_{\text{geo}}}\}$ are the $\mathbb{P}_2$ Lagrange local shape functions. □

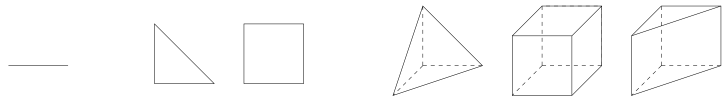


Fig. 1.15. Examples of reference elements.

Note that a mesh consisting of curved triangles may not necessarily cover the domain Ω; see Figure 1.14. In other words, the open set Ω_h such that

$$\overline{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} K, \tag{1.52}$$

does not necessarily coincide with Ω; the domain Ω_h is called a *geometric interpolation* of Ω. For the sake of simplicity, $\mathcal{T}_h$ is said to be a mesh of Ω even though it may happen that $\Omega \neq \Omega_h$.

1.3.3 Geometrically conforming meshes

Henceforth we assume that the reference element $\widehat{K}$ used to generate the mesh is a polyhedron. Classical examples include the following (see Figure 1.15):

(i) $\widehat{K}$ is the *unit interval* $[0,1]$ in dimension 1.
(ii) $\widehat{K}$ is either the *unit simplex* with vertices $(0,0)$, $(1,0)$, $(0,1)$ or the *unit square* $[0,1]^2$ in dimension 2.
(iii) $\widehat{K}$ is either the *unit simplex* with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, or the *unit cube* $[0,1]^3$, or the *unit prism* with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ in dimension 3.

For a given mesh cell $K = T_K(\widehat{K})$, the vertices, edges, and faces are defined to be the image by the geometric transformation T_K of the vertices, edges, and faces of the reference element $\widehat{K}$.

Definition 1.55 (Geometrically conforming meshes). *Let Ω be a domain in $\mathbb{R}^d$ and let $\mathcal{T}_h = \{K_m\}_{1 \le m \le N_{\mathrm{el}}}$ be a mesh of Ω. The mesh $\mathcal{T}_h$ is said to be* geometrically conforming *if the following matching criterion is satisfied: For all K_m and K_n having a non-empty $(d-1)$-dimensional intersection, say $F = K_m \cap K_n$, there is a face $\widehat{F}$ of $\widehat{K}$ and renumberings of the geometric nodes of K_m and K_n such that $F = T_m(\widehat{F}) = T_n(\widehat{F})$ and*

$$T_{m|\widehat{F}} = T_{n|\widehat{F}}. \tag{1.53}$$

If more than one geometric reference element is used to generate the mesh, say $\widehat{K}_1$ and $\widehat{K}_2$, (1.53) is replaced by the following statement: $T_{m|F}^{-1}(F)$ is a face of $\widehat{K}_1$, $T_{n|F}^{-1}(F)$ is a face of $\widehat{K}_2$, and there is a bijective affine transformation mapping $T_{m|F}^{-1}(F)$ to $T_{n|F}^{-1}(F)$.

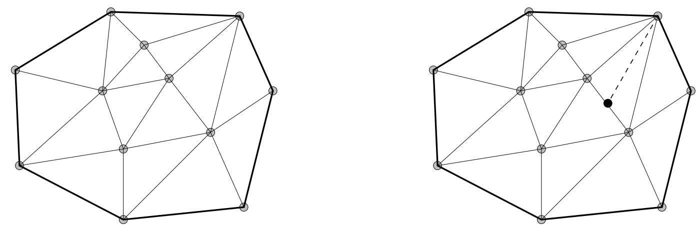


Fig. 1.16. Example and counterexample of a geometrically conforming mesh.

Remark 1.56. If Ω_h is connected, Definition 1.55 implies that for any cell pair $\{K_m, K_n\}$ with $m \neq n$, the intersection $K_m \cap K_n$ is:

(i) either empty or a common vertex in dimension 1;
(ii) either empty, or a common vertex, or a common edge in dimension 2;
(iii) either empty, or a common vertex, or a common edge, or a common face in dimension 3.

An example and a counterexample of a geometrically conforming mesh are shown in Figure 1.16. □

Geometrically conforming meshes form a particular class of meshes that are convenient to generate H^1-conforming approximation spaces; see §1.4.5. Moreover, on such meshes, the Euler relations provide useful means to count global degrees of freedom.

Lemma 1.57 (Euler relations). *Let $\mathcal{T}_h$ be a geometrically conforming mesh and let Ω_h be defined in* (1.52).

(i) *In dimension* 2*, let I be the degree of multiple-connectedness*[1] *of Ω_h, N_{el} the number of cells (or elements), N_{ed} the number of edges, N_{v} the number of vertices, $N_{\mathrm{ed}}^{\partial}$ the number of boundary edges, and $N_{\mathrm{v}}^{\partial}$ the number of boundary vertices; then,*

$$\begin{cases} N_{\mathrm{el}} - N_{\mathrm{ed}} + N_{\mathrm{v}} = 1 - I, \\ N_{\mathrm{v}}^{\partial} - N_{\mathrm{ed}}^{\partial} = 0. \end{cases}$$

Furthermore, if the mesh cells are polygons with ν vertices,

$$2N_{\mathrm{ed}} - N_{\mathrm{ed}}^{\partial} = \nu N_{\mathrm{el}}.$$

In particular, $2N_{\mathrm{ed}} - N_{\mathrm{ed}}^{\partial} = 3N_{\mathrm{el}}$ for triangles and $2N_{\mathrm{ed}} - N_{\mathrm{ed}}^{\partial} = 4N_{\mathrm{el}}$ for quadrangles.

(ii) *In dimension* 3*, let I be the degree of multiple-connectedness of Ω_h, J the number of connected components of the boundary of Ω_h, N_{el} the number of elements, N_{f} the number of faces, N_{ed} the number of edges, N_{v} the*

[1] I is the number of holes in Ω_h.

number of vertices, $N_{\mathrm{f}}^{\partial}$ *the number of boundary faces,* $N_{\mathrm{ed}}^{\partial}$ *the number of boundary edges, and* $N_{\mathrm{v}}^{\partial}$ *the number of boundary vertices; then,*

$$\begin{cases} N_{\mathrm{el}} - N_{\mathrm{f}} + N_{\mathrm{ed}} - N_{\mathrm{v}} = -1 + I - J, \\ N_{\mathrm{f}}^{\partial} - N_{\mathrm{ed}}^{\partial} + N_{\mathrm{v}}^{\partial} = 2(J - I). \end{cases}$$

Furthermore, if the mesh cells are polyhedra with ν *faces,*

$$2N_{\mathrm{f}} - N_{\mathrm{f}}^{\partial} = \nu N_{\mathrm{el}}.$$

In particular, $2N_{\mathrm{f}} - N_{\mathrm{f}}^{\partial} = 4N_{\mathrm{el}}$ *for tetrahedra and* $2N_{\mathrm{f}} - N_{\mathrm{f}}^{\partial} = 6N_{\mathrm{el}}$ *for hexahedra.*

1.3.4 Faces, edges, and jumps

Henceforth, we denote by $\mathcal{F}_h^{\mathrm{i}}$ the set of interior faces (or interfaces), i.e., $F \in \mathcal{F}_h^{\mathrm{i}}$ if F is a $(d{-}1)$-manifold and there are $K_1, K_2 \in \mathcal{T}_h$ such that $F = K_1 \cap K_2$. We denote by $\mathcal{F}_h^{\partial}$ the set of the faces that separate the mesh from the exterior of Ω_h, i.e., $F \in \mathcal{F}_h^{\partial}$ if F is a $(d{-}1)$-manifold and there is $K \in \mathcal{T}_h$ such that $F = K \cap \partial\Omega_h$. Finally, we set $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\partial}$. In all dimensions we refer to the elements of $\mathcal{F}_h$ as faces. In dimension 2, faces are also called edges, but this distinction will not be made unless necessary. In dimension $d \geq 3$, we define $\mathcal{E}_h^{\mathrm{i}}$, $\mathcal{E}_h^{\partial}$, and $\mathcal{E}_h = \mathcal{E}_h^{\mathrm{i}} \cup \mathcal{E}_h^{\partial}$ to be the sets of internal edges (i.e., one-dimensional manifolds), boundary edges, and edges, respectively.

Let $F \in \mathcal{F}_h^{\mathrm{i}}$ with $F = K_1 \cap K_2$, and denote by n_1 and n_2 the outward normal to K_1 and K_2, respectively. Let v be a *scalar-valued* function defined on all cells K of the mesh. Assume that v is smooth enough to have limits on both sides of F (these limits being not necessarily the same). Set $v_1 = v_{|K_1}$ and $v_2 = v_{|K_2}$. Then, the *jump* of v across F is defined to be

$$[\![v]\!]_F = v_1 n_1 + v_2 n_2. \tag{1.54}$$

Note that $[\![v]\!]_F$ is an $\mathbb{R}^d$-valued function defined on F. When there is no ambiguity, the subscript F is dropped. When v is an $\mathbb{R}^d$*-valued* function, we use the notation

$$[\![v{\cdot}n]\!]_F = v_1{\cdot}n_1 + v_2{\cdot}n_2, \tag{1.55}$$

for the jump of the *normal component* of v. In dimension 3, we also use the notation

$$[\![v{\times}n]\!]_F = v_1{\times}n_1 + v_2{\times}n_2, \tag{1.56}$$

for the jump of the *tangential component* of v.

1.4 Approximation Spaces and Interpolation Operators

This section reviews approximation spaces and global interpolation operators that can be used in conjunction with Galerkin methods to approximate PDEs.

1.4.1 Finite element generation

Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a fixed finite element. Denote by $\{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{n_{\mathrm{sh}}}\}$ the local degrees of freedom and by $\{\widehat{\theta}_1, \ldots, \widehat{\theta}_{n_{\mathrm{sh}}}\}$ the local ($\mathbb{R}^m$-valued) shape functions. Let $V(\widehat{K})$ be the domain of the *local interpolation operator* $\mathcal{I}_{\widehat{K}}$ associated with $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$, i.e.,

$$\mathcal{I}_{\widehat{K}} : V(\widehat{K}) \ni \widehat{v} \longmapsto \sum_{i=1}^{n_{\mathrm{sh}}} \widehat{\sigma}_i(\widehat{v})\,\widehat{\theta}_i \in \widehat{P}. \tag{1.57}$$

Definition 1.58. $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ *is called the* reference finite element *and* $\mathcal{I}_{\widehat{K}}$ *the* reference interpolation operator.

Let $\mathcal{T}_h$ be a mesh generated as described in §1.3.2. Recall that a cell $K \in \mathcal{T}_h$ is constructed using the $\mathcal{C}^1$-diffeomorphism $T_K : \widehat{K} \to K$ defined in (1.51).

Definition 1.59 (Iso- and subparametric interpolation). *Let* $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ *be the reference finite element and let* $\{\widehat{K}, \widehat{P}_{\mathrm{geo}}, \widehat{\Sigma}_{\mathrm{geo}}\}$ *be the geometric reference finite element used to define* T_K. *When the two finite elements are identical, the interpolation is said to be* isoparametric, *whereas it is said to be* subparametric *whenever* $\widehat{P}_{\mathrm{geo}} \subsetneq \widehat{P}$.

Example 1.60. For scalar-valued finite elements, the most common example of subparametric interpolation is $\mathbb{P}_1 \subset \widehat{P}_{\mathrm{geo}} \neq \mathbb{P}_2 \subset \widehat{P}$. □

Elementary generation of finite elements. For all $K \in \mathcal{T}_h$, one must first define the counterpart of $V(\widehat{K})$, i.e., a Banach space $V(K)$ of $\mathbb{R}^m$-valued functions and a linear bijective mapping

$$\psi_K : V(K) \longrightarrow V(\widehat{K}).$$

Then, a set of $\mathcal{T}_h$-based finite elements can be defined as follows:

Proposition 1.61. *For* $K \in \mathcal{T}_h$, *the triplet* $\{K, P_K, \Sigma_K\}$ *defined by*

$$\begin{cases} K = T_K(\widehat{K}); \\ P_K = \{\psi_K^{-1}(\widehat{p});\, \widehat{p} \in \widehat{P}\}; \\ \Sigma_K = \{\{\sigma_{K,i}\}_{1 \le i \le n_{\mathrm{sh}}};\, \sigma_{K,i}(p) = \widehat{\sigma}_i(\psi_K(p)),\ \forall p \in P_K\}; \end{cases} \tag{1.58}$$

is a finite element. The local shape functions *are* $\theta_{K,i} = \psi_K^{-1}(\widehat{\theta}_i)$, $1 \le i \le n_{\mathrm{sh}}$, *and the associated* local interpolation operator *is*

$$\mathcal{I}_K : V(K) \ni v \longmapsto \mathcal{I}_K v = \sum_{i=1}^{n_{\mathrm{sh}}} \sigma_{K,i}(v)\,\theta_{K,i} \in P_K. \tag{1.59}$$

Proof. Left as an exercise. □

Proposition 1.62. *Let $\mathcal{I}_K$ be defined in* (1.59). *Then, the following diagram commutes:*

$$\begin{tikzcd} V(K) \arrow[r, "\psi_K"] \arrow[d, "\mathcal{I}_K"] & V(\widehat{K}) \arrow[d, "\mathcal{I}_{\widehat{K}}"] \\ P_K \arrow[r, "\psi_K"] & \widehat{P} \end{tikzcd}$$

Proof. Let v in $V(K)$. The definition (1.58) for $\{K, P_K, \Sigma_K\}$ implies

$$\mathcal{I}_{\widehat{K}}(\psi_K(v)) = \sum_{i=1}^{n_{\rm sh}} \widehat{\sigma}_i(\psi_K(v))\,\widehat{\theta}_i = \sum_{i=1}^{n_{\rm sh}} \sigma_{K,i}(v)\,\psi_K(\theta_{K,i}) = \psi_K(\mathcal{I}_K(v)),$$

owing to the linearity of ψ_K. $\square$

Proposition 1.62 plays an important role in the analysis of the interpolation error; see, e.g., the proof of Theorem 1.103. This result is the main motivation for the construction (1.58).

Example 1.63.

(i) Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a Lagrange finite element. Then, one may choose $V(\widehat{K}) = [\mathcal{C}^0(\widehat{K})]^m$. Defining $V(K)$ similarly and setting

$$\psi_K : V(K) \ni v \longmapsto \psi_K(v) = v \circ T_K \in V(\widehat{K}), \tag{1.60}$$

yields a linear bijective mapping. Then, for all $K \in \mathcal{T}_h$, the finite element $\{K, P_K, \Sigma_K\}$ constructed in Proposition 1.61 is a Lagrange finite element. Indeed, owing to

$$\sigma_i(v) = \widehat{\sigma}_i(\psi_K(v)) = \psi_K(v)(\widehat{a}_i) = v \circ T_K(\widehat{a}_i),$$

and setting $a_{K,i} = T_K(\widehat{a}_i)$ for $1 \le i \le n_{\rm sh}$, we infer that $\{a_{K,i}\}_{1\le i\le n_{\rm sh}}$ are the nodes of $\{K, P_K, \Sigma_K\}$.

(ii) For the Raviart–Thomas finite element (see §1.2.7), set $V(\widehat{K}) = \{v \in [L^p(\widehat{K})]^d;\ \nabla\cdot v \in L^s(\widehat{K})\}$ for $p > 2$, $s \ge q$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$, and define $V(K)$ similarly. The transformation $p \mapsto p \circ T_K$ does not map $V(K)$ to $V(\widehat{K})$. A suitable choice for ψ_K is the so-called Piola transformation; see §1.4.7. $\square$

Remark 1.64. In the literature the notation $\psi_K(v) = \widehat{v}$ is often used. Then, the relation $\mathcal{I}_{\widehat{K}}(\psi_K(v)) = \psi_K(\mathcal{I}_K(v))$ resulting from Proposition 1.62 can be rewritten in the form $\mathcal{I}_{\widehat{K}}\widehat{v} = \widehat{\mathcal{I}_K(v)}$. This notation can sometimes be misleading; in particular, it must not be confused with the notation $\widehat{x} = T_K^{-1}(x)$. $\square$

Finite element generation with rescaling. The technique described in Proposition 1.61 to generate finite elements is generally sufficient to construct approximation spaces. However, in the most general situation, a more sophisticated technique must be designed. To understand the nature of the problem, observe that the degrees of freedom in $\widehat{\Sigma}$ are constrained only locally by unisolvence. When constructing approximation spaces, one often wishes to enforce interface conditions between adjacent elements, thus introducing a new constraint on the degrees of freedom. Accordingly, we allow for a rescaling of the degrees of freedom in Σ_K.

Proposition 1.65. *For $K \in \mathcal{T}_h$, let $\alpha_K \in \mathbb{R}^{n_{\rm sh}}$ be such that $\alpha_{K,i} \neq 0$ for all $1 \leq i \leq n_{\rm sh}$. Define the triplet $\{K, P_K, \Sigma_K^\alpha\}$ by taking K and P_K as in* (1.58) *and by choosing the* local degrees of freedom *$\Sigma_K^\alpha = \{\sigma_{K,1}, \dots, \sigma_{K,n_{\rm sh}}\}$ such that, for all $1 \leq i \leq n_{\rm sh}$,*

$$\sigma_{K,i} : P_K \ni p \longmapsto \sigma_{K,i}(p) = \alpha_{K,i}\widehat{\sigma}_i(\psi_K(p)). \tag{1.61}$$

Then, $\{K, P_K, \Sigma_K^\alpha\}$ is a finite element. Furthermore, the local shape functions *on K are given by $\theta_{K,i} = \frac{1}{\alpha_{K,i}}\psi_K^{-1}(\widehat{\theta}_i)$, $1 \leq i \leq n_{\rm sh}$, and the associated local interpolation operator $\mathcal{I}_K^\alpha$ is defined as in* (1.59).

Proof. Left as an exercise. □

Proposition 1.66. *Let $\mathcal{I}_K^\alpha$ be the local interpolation operator associated with $\{K, P_K, \Sigma_K^\alpha\}$. Then, the diagram in Proposition* 1.62 *commutes.*

Proof. Straightforward verification. □

Example 1.67. An example where a rescaling of the degrees of freedom is needed is the Hermite finite element discussed in §1.4.6; see also Remark 1.72(i), Remark 1.88, and Remark 1.94 for further examples. □

1.4.2 Global interpolation operator

Using the $\mathcal{T}_h$-based family of finite elements $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ generated in Proposition 1.61 or Proposition 1.65, a global interpolation operator $\mathcal{I}_h$ can be constructed as follows: First, choose its domain to be

$$D(\mathcal{I}_h) = \{v \in [L^1(\Omega_h)]^m;\ \forall K \in \mathcal{T}_h,\ v_{|K} \in V(K)\}, \tag{1.62}$$

where Ω_h is the geometric interpolation of Ω defined in (1.52). For a function $v \in D(\mathcal{I}_h)$, the quantities $\sigma_{K,i}(v_{|K})$ are meaningful on all the mesh elements and for all $1 \leq i \leq n_{\rm sh}$. Then, the global interpolant $\mathcal{I}_h v$ can be specified elementwise using the local interpolation operators defined in (1.59), i.e.,

$$\forall K \in \mathcal{T}_h, \quad (\mathcal{I}_h v)_{|K} = \mathcal{I}_K(v_{|K}) = \sum_{i=1}^{n_{\rm sh}} \sigma_{K,i}(v_{|K})\theta_{K,i}.$$

Note that the function $\mathcal{I}_h v$ is defined on Ω_h. It may happen that $\mathcal{I}_h v$ is multivalued at the interfaces of the elements. This is not a major difficulty since $\mathcal{F}_h$ is of zero Lebesgue measure. The *global interpolation operator* is defined as follows:

$$\mathcal{I}_h : D(\mathcal{I}_h) \ni v \longmapsto \sum_{K\in\mathcal{T}_h} \sum_{i=1}^{n_{\mathrm{sh}}} \sigma_{K,i}(v_{|K})\theta_{K,i} \in W_h, \tag{1.63}$$

where W_h, the codomain of $\mathcal{I}_h$, is

$$W_h = \{v_h \in [L^1(\Omega_h)]^m; \forall K \in \mathcal{T}_h,\ v_{|K} \in P_K\}. \tag{1.64}$$

The space W_h is called an *approximation space*. In (1.63), we abuse the notation by implicitly extending $\theta_{K,i}$ by zero outside K.

One often wishes to impose additional regularity properties on the functions of W_h. At this stage, we only state the following general definition:

Definition 1.68 (Conforming approximation). *Let W_h be defined in* (1.64) *and let V be a Banach space. W_h is said to be V*-conforming *if $W_h \subset V$.*

Practical examples are investigated in §1.4.5, §1.4.6, §1.4.7, and §1.4.8.

1.4.3 Totally discontinuous spaces

Totally discontinuous spaces play an important role in the so-called Discontinuous Galerkin (DG) method; see §3.2.4, §5.6, and §6.3.2. Functions in such spaces only satisfy the simplest regularity requirement, namely to be integrable over Ω_h.

For the sake of simplicity, assume that $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ is such that the local degrees of freedom are of the form

$$\widehat{\sigma}_i : V(\widehat{K}) \ni \widehat{v} \longmapsto \widehat{\sigma}_i(\widehat{v}) = \tfrac{1}{\mathrm{meas}(\widehat{K})} \int_{\widehat{K}} \widehat{v}\,\widehat{\mathcal{K}}_i, \quad 1 \le i \le n_{\mathrm{sh}},$$

where $n_{\mathrm{sh}} = \dim(\widehat{P})$ and $\widehat{\mathcal{K}}_i$ is a smooth function on $\widehat{K}$; hence, $V(\widehat{K}) = L^1(\widehat{K})$ is an admissible choice. Define $V(K)$ similarly and choose the mapping defined in (1.60), i.e., $\psi_K(v) = v \circ T_K$. Construct the family $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ using Proposition 1.61. Then, for each $K \in \mathcal{T}_h$, setting $\mathcal{K}_{K,i} = \widehat{\mathcal{K}}_i \circ T_K^{-1}$, we infer

$$\sigma_{K,i}(v) = \widehat{\sigma}_i(\psi_K(v)) = \tfrac{1}{\mathrm{meas}(K)} \int_K v\,\mathcal{K}_{K,i}.$$

The local shape functions are $\theta_{K,i} = \widehat{\theta}_i \circ T_K^{-1}$, $1 \le i \le n_{\mathrm{sh}}$, where $\{\widehat{\theta}_1, \ldots, \widehat{\theta}_{n_{\mathrm{sh}}}\}$ are the local shape functions associated with $\{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{n_{\mathrm{sh}}}\}$.

Consider the approximation space

$$Z_{\mathrm{td},h} = \{v_h \in L^1(\Omega_h); \forall K \in \mathcal{T}_h,\ v_{|K} \in P_K\}. \tag{1.65}$$

Because local degrees of freedom can be taken independently on each mesh cell, $Z_{\mathrm{td},h}$ is of dimension $N_{\mathrm{el}} \times n_{\mathrm{sh}}$ where N_{el} is the number of mesh cells. For $v \in L^1(\Omega_h)$, the quantities $\sigma_{K,i}(v_{|K})$ are meaningful for $K \in \mathcal{T}_h$ and $1 \leq i \leq n_{\mathrm{sh}}$. Then, the global interpolation operator is constructed as follows:

$$\mathcal{I}_{\mathrm{td},h} : L^1(\Omega_h) \ni v \longmapsto \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{n_{\mathrm{sh}}} \tfrac{1}{\mathrm{meas}(K)} \left(\int_K v\, \mathcal{K}_{K,i} \right) \theta_{K,i} \in Z_{\mathrm{td},h}. \quad (1.66)$$

Example 1.69.

(i) Choosing $\widehat{P} = \mathbb{P}_k$ and assuming that the mesh is affine, we infer $P_K = \mathbb{P}_k$, so that the approximation space defined in (1.65) is

$$P^k_{\mathrm{td},h} = \{v_h \in L^1(\Omega_h);\, \forall K \in \mathcal{T}_h,\, v_{h|K} \in \mathbb{P}_k\}. \quad (1.67)$$

For instance, the space $P^0_{\mathrm{td},h} = \{v_h \in L^1(\Omega_h);\, \forall K \in \mathcal{T}_h,\, v_{h|K} \in \mathbb{P}_0\}$ is of dimension N_{el} and spanned by $\{1_K\}_{K \in \mathcal{T}_h}$ where 1_K is the characteristic function of K. The global interpolation operator associated with $P^0_{\mathrm{td},h}$ is

$$\mathcal{I}^0_{\mathrm{td},h} : L^1(\Omega_h) \ni v \longmapsto \mathcal{I}^0_{\mathrm{td},h} v = \sum_{K \in \mathcal{T}_h} \left(\tfrac{1}{\mathrm{meas}(K)} \int_K v \right) 1_K \in P^0_{\mathrm{td},h}.$$

(ii) A similar construction is possible with $\mathbb{Q}_k$ polynomials. For instance, on quadrangular meshes, the local shape functions can be taken to be tensor products of Legendre polynomials, i.e.,

$$\widehat{\theta}_{l_1 \ldots l_d}(\widehat{x}) = \widehat{\mathcal{E}}_{l_1}(\widehat{x}_1) \ldots \widehat{\mathcal{E}}_{l_d}(\widehat{x}_d), \quad 0 \leq l_1, \ldots, l_d \leq k.$$

This choice naturally yields hierarchical bases; see, e.g., §1.1.5 and §1.2.9. □

1.4.4 Discontinuous spaces with patch-test

In this section, we assume that $\mathcal{T}_h$ is a simplicial, affine, and geometrically conforming mesh.

The Crouzeix–Raviart approximation space. Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be the Crouzeix–Raviart finite element introduced in §1.2.6. Set $V(\widehat{K}) = W^{1,1}(\widehat{K})$, define $V(K)$ similarly, and choose the mapping defined in (1.60), i.e., $\psi_K(v) = v \circ T_K$. Construct the family $\{K, P_K, \Sigma_K\}_{K \in \mathcal{T}_h}$ using Proposition 1.61. Then, for each $K \in \mathcal{T}_h$, letting $F_{K,i} = T_K(\widehat{F}_i)$, $0 \leq i \leq d$, where $\{\widehat{F}_0, \ldots, \widehat{F}_d\}$ are the faces of $\widehat{K}$, the local degrees of freedom are

$$\sigma_{K,i}(v) = \widehat{\sigma}_i(\psi_K(v)) = \tfrac{1}{\mathrm{meas}(\widehat{F}_i)} \int_{\widehat{F}_i} \psi_K(v) = \tfrac{1}{\mathrm{meas}(F_{K,i})} \int_{F_{K,i}} v. \quad (1.68)$$

In addition, since the mesh is affine, $P_K = \mathbb{P}_1$. As a result, $\{K, P_K, \Sigma_K\}$ is a Crouzeix–Raviart finite element.

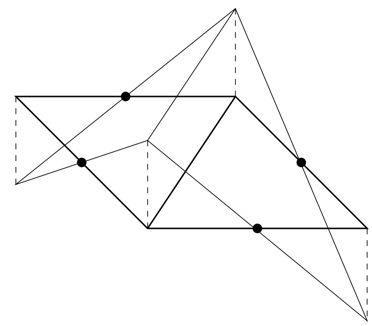


Fig. 1.17. Global shape function for the Crouzeix–Raviart approximation space. The support is materialized by thick lines and the graph by thin lines.

Consider the so-called *Crouzeix–Raviart approximation space*

$$P^1_{\mathrm{pt},h} = \{v_h \in L^1(\Omega_h);\, \forall K \in \mathcal{T}_h,\ v_{h|K} \in \mathbb{P}_1;\\ \forall F \in \mathcal{F}^{\mathrm{i}}_h,\ \int_F [\![v_h]\!] = 0\}. \tag{1.69}$$

Recall that $\mathcal{F}^{\mathrm{i}}_h$ denotes the set of interior faces (interfaces) in the mesh and $[\![v_h]\!]$ the jump of v_h across interfaces. For $F \in \mathcal{F}_h$, consider the function φ_F with support consisting of the one or two simplices to which F belongs and such that on each of these simplices, say K, the function $\varphi_{F|K}$ is the local shape function of $\{K, P_K, \Sigma_K\}$ associated with the face F. The graph of a function φ_F is shown in Figure 1.17.

Lemma 1.70. $\varphi_F \in P^1_{\mathrm{pt},h}$.

Proof. Let $F \in \mathcal{F}^{\mathrm{i}}_h$, say $F = K_1 \cap K_2$. Since $\varphi_{F|K_1}$ is the local shape function of $\{K_1, P_{K_1}, \Sigma_{K_1}\}$ associated with F, (1.68) implies

$$\int_F \varphi_{F|K_1} = \mathrm{meas}(F).$$

Similarly, $\int_F \varphi_{F|K_2} = \mathrm{meas}(F)$, proving that $\int_F [\![\varphi_F]\!] = 0$. Use the same argument to prove $\int_{F'} [\![\varphi_F]\!] = 0$ for all faces $F' \neq F$. Since the restriction of φ_F to any mesh element is in $\mathbb{P}_1$, $\varphi_F \in P^1_{\mathrm{pt},h}$. $\square$

For $F \in \mathcal{F}_h$, define the linear form $\gamma_F : P^1_{\mathrm{pt},h} \ni v_h \mapsto \frac{1}{\mathrm{meas}(F)} \int_F v_h$. Although $v_h \in P^1_{\mathrm{pt},h}$ may be multi-valued at F, the quantity $\gamma_F(v_h)$ is single-valued since $\int_F [\![v_h]\!] = 0$.

Proposition 1.71. *$\{\varphi_F\}_{F\in\mathcal{F}_h}$ is a basis for $P^1_{\mathrm{pt},h}$, and $\{\gamma_F\}_{F\in\mathcal{F}_h}$ is a basis for $\mathcal{L}(P^1_{\mathrm{pt},h};\mathbb{R})$.*

Proof. The proof is based on the fact that (with obvious notation) $\gamma_{F'}(\varphi_F) = \delta_{FF'}$ for $F, F' \in \mathcal{F}_h$. Consider a set of real numbers $\{\alpha_F\}_{F\in\mathcal{F}_h}$, and assume that the function $w = \sum_{F\in\mathcal{F}_h} \alpha_F \varphi_F$ vanishes identically. Then, $\alpha_F = \gamma_F(w) = 0$; hence, the set $\{\varphi_F\}_{F\in\mathcal{F}_h}$ is linearly independent. Let $v_h \in P^1_{\mathrm{pt},h}$ and set

$$w_h = \sum_{F\in\mathcal{F}_h} \left(\tfrac{1}{\mathrm{meas}(F)} \int_F v_h \right) \varphi_F.$$

Then, for all $K \in \mathcal{T}_h$, $v_{h|K}$ and $w_{h|K}$ are in P_K, and for all $\sigma \in \Sigma_K$, $\sigma(v_{h|K}) = \sigma(w_{h|K})$. Unisolvence implies $v_{h|K} = w_{h|K}$. This shows that $\{\varphi_F\}_{F\in\mathcal{F}_h}$ is a basis for $P^1_{\mathrm{pt},h}$. The proof is easily completed. □

Proposition 1.71 implies that $P^1_{\mathrm{pt},h}$ is a space of dimension N_{ed} in two dimensions and N_{f} in three dimensions. The linear forms $\{\gamma_F\}_{F\in\mathcal{F}_h}$ are called the *global degrees of freedom* in $P^1_{\mathrm{pt},h}$, and $\{\varphi_F\}_{F\in\mathcal{F}_h}$ are called the *global shape functions*.

For a function $v \in W^{1,1}(\Omega_h)$, the quantity $\gamma_F(v)$ is meaningful (and single-valued) for all $F \in \mathcal{F}_h$. The so-called *global Crouzeix–Raviart interpolation operator* is constructed as follows:

$$\mathcal{I}_h^{\mathrm{CR}} : W^{1,1}(\Omega_h) \ni v \longmapsto \mathcal{I}_h^{\mathrm{CR}} v = \sum_{F\in\mathcal{F}_h} \left(\tfrac{1}{\mathrm{meas}(F)} \int_F v \right) \varphi_F \in P^1_{\mathrm{pt},h}. \quad (1.70)$$

Note that $P^1_{\mathrm{pt},h}$ is the codomain of $\mathcal{I}_h^{\mathrm{CR}}$.

Remark 1.72.

(i) If the degrees of freedom in $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ are chosen to be the integral over the faces instead of the mean-value (see Remark 1.38(ii)), Proposition 1.65 must be used to generate the family $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$. Indeed, taking $\alpha_{K,i} = \frac{\mathrm{meas}(F_{K,i})}{\mathrm{meas}(\widehat{F}_i)}$ for $0 \le i \le d$ in (1.61) yields $\sigma_{K,i}(v) = \int_{F_{K,i}} v$. Then, constructing φ_F as before, Lemma 1.70 holds. If Proposition 1.61 had been used instead, then $\sigma_{K,i}(v) = \frac{\mathrm{meas}(\widehat{F}_i)}{\mathrm{meas}(F_{K,i})} \int_{F_{K,i}} v$, yielding $\theta_{K,i} = \frac{1}{\mathrm{meas}(\widehat{F}_i)}(1 - \frac{\lambda_{K,i}}{d})$ where $\lambda_{K,i}$ is the i-th barycentric coordinate of K. Then, $\theta_{K,i} = \frac{1}{\mathrm{meas}(\widehat{F}_i)}$ on $F_{K,i}$; hence, $\int_F [\![\varphi_F]\!] \neq 0$, i.e., $\varphi_F \notin P^1_{\mathrm{pt},h}$ (unless $\widehat{K}$ is equilateral).

(ii) Since $[\![v_h]\!]_F : F \ni x \mapsto [\![v_h]\!](x) \in \mathbb{R}$ is linear, the condition $\int_F [\![v_h]\!] = 0$ in (1.69) is equivalent to the continuity of v_h at the center of gravity of F. □

Extension to high-degree polynomials. The extension of the Crouzeix–Raviart approximation space $P^1_{\mathrm{pt},h}$ to higher-degree polynomials is somewhat technical. When approximating PDEs, one often wishes to impose the continuity of the moments, up to order $k-1$, of the functions in $P^k_{\mathrm{pt},h}$ on any interface of the mesh. This condition is known as the *patch-test*; see [IrR72]. The space $P^k_{\mathrm{pt},h}$ is thus defined as follows:

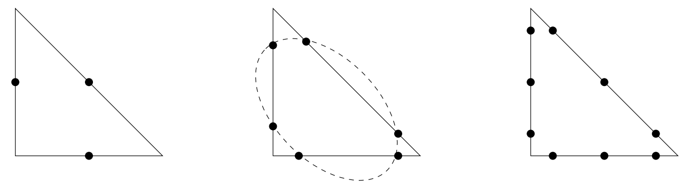


Fig. 1.18. Continuity points for functions in $P^k_{\mathrm{pt},h}$: $k=1$ (left); $k=2$ (center); and $k=3$ (right). For $k=2$, the six points lie on an ellipse.

$$P^k_{\mathrm{pt},h} = \{v_h \in L^1(\Omega_h);\, \forall K \in \mathcal{T}_h,\ v_{h|K} \in \mathbb{P}_k;\\ \forall F \in \mathcal{F}^{\mathrm{i}}_h,\, \forall q \in \mathbb{P}_{k-1},\ \int_F [\![v_h]\!]\, q = 0\}. \tag{1.71}$$

In two dimensions, the patch-test is equivalent to the continuity of v_h at the k Gauß points located on each face of K; see Figure 1.18 and Definition 8.1. These points (completed with internal points for $k \geq 3$) can be used to define local Lagrange degrees of freedom, say Σ, on the simplex K if k is odd, but this construction is not possible if k is even. For instance, if $k=2$, the six Gauß points lie on the ellipse of equation $2 - 3(\lambda^2_{0,K} + \lambda^2_{1,K} + \lambda^2_{2,K}) = 0$, where $\{\lambda_{0,K}, \lambda_{1,K}, \lambda_{2,K}\}$ are the barycentric coordinates of the simplex K. This means that the so-called *Fortin–Soulié bubble*

$$b_K = 2 - 3(\lambda^2_{0,K} + \lambda^2_{1,K} + \lambda^2_{2,K}), \tag{1.72}$$

vanishes at these six Gauß points. Then, because $b_K \in \mathbb{P}_2$, the linear mapping (1.34) associated with the triplet $\{K, \mathbb{P}_2, \Sigma\}$ is not bijective; hence, $\{K, \mathbb{P}_2, \Sigma\}$ is not a finite element.

In the three-dimensional case, a similar construction is possible. However, the patch-test no longer implies point-continuity except for $k=1$.

Remark 1.73. The space $P^2_{\mathrm{pt},h}$ can be used to approximate PDEs owing to a decomposition involving H^1-conforming quadratics and Fortin-Soulié bubbles; see [FoS83] and Exercise 1.12. □

1.4.5 H^1-conforming spaces based on Lagrange finite elements

The goal of this section is to construct a H^1-conforming subspace of the approximation space W_h defined in (1.64). We assume that the mesh is *geometrically conforming* (but not necessarily affine) and that the reference finite element is a *Lagrange finite element*. Hence, setting $V(\widehat{K}) = [\mathcal{C}^0(\widehat{K})]^m$, defining $V(K)$ similarly, and choosing the mapping ψ_K defined in (1.60), the family $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ constructed as in Proposition 1.61 is a family of Lagrange finite elements; see Example 1.63(i).

Consider the space

$$V_h = \{v_h \in W_h; \forall F \in \mathcal{F}_h^{\mathrm{i}},\ [\![v_h]\!]_F = 0\}. \tag{1.73}$$

The main motivation for introducing V_h is the following:

Proposition 1.74. $V_h \subset [H^1(\Omega_h)]^m$.

Proof. Assume $m = 1$. For vector-valued functions, the proof below is simply applied component by component. Let $v_h \in V_h$. Since its restriction to every $K \in \mathcal{T}_h$ is a polynomial, it is differentiable in the classical sense. For $1 \le j \le d$, consider the function $w_j \in L^2(\Omega_h)$ defined on $K \in \mathcal{T}_h$ by $w_{j|K} = \partial_j\big(v_{h|K}\big)$. Let $\phi \in \mathcal{D}(\Omega_h)$. Using the Green formula yields

$$\int_{\Omega_h} w_j\phi = \sum_{K\in\mathcal{T}_h}\int_K w_j\phi = -\sum_{K\in\mathcal{T}_h}\int_K v_{h|K}\,\partial_j\phi + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\phi v_{h|K} n_{K,j},$$

where ∂K is the boundary of K and $n_{K,j}$ is the j-th component of the outer normal to K. Use the fact that ϕ vanishes at the boundary of Ω_h, regroup interface terms, and employ the notation of §1.3.4 to infer

$$\int_{\Omega_h} w_j\phi = -\int_{\Omega_h} v_h\partial_j\phi + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}}\int_F \phi e_j\cdot[\![v_h]\!]_F,$$

where $\{e_1,\dots,e_d\}$ is the canonical basis of $\mathbb{R}^d$. Owing to $[\![v_h]\!]_F = 0$, $\int_{\Omega_h} w_j\phi = -\int_{\Omega_h} v_h\partial_j\phi$. Therefore, for $1 \le j \le d$, the distributional derivative of v_h with respect to the j-th coordinate is w_j. Since $w_j \in L^2(\Omega_h)$, $v_h \in H^1(\Omega_h)$. $\square$

The next question is to determine how the zero-jump condition in (1.73) can be enforced using the local degrees of freedom of adjacent cells. For $K \in \mathcal{T}_h$, denote by $\{a_{K,1},\dots,a_{K,n_{\mathrm{sh}}}\}$ the Lagrange nodes (not to be confused with the geometric nodes of K). Assume that:

(SC1) All the faces of $\widehat{K}$ have the same number of nodes, say $n^\partial_{n_{\mathrm{sh}}}$.

(SC2) Consider a face $\widehat{F}$ of $\widehat{K}$ and let $\{a_{1,\widehat{F}},\dots,a_{n^\partial_{n_{\mathrm{sh}}},\widehat{F}}\}$ be its nodes. Define $\widehat{P}_{\widehat{F}} = \{\widehat{q};\ \exists\widehat{p}\in\widehat{P},\ \widehat{q} = \widehat{p}_{|\widehat{F}}\}$ and $\Sigma_{\widehat{F}} = \{\widehat{\sigma}_1,\dots,\widehat{\sigma}_{n^\partial_{n_{\mathrm{sh}}}}\}$ such that $\widehat{\sigma}_i(\widehat{q}) = \widehat{q}(a_{i,\widehat{F}})$ for $\widehat{q}\in\widehat{P}_{\widehat{F}}$ and $1 \le i \le n^\partial_{n_{\mathrm{sh}}}$. Then, $\{\widehat{F},\widehat{P}_{\widehat{F}},\widehat{\Sigma}_{\widehat{F}}\}$ is a finite element.

(SC3) For all $F \in \mathcal{F}_h^{\mathrm{i}}$ with $F = K_1 \cap K_2$, assume that there are renumberings of the Lagrange nodes of K_1 and K_2 such that (see Figure 1.19):

$$\forall i \in \{1,\dots,n^\partial_{n_{\mathrm{sh}}}\},\quad a_{K_1,i} = a_{K_2,i}.$$

Lemma 1.75. *Assume* (SC1)–(SC3)*. Let* $v_h \in W_h$*. Then,* $[\![v_h]\!]_F = 0$ *for all* $F \in \mathcal{F}_h^{\mathrm{i}}$ *if and only if, for all* $F \in \mathcal{F}_h^{\mathrm{i}}$ *such that* $F = K_1 \cap K_2$*,*

$$\forall i \in \{1,\dots,n^\partial_{n_{\mathrm{sh}}}\},\quad v_{h|K_1}(a_{K_1,i}) = v_{h|K_2}(a_{K_2,i}). \tag{1.74}$$

Fig. 1.19. Compatible (left) and incompatible (right) position of nodes at an interface for a geometrically conforming mesh.

Proof. The direct statement is evident. To prove the converse, let $v_h \in W_h$, let $F \in \mathcal{F}_h^{\mathrm{i}}$ with $F = K_1 \cap K_2$, and assume (1.74). Let T_1 and T_2 be the geometric transformations associated with K_1 and K_2, respectively. Set $v_1 = v_{h|K_1}$ and $v_2 = v_{h|K_2}$. Since the mesh is geometrically conforming, there are renumberings of the geometric nodes of K_1 and K_2 such that (1.53) holds. Owing to (SC3), $\widehat{a}_{i,\widehat{F}} = T_{1|F}^{-1}(a_{K_1,i}) = T_{2|F}^{-1}(a_{K_2,i})$ for $1 \leq i \leq n_{n_{\mathrm{sh}}}^{\partial}$. Define $\widehat{v}_{1|\widehat{F}} = v_{1|F} \circ T_{1|F}$ and $\widehat{v}_{2|\widehat{F}} = v_{2|F} \circ T_{2|F}$. Then, (1.74) implies

$$\forall i \in \{1, \ldots, n_{n_{\mathrm{sh}}}^{\partial}\}, \quad \widehat{v}_{1|\widehat{F}}(\widehat{a}_{i,\widehat{F}}) = \widehat{v}_{2|\widehat{F}}(\widehat{a}_{i,\widehat{F}}).$$

Owing to (SC2), $\widehat{v}_{1|\widehat{F}} = \widehat{v}_{2|\widehat{F}}$, and since the geometric transformations are bijective, this readily implies $v_{1|F} = v_{2|F}$. □

Remark 1.76. All the Lagrange finite elements introduced in §1.2.3–§1.2.5 satisfy assumption (SC2). This is not the case for the Crouzeix–Raviart finite element considered as a Lagrange finite element. □

Let $\{a_1, \ldots, a_N\} = \bigcup_{K \in \mathcal{T}_h} \{a_{K,1}, \ldots, a_{K,n_{\mathrm{sh}}}\}$ be the set of all the *Lagrange nodes*. For $K \in \mathcal{T}_h$ and $m \in \{1, \ldots, n_{\mathrm{sh}}\}$, let $\mathtt{j}(K,m) \in \{1, \ldots, N\}$ be the corresponding index of the Lagrange node. Let $\{\varphi_1, \ldots, \varphi_N\}$ be the set of functions in W_h defined elementwise by $\varphi_{i|K}(a_{K,m}) = \delta_{mn}$ if there is $n \in \{1, \ldots, n_{\mathrm{sh}}\}$ such that $i = \mathtt{j}(K,n)$ and 0 otherwise. This implies $\varphi_i(a_j) = \delta_{ij}$ for $1 \leq i, j \leq N$.

Lemma 1.77. *Under the assumptions of Lemma* 1.75, $\varphi_i \in V_h$.

Proof. Use the converse statement in Lemma 1.75. □

For $1 \leq i \leq N$, define the linear form $\gamma_i : V_h \ni v_h \mapsto v_h(a_i) \in \mathbb{R}$.

Proposition 1.78. $\{\varphi_1, \ldots, \varphi_N\}$ *is a basis for* V_h, *and* $\{\gamma_1, \ldots, \varphi_N\}$ *is a basis for* $\mathcal{L}(V_h; \mathbb{R})$.

Proof. The family $\{\varphi_1, \ldots, \varphi_N\}$ is linearly independent; indeed, if the function $\sum_{j=1}^N \alpha_j \varphi_j$ vanishes identically, evaluating it at the node a_i yields $\alpha_i = 0$. Now, let $v_h \in V_h$. Owing to the direct statement of Lemma 1.75, v_h is single-valued at all the Lagrange nodes. Set $w_h = \sum_{i=1}^N v_h(a_i)\varphi_i$. Then, for all $K \in \mathcal{T}_h$, $v_{h|K}$ and $w_{h|K}$ are in P_K and coincide at the nodes $\{a_{K,1}, \ldots, a_{K,n_{\mathrm{sh}}}\}$. Unisolvence implies $v_{h|K} = w_{h|K}$. Hence, $\{\varphi_1, \ldots, \varphi_N\}$ is a basis for V_h. Proving that $\{\gamma_1, \ldots, \varphi_N\}$ is a basis for $\mathcal{L}(V_h; \mathbb{R})$ is then straightforward. □

	dim.	$k=1$	$k=2$	$k=3$
$P^k_{\mathrm{c},h}$	2	N_{v}	$N_{\mathrm{v}}+N_{\mathrm{ed}}$	$N_{\mathrm{v}}+2N_{\mathrm{ed}}+N_{\mathrm{el}}$
$Q^k_{\mathrm{c},h}$	2	N_{v}	$N_{\mathrm{v}}+N_{\mathrm{ed}}+N_{\mathrm{el}}$	$N_{\mathrm{v}}+2N_{\mathrm{ed}}+4N_{\mathrm{el}}$
$P^k_{\mathrm{c},h}$	3	N_{v}	$N_{\mathrm{v}}+N_{\mathrm{ed}}$	$N_{\mathrm{v}}+2N_{\mathrm{ed}}+N_{\mathrm{f}}$
$Q^k_{\mathrm{c},h}$	3	N_{v}	$N_{\mathrm{v}}+N_{\mathrm{ed}}+N_{\mathrm{f}}+N_{\mathrm{el}}$	$N_{\mathrm{v}}+2N_{\mathrm{ed}}+4N_{\mathrm{f}}+8N_{\mathrm{el}}$

Table 1.4. Dimension of H^1-conforming spaces constructed using a geometrically conforming mesh and various Lagrange finite elements. The second column indicates the space dimension. N_{el} denotes the number of cells in the mesh, N_{f} the number of faces, N_{ed} the number of edges, and N_{v} the number of vertices.

Proposition 1.78 implies that V_h is a space of dimension N. The linear forms $\{\gamma_1,\ldots,\gamma_N\}$ are called the *global degrees of freedom* in V_h, and $\{\varphi_1,\ldots,\varphi_N\}$ are called the *global shape functions*. The *global Lagrange interpolation operator* is defined as follows:

$$\mathcal{I}_h : \mathcal{C}^0(\overline{\Omega}_h) \ni v \longmapsto \sum_{i=1}^{N} v(a_i)\varphi_i \in V_h. \tag{1.75}$$

Note that the domain of $\mathcal{I}_h$ can also be taken to be $H^s(\Omega_h)$ for $s > \frac{d}{2}$.

We shall often consider the approximation spaces

$$P^k_{\mathrm{c},h} = \{v_h \in \mathcal{C}^0(\overline{\Omega}_h);\, \forall K \in \mathcal{T}_h,\, v_h \circ T_K \in \mathbb{P}_k\}, \tag{1.76}$$
$$Q^k_{\mathrm{c},h} = \{v_h \in \mathcal{C}^0(\overline{\Omega}_h);\, \forall K \in \mathcal{T}_h,\, v_h \circ T_K \in \mathbb{Q}_k\}. \tag{1.77}$$

The dimension of these spaces is given in Table 1.4 for the first values of k. The subscript 'c' refers to the continuity condition across mesh interfaces (for simplicity, it was not used in the one-dimensional cases treated in §1.1).

Example 1.79. Assume that $\mathcal{T}_h$ is composed of triangles in dimension 2.

(i) Let $\{S_1,\ldots,S_{N_{\mathrm{v}}}\}$ be the mesh vertices. For $1 \le i \le N_{\mathrm{v}}$, the global shape functions in $P^1_{\mathrm{c},h}$ satisfy $\varphi_i(S_j) = \delta_{ij}$ for $1 \le i,j \le N_{\mathrm{v}}$; see the left panel of Figure 1.20. Owing to Proposition 1.78, the set $\{\varphi_1,\ldots,\varphi_{N_{\mathrm{v}}}\}$ is a basis for $P^1_{\mathrm{c},h}$.

(ii) Let $\{T_1,\ldots,T_{N_{\mathrm{ed}}}\}$ be the edge midpoints. For $1 \le i \le N_{\mathrm{v}}$, let $\varphi_{i,0} \in P^2_{\mathrm{c},h}$ be such that $\varphi_{i,0}(S_j) = \delta_{ij}$ and $\varphi_{i,0}(T_j) = 0$. In addition, for $1 \le i \le N_{\mathrm{ed}}$, let $\varphi_{i,1} \in P^2_{\mathrm{c},h}$ be such that $\varphi_{i,1}(S_j) = 0$ and $\varphi_{i,1}(T_j) = \delta_{ij}$. The functions $\varphi_{i,0}$ and $\varphi_{i,1}$ are illustrated in the central and right panels of Figure 1.20. Owing to Proposition 1.78, $\{\varphi_{1,0},\ldots,\varphi_{N_{\mathrm{v}},0},\varphi_{1,1},\ldots,\varphi_{N_{\mathrm{ed}},1}\}$ is a basis for $P^2_{\mathrm{c},h}$. □

Remark 1.80. Lemma 1.77 can be easily extended to $\mathbb{R}^m$-valued functions by considering the functions $\varphi_{i,n}$ for $1 \le i \le N$ and $1 \le n \le m$, such that $\varphi_{i,n}(a_j) = \delta_{ij} e_n$, where e_n is the n-th vector of the canonical basis of $\mathbb{R}^m$. □

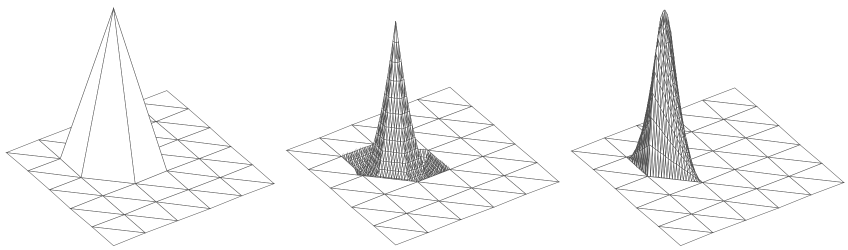


Fig. 1.20. Global shape functions for H^1-conforming spaces in two dimensions: $P^1_{c,h}$ (left) and $P^2_{c,h}$ (center and right).

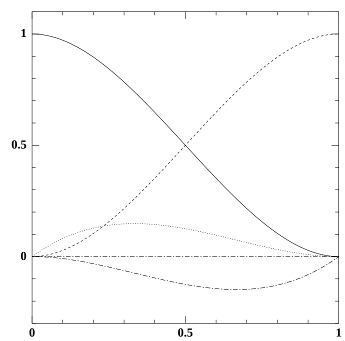


Fig. 1.21. Local shape functions for the Hermite finite element in the reference interval $[0, 1]$.

1.4.6 H^2-conforming spaces

In dimension 1, a H^2-conforming space can be constructed using Hermite finite elements. Let $\widehat{K} = [0, 1]$ be the reference interval, set $\widehat{P} = \mathbb{P}_3$, and define the local degrees of freedom $\widehat{\Sigma} = \{\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3, \widehat{\sigma}_4\}$ to be

$$\widehat{\sigma}_1(\widehat{p}) = \widehat{p}(0), \qquad \widehat{\sigma}_2(\widehat{p}) = \widehat{p}'(0), \qquad \widehat{\sigma}_3(\widehat{p}) = \widehat{p}(1), \qquad \widehat{\sigma}_4(\widehat{p}) = \widehat{p}'(1).$$

One readily verifies that $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ is a finite element; it is called a *Hermite finite element*. The local shape functions $\{\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{\theta}_4\}$ are (see Figure 1.21)

$$\widehat{\theta}_1(t) = (2t + 1)(t - 1)^2, \qquad \widehat{\theta}_2(t) = t(t - 1)^2,$$
$$\widehat{\theta}_3(t) = (3 - 2t)t^2, \qquad \widehat{\theta}_4(t) = (t - 1)t^2.$$

Owing to the choice of the local degrees of freedom, an admissible choice for $V(\widehat{K})$ is $\mathcal{C}^1(\widehat{K})$ (or $H^s(\widehat{K})$ with $s > \frac{3}{2}$).

Let $\Omega =]a, b[$ and let $\mathcal{T}_h = \{I_i\}_{0 \le i \le N}$ be the one-dimensional mesh of Ω introduced in §1.1.1. Consider the affine transformation T_i defined in

(1.21), i.e., $T_i : \widehat{K} \ni t \mapsto x = x_i + th \in I_i$. The goal is to generate a family of Hermite finite elements over the mesh intervals. To this end, one must use Proposition 1.65 since the degrees of freedom in $\widehat{\Sigma}$ are of different dimensionality. Specifically, set $V(I_i) = \mathcal{C}^1(I_i)$ and choose the mapping $\psi_{I_i} : V(I_i) \ni v \mapsto \psi_{I_i}(v) = v \circ T_i \in V(\widehat{K})$. Set $\alpha_{i,1} = \alpha_{i,3} = 1$, $\alpha_{i,2} = \alpha_{i,4} = \frac{1}{h_i}$, and $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3}, \alpha_{i,4})$. Using Proposition 1.65 to generate the family $\{I_i, P_i, \Sigma_i\}_{0 \le i \le N}$, we infer $P_i = \mathbb{P}_3$ and that the local degrees of freedom are

$$\begin{aligned} \sigma_{i,1}(p) &= p(x_i), & \sigma_{i,2}(p) &= p'(x_i), \\ \sigma_{i,3}(p) &= p(x_{i+1}), & \sigma_{i,4}(p) &= p'(x_{i+1}). \end{aligned}$$

The local shape functions are

$$\begin{aligned} \theta_{i,1} &= \widehat{\theta}_1 \circ T_i^{-1}, & \theta_{i,2} &= h_i \widehat{\theta}_2 \circ T_i^{-1}, \\ \theta_{i,3} &= \widehat{\theta}_3 \circ T_i^{-1}, & \theta_{i,4} &= h_i \widehat{\theta}_4 \circ T_i^{-1}, \end{aligned}$$

and the *local Hermite interpolation operator* is defined as follows:

$$\mathcal{I}_{I_i}^{\mathrm{H}} : \mathcal{C}^1(I_i) \ni v \longmapsto \sum_{m=1}^{4} \sigma_{i,m}(v) \theta_{i,m} \in \mathbb{P}_3. \tag{1.78}$$

Consider the so-called *Hermite approximation space*

$$H_h = \{v_h \in \mathcal{C}^1(\overline{\Omega}); \forall i \in \{0, \dots, N\}, \, v_{h|I_i} \in \mathbb{P}_3\}. \tag{1.79}$$

The main motivation for introducing H_h is the following:

Proposition 1.81. $H_h \subset H^2(\Omega)$.

Proof. Adapt the proof of Lemma 1.3. □

Introduce the functions $\{\varphi_{0,0}, \dots, \varphi_{N+1,0}, \varphi_{0,1}, \dots, \varphi_{N+1,1}\}$ such that

$$\varphi_{i,0}(x) = \begin{cases} \theta_{i-1,3}(x) & \text{if } x \in I_{i-1}, \\ \theta_{i,1}(x) & \text{if } x \in I_i, \\ 0 & \text{otherwise}, \end{cases} \qquad \varphi_{i,1}(x) = \begin{cases} \theta_{i-1,4}(x) & \text{if } x \in I_{i-1}, \\ \theta_{i,2}(x) & \text{if } x \in I_i, \\ 0 & \text{otherwise}, \end{cases}$$

with obvious modifications if $i = 0$ or $N + 1$.

Lemma 1.82. *$\varphi_{i,0} \in H_h$ and $\varphi_{i,1} \in H_h$.*

Proof. Left as an exercise. □

For $i \in \{0, \dots, N\}$, consider the linear forms

$$\begin{aligned} \gamma_{i,0} &: \mathcal{C}^1(\overline{\Omega}) \ni v \longmapsto \gamma_{i,0}(v) = v(x_i), \\ \gamma_{i,1} &: \mathcal{C}^1(\overline{\Omega}) \ni v \longmapsto \gamma_{i,1}(v) = v'(x_i). \end{aligned}$$

Proposition 1.83. *$\{\varphi_{i,l}\}_{0\le i\le N, 0\le l\le 1}$ is a basis for H_h, and $\{\gamma_{i,l}\}_{0\le i\le N, 0\le l\le 1}$ is a basis for $\mathcal{L}(H_h;\mathbb{R})$.*

Proof. Use the fact that $\gamma_{i,l}(\varphi_{i'l'}) = \delta_{ii'}\delta_{ll'}$ and that on each interval I_i, a function in H_h is a polynomial of degree at most 3 and is, therefore, uniquely determined by its value and that of its first derivative at the endpoints x_i and x_{i+1}; details are left as an exercise. $\square$

Proposition 1.83 implies that H_h is a space of dimension $2(N+2)$. The linear forms $\{\gamma_{i,l}\}_{0\le i\le N, 0\le l\le 1}$ are called the *global degrees of freedom* in H_h, and the functions $\{\varphi_{i,l}\}_{0\le i\le N, 0\le l\le 1}$ are called the *global shape functions*.

Define the *global Hermite interpolation operator* $\mathcal{I}_h^{\mathrm{H}}$ with codomain H_h as follows:

$$\mathcal{I}_h^{\mathrm{H}} : \mathcal{C}^1(\overline{\Omega}) \ni v \longmapsto \sum_{i=0}^{N+1} \gamma_{i,0}(v)\varphi_{i,0} + \sum_{i=0}^{N+1} \gamma_{i,1}(v)\varphi_{i,1} \in H_h. \tag{1.80}$$

$\mathcal{I}_h^{\mathrm{H}}$ is a linear operator, and $\mathcal{I}_h^{\mathrm{H}} v$ is the unique function in H_h that coincides with v and its derivatives at all the mesh points.

In dimension 2, the construction of H^2-conforming spaces is more technical. A classical example uses Argyris finite elements; see, e.g., [Cia91, p. 88].

1.4.7 $H(\mathrm{div})$-conforming spaces

Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be the Raviart–Thomas finite element introduced in §1.2.7. Choose $V(\widehat{K}) = \{v \in [L^p(\widehat{K})]^d;\ \nabla\cdot v \in L^s(\widehat{K})\}$, with $p > 2$ and $s \ge q$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$, and define $V(K)$ similarly. Since $\psi_K(v) = v \circ T_K$ does not map $V(K)$ to $V(\widehat{K})$, one introduces the so-called *Piola transformation*

$$\psi_K : V(K) \ni v \longmapsto \psi_K(v)(\widehat{x}) = \det(J_K)\, J_K^{-1} \left[v \circ T_K(\widehat{x})\right] \in \widehat{V}(\widehat{K}), \tag{1.81}$$

where J_K is the Jacobian matrix of T_K.

Lemma 1.84. *Let $v \in V(K)$ and set $\widehat{v} = \psi_K(v)$. Then, whenever the left-hand sides are meaningful, the following identities hold:*

(i) $\nabla_x v = \frac{1}{\det(J_K)} J_K \left[\nabla_{\widehat{x}} \psi_K(v)\right] J_K^{-1}$ *and* $\int_F v\cdot n = \int_{\widehat{F}} \widehat{v}\cdot\widehat{n}$.
(ii) $\int_K q\, \nabla_x\cdot v = \int_{\widehat{K}} \widehat{q}\, \nabla_{\widehat{x}}\cdot\widehat{v}$ *and* $\int_K v\cdot\nabla_x q = \int_{\widehat{K}} \widehat{v}\cdot\nabla_{\widehat{x}}\widehat{q}$ *with* $\widehat{q} = q \circ T_K$.

Proof. Observe that $\nabla_x q = (J_K^{-1})^T \nabla_{\widehat{x}}\widehat{q}$ and $\nabla_x\cdot v(x) = \frac{1}{\det(J_K)} \nabla_{\widehat{x}}\cdot\widehat{v}(\widehat{x})$. $\square$

Construct the family $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ using Proposition 1.61. Then, for each $K \in \mathcal{T}_h$, letting $F_{K,i} = T_K(\widehat{F}_i)$ with $0 \le i \le d$ where $\{\widehat{F}_0, \dots, \widehat{F}_d\}$ are the faces of $\widehat{K}$, Lemma 1.84(i) implies that the local degrees of freedom are

$$\sigma_{K,i}(v) = \int_{F_{K,i}} v\cdot n_i, \tag{1.82}$$

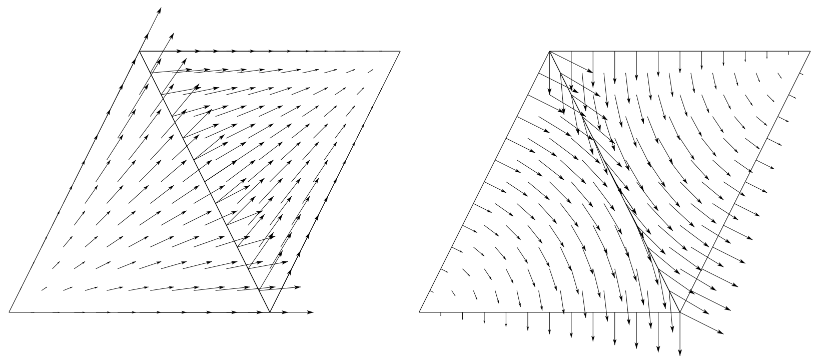


Fig. 1.22. Global shape functions associated with the Raviart–Thomas (left) and the Nédélec (right) finite elements in dimension 2. The normal (resp., tangential) component of the Raviart–Thomas (resp., Nédélec) global shape function is continuous across the interface, but since the triangles are not isosceles, the tangential (resp., normal) component is not antisymmetric.

where n_i is the outward normal to $F_{K,i}$. Furthermore, since the mesh is affine, $T_K(\widehat{x}) = J_K\widehat{x} + b_K$ where $J_K \in \mathbb{R}^{d,d}$ and $b_K \in \mathbb{R}^d$. Hence, for $p \in P_K$, $\psi_K(p) = \widehat{x}_0 + \alpha\widehat{x}$, where $\widehat{x}_0 \in [\mathbb{P}_0]^d$ and $\alpha \in \mathbb{R}$, yielding $p = \psi_K^{-1}(\widehat{x}_0 + \alpha\widehat{x}) = \frac{1}{\det(J_K)}J_K(\widehat{x}_0+\alpha\widehat{x})$. Then, using $\widehat{x} = J_K^{-1}(x-b_K)$ yields $p \in \mathbb{RT}_0$. As a result, $P_K = \mathbb{RT}_0$ and $\{K, P_K, \Sigma_K\}$ is a Raviart–Thomas finite element.

Consider the so-called *Raviart–Thomas approximation space*

$$D_h = \{v_h \in [L^1(\Omega_h)]^d;\ \forall K \in \mathcal{T}_h,\ v_{h|K} \in \mathbb{RT}_0,\\ \forall F \in \mathcal{F}_h^{\mathrm{i}},\ [\![v_h{\cdot}n]\!]_F = 0\}, \tag{1.83}$$

where $[\![v_h{\cdot}n]\!]_F$ denotes the jump of the normal component of v_h across the interface F. The main motivation for introducing D_h is the following:

Proposition 1.85. $D_h \subset H(\mathrm{div};\Omega_h) = \{v \in [L^2(\Omega_h)]^d;\ \nabla{\cdot}v \in L^2(\Omega_h)\}$.

Proof. Proceed as in the proof of Proposition 1.74. □

Now, let us specify the global shape functions in D_h. For $F \in \mathcal{F}_h$, let n_F be a normal unit vector to F (its direction is irrelevant). Consider the function φ_F with support consisting of the one or two simplices to which F belongs and such that on one simplex, say K, the function $\varphi_{F|K}$ is the local shape function of $\{K, P_K, \Sigma_K\}$ associated with the face F and on the other simplex, say K', $\varphi_{F|K'}$ is the opposite of the local shape function associated with F on K'; see the left panel of Figure 1.22.

Lemma 1.86. $\varphi_F \in D_h$.

Proof. Adapt the proof of Lemma 1.70 and use the fact that $\varphi{\cdot}n_F$ is constant on F. □

Proposition 1.87. *$\{\varphi_F\}_{F\in\mathcal{F}_h}$ is a basis for D_h, and defining the linear forms $\gamma_F : D_h \ni v_h \mapsto \int_F v_h{\cdot}n_F \in \mathbb{R}$, $\{\gamma_F\}_{F\in\mathcal{F}_h}$ is a basis for $\mathcal{L}(D_h;\mathbb{R})$.*

Proof. Left as an exercise. ☐

Proposition 1.87 implies that D_h is a space of dimension N_{ed} in two dimensions and N_{f} in three dimensions. The linear forms $\{\gamma_F\}_{F\in\mathcal{F}_h}$ are called the *global degrees of freedom* in D_h, and $\{\varphi_F\}_{F\in\mathcal{F}_h}$ the *global shape functions.*

For a function v in the space

$$V^{\mathrm{div}} = \{v \in [L^p(\Omega_h)]^d;\ \nabla{\cdot}v \in L^s(\Omega_h)\}, \tag{1.84}$$

with $p > 2$ and $s \geq q$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$, the quantity $\gamma_F(v)$ is meaningful (and single-valued) for all $F \in \mathcal{F}_h$. The so-called *global Raviart–Thomas interpolation operator* is constructed as follows:

$$\mathcal{I}_h^{\mathrm{RT}} : V^{\mathrm{div}} \ni v \longmapsto \mathcal{I}_h^{\mathrm{RT}} v = \sum_{F\in\mathcal{F}_h} \left(\int_F v{\cdot}n_F \right) \varphi_F \in D_h. \tag{1.85}$$

Note that D_h is the codomain of $\mathcal{I}_h^{\mathrm{RT}}$. See [BrF91^b, RaT77] for further results on $H(\mathrm{div})$-conforming spaces.

Remark 1.88. If the degrees of freedom in $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ are chosen to be the mean-value of the flux (see Remark 1.40(ii)), Proposition 1.65 must be used to construct the family $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$; see Remark 1.72(i). ☐

1.4.8 $H(\mathrm{curl})$-conforming spaces

We consider a three-dimensional setting, but a similar construction is possible in two dimensions. Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be the Nédélec finite element introduced in §1.2.8. Choose $V(\widehat{K}) = \{v \in [L^p(\widehat{K})]^3;\ \nabla{\times}v \in [L^s(\widehat{K})]^3\}$ with $p > 2$ and $s > \frac{1}{2}$, and define $V(K)$ similarly. Introduce the mapping

$$\psi_K : V(K) \ni v \longmapsto \psi_K(v)(\widehat{x}) = J_K^T[v \circ T_K(\widehat{x})] \in \widehat{V}(\widehat{K}). \tag{1.86}$$

Lemma 1.89. *Let $\mathcal{C}(v) = \nabla v - (\nabla v)^T$. For all $v \in V(K)$, the following identities hold:*

(i) *For all $\beta \in \mathbb{R}^3$, $\mathcal{C}(v){\cdot}\beta = (\nabla{\times}v){\times}\beta$.*
(ii) *$\|\mathcal{C}(v)\|_{\mathbb{R}^{3,3}} = \|\nabla{\times}v\|_{\mathbb{R}^3}$.*
(iii) *$\mathcal{C}[\psi_K(v)] = (J_K)^T\mathcal{C}(v)J_K$.*

Construct the family $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$ using Proposition 1.61. Denote by $\{\widehat{e}_1, \ldots, \widehat{e}_6\}$ the edges of $\widehat{K}$ and, for $1 \leq i \leq 6$, let $e_{K,i} = T_K(\widehat{e}_i)$ be the corresponding edge of K. Let $\widehat{t}_i$ (resp. $t_{K,i}$) be one of the two unit vectors parallel to $\widehat{e}_i$ (resp. $e_{K,i}$). Since $J_K\widehat{t}_i = \frac{\mathrm{meas}(e_{K,i})}{\mathrm{meas}(\widehat{e}_i)} t_{K,i}$,

$$\sigma_i(v) = \int_{\widehat{e}_i} \psi_K(v)\cdot\widehat{t}_i = \int_{e_{K,i}} v\cdot t_{K,i}. \tag{1.87}$$

Furthermore, since the mesh is affine, $T_K(\widehat{x}) = J_K\widehat{x} + b_K$ where $J_K \in \mathbb{R}^{d,d}$ and $b_K \in \mathbb{R}^d$. Hence, for $p \in P_K$, $\psi_K(p) = J_K^T[p \circ T_K] = \alpha + \beta\times\widehat{x}$, yielding $p = \alpha' + (J_K^T)^{-1}[\beta\times J_K^{-1}x]$. Then, it is clear that $((J_K^T)^{-1}[\beta\times J_K^{-1}x])\cdot x = (\beta\times J_K^{-1}x)\cdot J_K^{-1}x = 0$, i.e., $p \in \mathbb{N}_0$. As a result, $P_K = \mathbb{N}_0$ and $\{K, P_K, \Sigma_K\}$ is a Nédélec finite element.

Consider the so-called *Nédélec approximation space*

$$R_h = \{v_h \in [L^1(\Omega_h)]^3;\, \forall K \in \mathcal{T}_h,\, v_{h|K} \in \mathbb{N}_0;\\ \forall F \in \mathcal{F}_h^{\mathrm{i}},\, [\![v_h\times n]\!]_F = 0\}, \tag{1.88}$$

where $[\![v_h\times n]\!]_F$ denotes the jump of the tangential component of v_h across the interface F. The main motivation for introducing R_h is the following:

Proposition 1.90. $R_h \subset H(\mathrm{curl};\Omega_h) = \{v \in [L^2(\Omega_h)]^3;\, \nabla\times v \in [L^2(\Omega_h)]^3\}$.

Proof. Proceed as in the proof of Proposition 1.74. □

To derive the global shape functions in R_h, we first state the following:

Lemma 1.91. *Let $F = K_1 \cap K_2$ and let v_h be such that $v_{h|K_1} \in \mathbb{N}_0$ and $v_{h|K_2} \in \mathbb{N}_0$. Then, $[\![v_h\times n]\!]_F = 0$ if and only if $\int_e v_{h|K_1}\cdot t_e = \int_e v_{h|K_2}\cdot t_e$ for the three edges of F.*

Proof. Write $v_{h|K_1} = \alpha_1 + \beta_1\times x$ and $v_{h|K_2} = \alpha_2 + \beta_2\times x$. Let n_F be one of the two unit vectors that are normal to F. Clearly, $v_{h|K_1}\times n_F = \alpha_1\times n_F + (\beta_1\cdot n_F)x - (x\cdot n_F)\beta_1$. Since $x\cdot n_F$ is constant on F, $v_{h|K_1}\times n_F = s + tx$ where $s \in \mathbb{R}^3$ and $t \in \mathbb{R}$; that is to say, $v_{h|K_1}\times n_F \in \mathbb{RT}_0$; see (1.40). Let e_1, e_2, and e_3 be the three edges of F. Denote by n_1, n_2, and n_3 the three unit vectors that are parallel to F, are normal to e_1, e_2, and e_3, and point outward. It is clear that $t_i = n_F \times n_i$ is a unit vector parallel to the edge e_i. Let $\{\theta_1, \theta_2, \theta_3\}$ be the two-dimensional Raviart-Thomas shape functions on F. It is readily checked that

$$v_{h|K_1}\times n_F = \sum_{i=1}^{3}\left(\int_{e_i}(v_{h|K_1}\times n_F)\cdot n_i\right)\theta_i = \sum_{i=1}^{3}\left(\int_{e_i} v_{h|K_1}\cdot t_i\right)\theta_i.$$

Since the set $\{\theta_1, \theta_2, \theta_3\}$ is linearly independent, it is clear that $[\![v_h\times n]\!]_F = 0$ if and only if $\int_{e_i} v_{h|K_1}\cdot t_i = \int_{e_i} v_{h|K_2}\cdot t_i$ for all $i \in \{1, 2, 3\}$. □

For an edge $e \in \mathcal{E}_h$, choose one of the two unit vectors parallel to e, say t_e. Consider the function φ_e with support consisting of the simplices to which e belongs and such that on each of these simplices, say K, the function $\varphi_{e|K}$ is the local shape function of $\{K, P_K, \Sigma_K\}$ associated with the edge e oriented by t_e; see the right panel of Figure 1.22.

Lemma 1.92. $\varphi_e \in R_h$.

Proof. Let $e \in \mathcal{E}_h$.
(1) Let K_1 and K_2 be two elements sharing the edge e. Then, owing to (1.87), $\int_e \varphi_{e|K_1}\cdot t_e = 1 = \int_e \varphi_{e|K_2}\cdot t_e$ and for $e' \neq e$, $\int_{e'} \varphi_{e|K_1}\cdot t_{e'} = 0 = \int_{e'} \varphi_{e|K_2}\cdot t_{e'}$.
(2) Let $F \in \mathcal{F}_h^{\mathrm{i}}$, say $F = K_1 \cap K_2$. Owing to step 1, the converse statement of Lemma 1.91 implies $[\![\varphi_e \times n]\!]_F = 0$. The conclusion follows easily. □

Proposition 1.93.

(i) *For all $e \in \mathcal{E}_h$, the linear form $\gamma_e : R_h \ni v_h \mapsto \int_e v_h\cdot t_e$ is single-valued.*
(ii) *$\{\varphi_e\}_{e\in\mathcal{E}_h}$ is a basis for R_h, and $\{\gamma_e\}_{e\in\mathcal{E}_h}$ is a basis for $\mathcal{L}(R_h;\mathbb{R})$.*

Proof. (1) Let $e \in \mathcal{E}_h$ and let K_1 and K_2 be two elements sharing the edge e. Then, there exists a finite family of elements $\{K_{j_1}, \dots, K_{j_J}\}$ such that $K_{j_1} = K_1$, $K_{j_J} = K_2$, and $K_{j_l} \cap K_{j_{l+1}}$ is a face containing e. Owing to the direct statement of Lemma 1.91 for each pair $\{K_{j_l}, K_{j_{l+1}}\}$, the quantity $\int_e v_h\cdot t_e$ is single-valued for all edge $e \in \mathcal{E}_h$ and all $v_h \in R_h$.
(2) The family $\{\varphi_e\}_{e\in\mathcal{E}_h}$ is linearly independent since $\gamma_{e'}(\varphi_e) = \delta_{ee'}$ (with obvious notation). Let $v_h \in R_h$. Owing to step 1, it is legitimate to consider the function

$$w_h = \sum_{e\in\mathcal{E}_h} \left(\int_e v_h\cdot t_e\right) \varphi_e.$$

Then, it is clear that for all $K \in \mathcal{T}_h$, $v_{h|K}$ and $w_{h|K}$ are in $\mathbb{N}_0$ and that $\int_e v_{h|K}\cdot t_e = \int_e w_{h|K}\cdot t_e$ for all edge $e \in \partial K$. Unisolvence implies $v_{h|K} = w_{h|K}$. Hence, $\{\varphi_e\}_{e\in\mathcal{E}_h}$ is a basis for R_h. Proving that $\{\gamma_e\}_{e\in\mathcal{E}_h}$ is a basis for $\mathcal{L}(R_h;\mathbb{R})$ is then straightforward. □

Proposition 1.93 implies that R_h is a space of dimension N_{ed}. The linear forms $\{\gamma_e\}_{e\in\mathcal{E}_h}$ are called the *global degrees of freedom* in R_h, and $\{\varphi_e\}_{e\in\mathcal{E}_h}$ are called the *global shape functions*.

For a function v in the space

$$V^{\mathrm{curl}} = \{v \in [L^p(\Omega_h)]^3;\ \nabla\times v \in [L^s(\Omega_h)]^3\}, \tag{1.89}$$

with $p > 2$ and $s > \frac{1}{2}$, the quantity $\gamma_e(v)$ is meaningful (and single-valued) for all $e \in \mathcal{E}_h$. The so-called *global Nédélec interpolation operator* is constructed as follows:

$$\mathcal{I}_h^{\mathrm{N}} : V^{\mathrm{curl}} \ni v \longmapsto \mathcal{I}_h^{\mathrm{N}} v = \sum_{e\in\mathcal{E}_h} \left(\int_e v\cdot t_e\right) \varphi_e \in R_h. \tag{1.90}$$

Note that R_h is the codomain of $\mathcal{I}_h^{\mathrm{N}}$. For further results on $H(\mathrm{curl})$-conforming spaces, see, e.g., [Néd80, Néd86, Mon92, Bos93].

Remark 1.94. If the degrees of freedom in $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ are chosen to be the mean-value of the integral over the edges (see Remark 1.43(ii)), Proposition 1.65 must be used to construct the family $\{K, P_K, \Sigma_K\}_{K\in\mathcal{T}_h}$; see Remark 1.72(i). □

1.4.9 A link between H^1-, $H(\mathrm{curl})$-, and $H(\mathrm{div})$-conforming spaces

When the mesh $\mathcal{T}_h$ consists of affine simplices, an interesting relation exists between the spaces $P^1_{\mathrm{c},h}$, R_h, and D_h in three dimensions. To formalize this relation, we introduce the concept of exact sequence; see, e.g., [God71]. Let $\{E_j\}_{j\in J}$ be a sequence of vector spaces on the same field and indexed by an interval J of $\mathbb{N}$. For $j \in J$ such that $j+1 \in J$, let $h_j : E_j \to E_{j+1}$ be a homomorphism.

Definition 1.95. *The sequence*

$$\dots \xrightarrow{h_{j-1}} E_j \xrightarrow{h_j} E_{j+1} \xrightarrow{h_{j+1}} E_{j+2} \xrightarrow{h_{j+2}} \dots$$

is said to be exact *if for all $j \in J$ such that $j+2 \in J$, $\mathrm{Ker}(h_{j+1}) = \mathrm{Im}(h_j)$.*

Consider a domain Ω in $\mathbb{R}^3$. Let $H_0(\mathrm{curl};\Omega)$ be the subspace of $H(\mathrm{curl};\Omega)$ consisting of the vector fields whose tangential components vanish at $\partial\Omega$. Let also $H_0(\mathrm{div};\Omega)$ be the subspace of $H(\mathrm{div};\Omega)$ consisting of the vector fields whose normal component vanishes at $\partial\Omega$. Let i be the canonical injection and let m be the averaging operator over Ω.

Proposition 1.96. *If Ω is simply connected and $\partial\Omega$ is connected, the following sequence is exact:*

$$\{0\} \xrightarrow{i} H^1_0(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl};\Omega) \xrightarrow{\nabla\times} H_0(\mathrm{div};\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \xrightarrow{m} \mathrm{span}\{1\}.$$

Let Ω_h be a geometric interpolate of the domain Ω based on a mesh $\mathcal{T}_h$. Define the approximation spaces $P^1_{\mathrm{c},h,0} = P^1_{\mathrm{c},h} \cap H^1_0(\Omega_h)$, $R_{h,0} = R_h \cap H_0(\mathrm{curl};\Omega_h)$, and $D_{h,0} = D_h \cap H_0(\mathrm{div};\Omega_h)$. Let also $P^0_{\mathrm{td},h}$ be the space of piecewise constant functions on the mesh $\mathcal{T}_h$. As a discrete counterpart of Proposition 1.96, one easily proves the following:

Proposition 1.97. *If Ω_h is simply connected and $\partial\Omega_h$ is connected, the following sequence is exact:*

$$\{0\} \xrightarrow{i} P^1_{\mathrm{c},h,0} \xrightarrow{\nabla} R_{h,0} \xrightarrow{\nabla\times} D_{h,0} \xrightarrow{\nabla\cdot} P^0_{\mathrm{td},h} \xrightarrow{m} \mathrm{span}\{1\}.$$

Assume $\Omega_h \subset \Omega$ for the sake of simplicity. Set $V^1 = H^s(\Omega)$ with $s > \frac{d}{2}$ and $V^0 = L^1(\Omega)$. Let V^{div} and V^{curl} be defined in (1.84) and (1.89), respectively. Let $\mathcal{I}^1_h$, $\mathcal{I}^{\mathrm{N}}_h$, $\mathcal{I}^{\mathrm{RT}}_h$, and $\mathcal{I}^0_{\mathrm{td},h}$ the interpolation operators associated with the finite element spaces $P^1_{\mathrm{c},h}$, R_h, D_h, and $P^0_{\mathrm{td},h}$, respectively. The following striking property holds:

Proposition 1.98. *The following diagram commutes:*

$$\begin{tikzcd}
V^1 \arrow[r, "\nabla"] \arrow[d, "\mathcal{I}^1_h"] & V^{\mathrm{curl}} \arrow[r, "\nabla\times"] \arrow[d, "\mathcal{I}^{\mathrm{N}}_h"] & V^{\mathrm{div}} \arrow[r, "\nabla\cdot"] \arrow[d, "\mathcal{I}^{\mathrm{RT}}_h"] & V^0 \arrow[d, "\mathcal{I}^0_{\mathrm{td},h}"] \\
P^1_{\mathrm{c},h} \arrow[r, "\nabla"] & R_h \arrow[r, "\nabla\times"] & D_h \arrow[r, "\nabla\cdot"] & P^0_{\mathrm{td},h}
\end{tikzcd}$$

Proof. This is a simple corollary of Lemmas 1.41, 1.44, and 1.45. □

Remark 1.99.

(i) Propositions 1.97 and 1.98 can be extended to higher-order finite element spaces; see the de Rham diagram theory developed in [DeM00, Bof01].

(ii) Proposition 1.97 provides an efficient means of constructing all the fields in $R_{h,0}$ with vanishing curl and all the solenoidal fields in $D_{h,0}$. For further results, see [Bos93]. □

1.5 Interpolation of Smooth Functions

Letting $\mathcal{I}_h$ be one of the interpolation operators constructed in §1.4, the goal of this section is to estimate the interpolation error $v - \mathcal{I}_h v$ assuming that the function v is smooth enough to be in the domain of $\mathcal{I}_h$. First, we investigate thoroughly the interpolation of scalar- and vector-valued functions on affine meshes. Then, we briefly discuss non-affine transformations.

1.5.1 Interpolation in $W^{s,p}(\Omega)$

In this section, we establish local and global interpolation error estimates on affine meshes for scalar-valued functions living in Sobolev spaces; see Appendix B for a definition of these spaces and the corresponding norms. Interpolation error estimates in vector-valued Sobolev spaces are readily derived by applying the scalar-valued interpolation error estimates componentwise.

Since the mesh is affine, the transformation T_K takes the form

$$T_K : \widehat{K} \ni \widehat{x} \longmapsto J_K \widehat{x} + b_K \in K, \tag{1.91}$$

where $J_K \in \mathbb{R}^{d,d}$ and $b_K \in \mathbb{R}^d$. The Jacobian matrix J_K is invertible since T_K is bijective. Let $\|\cdot\|_d$ be the Euclidean norm in $\mathbb{R}^d$ as well as the associated matrix norm. Throughout this section, we assume that the mapping $\psi_K : V(K) \to \widehat{V}(\widehat{K})$ in Proposition 1.62 is $\psi_K(v) = v \circ T_K$, and we set $\widehat{v} = v \circ T_K$.

Lemma 1.100. *Let ρ_K be the diameter of the largest ball that can be inscribed in K. Then,*

$$|\det(J_K)| = \tfrac{\mathrm{meas}(K)}{\mathrm{meas}(\widehat{K})}, \qquad \|J_K\|_d \le \tfrac{h_K}{\rho_{\widehat{K}}}, \qquad \textit{and} \qquad \|J_K^{-1}\|_d \le \tfrac{h_{\widehat{K}}}{\rho_K}. \tag{1.92}$$

Proof. The first property in (1.92) is classical. Furthermore,

$$\|J_K\|_d = \sup_{\widehat{x} \ne 0} \frac{\|J_K \widehat{x}\|_d}{\|\widehat{x}\|_d} = \frac{1}{\rho_{\widehat{K}}} \sup_{\|\widehat{x}\|_d = \rho_{\widehat{K}}} \|J_K \widehat{x}\|_d.$$

Write $\widehat{x} = \widehat{x}_1 - \widehat{x}_2$ with $\widehat{x}_1$ and $\widehat{x}_2$ in $\widehat{K}$ and use $J_K \widehat{x} = T_K \widehat{x}_1 - T_K \widehat{x}_2 = x_1 - x_2$ to obtain $\|J_K \widehat{x}\|_d \le h_K$. This proves the first inequality in (1.92). The second inequality is obtained by exchanging the roles of K and $\widehat{K}$. □

Lemma 1.101. *Let $s \geq 0$ and let $1 \leq p \leq \infty$. There exists c such that, for all K and $w \in W^{s,p}(K)$,*

$$|\widehat{w}|_{s,p,\widehat{K}} \leq c\,\|J_K\|_d^s\,|\det(J_K)|^{-\frac{1}{p}}\,|w|_{s,p,K}, \tag{1.93}$$

$$|w|_{s,p,K} \leq c\,\|J_K^{-1}\|_d^s\,|\det(J_K)|^{\frac{1}{p}}\,|\widehat{w}|_{s,p,\widehat{K}}, \tag{1.94}$$

with $\widehat{w} = w \circ T_K$ and with the convention that, for $p = \infty$ and any positive real x, $x^{\pm\frac{1}{p}} = 1$.

Proof. Let α be a multi-index with length $|\alpha| = s$. Use the chain-rule and the fact that the transformation T_K is affine to obtain

$$\|\partial^\alpha \widehat{w}\|_{L^p(\widehat{K})} \leq c\,\|J_K\|_d^s \sum_{|\beta|=s} \|\partial^\beta w \circ T_K\|_{L^p(\widehat{K})}.$$

Changing variables in the right-hand side yields

$$\|\partial^\alpha \widehat{w}\|_{L^p(\widehat{K})} \leq c\,\|J_K\|_d^s\,|\det(J_K)|^{-\frac{1}{p}}\,|w|_{s,p,K}.$$

We deduce (1.93) upon summing over α. The proof of (1.94) is similar. $\square$

Remark 1.102. The upper bounds in (1.93) and (1.94) involve only seminorms because affine transformations are considered. $\square$

Theorem 1.103 (Local interpolation). *Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element with associated normed vector space $V(\widehat{K})$. Let $1 \leq p \leq \infty$ and assume that there exists an integer k such that*

$$\mathbb{P}_k \subset \widehat{P} \subset W^{k+1,p}(\widehat{K}) \subset V(\widehat{K}). \tag{1.95}$$

Let $T_K : \widehat{K} \rightarrow K$ be an affine bijective mapping and let $\mathcal{I}_K^k$ be the local interpolation operator on K defined in (1.59)*. Let l be such that $0 \leq l \leq k$ and $W^{l+1,p}(\widehat{K}) \subset V(\widehat{K})$ with continuous embedding. Then, setting $\sigma_K = \frac{h_K}{\rho_K}$, there exists $c > 0$ such that, for all $m \in \{0, \dots, l+1\}$,*

$$\forall K,\ \forall v \in W^{l+1,p}(K), \quad |v - \mathcal{I}_K^k v|_{m,p,K} \leq c\,h_K^{l+1-m}\sigma_K^m\,|v|_{l+1,p,K}. \tag{1.96}$$

Proof. Let $\mathcal{I}_{\widehat{K}}^k$ be the local interpolation operator on $\widehat{K}$ defined in (1.57). Let $\widehat{w} \in W^{l+1,p}(\widehat{K})$. Since $W^{l+1,p}(\widehat{K}) \subset V(\widehat{K})$ with continuous embedding, the linear operator

$$\mathcal{F} : W^{l+1,p}(\widehat{K}) \ni \widehat{w} \longmapsto \widehat{w} - \mathcal{I}_{\widehat{K}}^k \widehat{w} \in W^{m,p}(\widehat{K}),$$

is continuous from $W^{l+1,p}(\widehat{K})$ to $W^{m,p}(\widehat{K})$ for all $m \in \{0, \dots, l+1\}$. Since $l \leq k$, $\mathbb{P}_l \subset \widehat{P}$ and, therefore, $\mathbb{P}_l$ is invariant under $\mathcal{I}_{\widehat{K}}^k$ owing to Proposition 1.30. Hence, $\mathcal{F}$ vanishes on $\mathbb{P}_l$. As a consequence,

$$
\begin{aligned}
|\widehat{w} - \mathcal{I}_{\widehat{K}}^k \widehat{w}|_{m,p,\widehat{K}} &= |\mathcal{F}(\widehat{w})|_{m,p,\widehat{K}} = \inf_{\widehat{p}\in\mathbb{P}_l} |\mathcal{F}(\widehat{w}+\widehat{p})|_{m,p,\widehat{K}} \\
&\le \|\mathcal{F}\|_{\mathcal{L}(W^{l+1,p}(\widehat{K});W^{m,p}(\widehat{K}))} \inf_{\widehat{p}\in\mathbb{P}_l} \|\widehat{w}+\widehat{p}\|_{l+1,p,\widehat{K}} \\
&\le c \inf_{\widehat{p}\in\mathbb{P}_l} \|\widehat{w}+\widehat{p}\|_{l+1,p,\widehat{K}} \le c\,|\widehat{w}|_{l+1,p,\widehat{K}},
\end{aligned}
$$

the last estimate resulting from the Deny–Lions Lemma; see Lemma B.67. Now let $v \in W^{l+1,p}(K)$ and set $\widehat{v} = \psi_K(v) = v \circ T_K$. Owing to Proposition 1.62, $[\mathcal{I}_K^k v] \circ T_K = \mathcal{I}_{\widehat{K}}^k \widehat{v}$. Using Lemma 1.101 yields

$$
\begin{aligned}
|v - \mathcal{I}_K^k v|_{m,p,K} &\le c\,\|J_K^{-1}\|_d^m\,|\det(J_K)|^{\frac{1}{p}}\,|\widehat{v} - \mathcal{I}_{\widehat{K}}^k \widehat{v}|_{m,p,\widehat{K}} \\
&\le c\,\|J_K^{-1}\|_d^m\,|\det(J_K)|^{\frac{1}{p}}\,|\widehat{v}|_{l+1,p,\widehat{K}} \\
&\le c\,\|J_K^{-1}\|_d^m\,\|J_K\|_d^{l+1}\,|v|_{l+1,p,K} \\
&\le c\,(\|J_K\|_d\,\|J_K^{-1}\|_d)^m\,\|J_K\|_d^{l+1-m}\,|v|_{l+1,p,K}.
\end{aligned}
$$

Conclude using (1.92). □

Definition 1.104 (Degree of a finite element). *The largest integer k such that* (1.95) *holds is called the* degree *of the finite element* $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$.

Remark 1.105. If the interpolated function is in $W^{k+1,p}(K)$, one can take $l = k$ in Theorem 1.103. The resulting error estimate is *optimal*, i.e., for $m \in \{0,\ldots,k+1\}$,

$$\forall K,\ \forall v \in W^{k+1,p}(K), \quad |v - \mathcal{I}_K^k v|_{m,p,K} \le c\,h_K^{k+1-m}\sigma_K^m\,|v|_{k+1,p,K}.$$ □

Example 1.106.

(i) For a Lagrange finite element of degree k, $V(\widehat{K}) = \mathcal{C}^0(\widehat{K})$; hence, the condition on l in Theorem 1.103 is $\frac{d}{p} - 1 < l \le k$. Indeed, owing to Theorem B.46, $W^{l+1,p}(\widehat{K}) \subset V(\widehat{K})$ provided $l+1 > \frac{d}{p}$. More generally, for a finite element with $V(\widehat{K}) = \mathcal{C}^t(\widehat{K})$ (for instance, $t = 1$ for the Hermite finite element), the condition on l is $\frac{d}{p} - 1 + t < l \le k$; see also [BrS94, p. 104].

(ii) For the Crouzeix–Raviart finite element, $k = 1$ and $V(\widehat{K}) = W^{1,1}(\widehat{K})$; as a result, the condition on l is $0 \le l \le k = 1$. □

To obtain global interpolation error estimates on Ω and to prove that these estimates converge to zero as $h \to 0$, the quantity σ_K appearing in (1.96) must be controlled independently of K and h. This leads to the following:

Definition 1.107 (Shape-regularity). *A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be* shape-regular *if there exists σ_0 such that*

$$\forall h,\ \forall K \in \mathcal{T}_h, \quad \sigma_K = \tfrac{h_K}{\rho_K} \le \sigma_0.$$

Remark 1.108.

(i) Let K be a triangle and denote by θ_K the smallest of its angles. One readily sees that

$$\frac{h_K}{\rho_K} \leq \frac{2}{\sin\theta_K}.$$

Therefore, in a shape-regular family of triangulations, the triangles cannot become too flat as $h \to 0$.

(ii) In dimension 1, $h_K = \rho_K$; hence, any mesh family is shape-regular.

(iii) Lemma 1.100 shows that for a shape-regular family of meshes, there is c such that, for all h and $K \in \mathcal{T}_h$, $\|J_K\|_d \|J_K^{-1}\|_d \leq c$. The quantity $\|J_K\|_d \|J_K^{-1}\|_d$ is called the Euclidean *condition number* of J_K. $\square$

Corollary 1.109 (Global interpolation). *Let p, k, and l satisfy the assumptions of Theorem* 1.103. *Let Ω be a polyhedron and let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine meshes of Ω. Denote by V_h^k the approximation space based on $\mathcal{T}_h$ and $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$. Let $\mathcal{I}_h^k$ be the corresponding global interpolation operator. Then, there exists c such that, for all h and $v \in W^{l+1,p}(\Omega)$,*

$$\|v - \mathcal{I}_h^k v\|_{L^p(\Omega)} + \sum_{m=1}^{l+1} h^m \left(\sum_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^k v|_{m,p,K}^p \right)^{\frac{1}{p}} \leq c\, h^{l+1} |v|_{l+1,p,\Omega}, \quad (1.97)$$

for $p < \infty$, and for $p = \infty$

$$\|v - \mathcal{I}_h^k v\|_{L^\infty(\Omega)} + \sum_{m=1}^{l+1} h^m \max_{K \in \mathcal{T}_h} |v - \mathcal{I}_h^k v|_{m,\infty,K} \leq c\, h^{l+1} |v|_{l+1,\infty,\Omega}. \quad (1.98)$$

Furthermore, for $p < \infty$ and $v \in L^p(\Omega)$, the following density result holds:

$$\lim_{h \to 0} \left(\inf_{v_h \in V_h^k} \|v - v_h\|_{L^p(\Omega)} \right) = 0. \quad (1.99)$$

Proof. Since the family $\{\mathcal{T}_h\}_{h>0}$ is shape-regular, estimates (1.97) and (1.98) result from (1.96). Let $v \in L^p(\Omega)$ and $\epsilon > 0$. Since $W^{l+1,p}(\Omega)$ is dense in $L^p(\Omega)$ for $p < \infty$, there is $v^\epsilon \in W^{l+1,p}(\Omega)$ such that $\|v - v^\epsilon\|_{L^p(\Omega)} \leq \epsilon$. Furthermore, (1.97) yields $\|v^\epsilon - \mathcal{I}_h^k v^\epsilon\|_{L^p(\Omega)} \leq c h^{l+1} |v^\epsilon|_{l+1,p,\Omega}$. Hence,

$$\inf_{v_h \in V_h^k} \|v - v_h\|_{L^p(\Omega)} \leq \|v - \mathcal{I}_h^k v^\epsilon\|_{L^p(\Omega)} \leq \|v - v^\epsilon\|_{L^p(\Omega)} + \|v^\epsilon - \mathcal{I}_h^k v^\epsilon\|_{L^p(\Omega)}.$$

That is to say, $\limsup_{h\to 0}(\inf_{v_h \in V_h^k} \|v - v_h\|_{L^p(\Omega)}) \leq \epsilon$, and (1.99) follows from the fact that ϵ is arbitrary. $\square$

Corollary 1.110 (Interpolation in $W^{s,p}(\Omega)$). *Let the hypotheses of Corollary* 1.109 *hold and assume that V_h^k is $W^{1,p}$-conforming. Then, there is c such that, for all h and $v \in W^{l+1,p}(\Omega)$,*

$$|v - \mathcal{I}_h^k v|_{1,p,\Omega} \le c\, h^l |v|_{l+1,p,\Omega}. \tag{1.100}$$

For $p < \infty$, the following density result holds:

$$\forall v \in W^{1,p}(\Omega), \quad \lim_{h\to 0}\left(\inf_{v_h\in V_h^k} |v - v_h|_{1,p,\Omega}\right) = 0. \tag{1.101}$$

Example 1.111.

(i) Consider a Lagrange finite element of degree k. Take $p = 2$ and assume $d \le 3$. Then, owing to Example 1.106(i), one can take $1 \le l \le k$, and (1.97) yields, for all $v \in H^{l+1}(\Omega)$,

$$\|v - \mathcal{I}_h^k v\|_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} \le c\, h^{l+1}|v|_{l+1,\Omega}. \tag{1.102}$$

This estimate is optimal if v is smooth enough, i.e., $v \in H^{k+1}(\Omega)$. However, if v is in $H^s(\Omega)$ and not in $H^{s+1}(\Omega)$ for some $s \ge 2$, increasing the degree of the finite element beyond $s - 1$ does not improve the interpolation error. This phenomenon is illustrated in §3.2.5. Note also that the same asymptotic order is obtained for $\mathbb{P}_k$ and $\mathbb{Q}_k$ Lagrange finite elements. For $\mathbb{Q}_k$ Lagrange finite elements, a sharper interpolation error estimate can be derived using a different norm for v in the right-hand side of (1.97); see, e.g., [BrS94, p. 112].

(ii) Consider the Hermite finite element; see §1.4.6. Take $p = 2$; since $d = 1$ and $k = 3$, Example 1.106(i) shows that one can take $2 \le l \le 3$. Owing to (1.97), we infer, for all $v \in H^{l+1}(\Omega)$,

$$\|v - \mathcal{I}_h^k v\|_{0,\Omega} + h|v - \mathcal{I}_h^k v|_{1,\Omega} + h^2|v - \mathcal{I}_h^k v|_{2,\Omega} \le c\, h^{l+1}|v|_{l+1,\Omega}. \tag{1.103}$$

If $l = 3$, i.e., if $v \in H^4(\Omega)$, the error estimate is optimal. □

Remark 1.112. Estimate (1.97) also applies when the parameter l is not an integer. As a simple example, consider a Lagrange finite element of degree $k \ge 1$ in dimension $d \le 3$. Since $W^{k+1-\frac{d}{2},\infty}(\widehat{K}) \subset \mathcal{C}^0(\widehat{K}) = V(\widehat{K})$ with continuous embedding (i.e., $k+1-\frac{d}{2} > 0$), (1.98) can be applied with $l = k - \frac{d}{2}$ and $p = \infty$ to obtain $\|v - \mathcal{I}_h^k v\|_{L^\infty(\Omega)} \le c\, h^{k+1-\frac{d}{2}}|v|_{k+1-\frac{d}{2},\infty,\Omega}$ for $v \in W^{k+1-\frac{d}{2},\infty}(\Omega)$. Therefore, using the fact that $H^{k+1}(\Omega) \subset W^{k+1-\frac{d}{2},\infty}(\Omega)$ with continuous embedding yields

$$\forall h,\ \forall v \in H^{k+1}(\Omega), \quad \|v - \mathcal{I}_h^k v\|_{L^\infty(\Omega)} \le c\, h^{k+1-\frac{d}{2}}|v|_{k+1,\Omega}.$$

Obviously, if $v \in W^{k+1,\infty}(\Omega)$, (1.98) implies the sharper estimate

$$\forall h,\ \forall v \in W^{k+1,\infty}(\Omega), \quad \|v - \mathcal{I}_h^k v\|_{L^\infty(\Omega)} \le c\, h^{k+1}|v|_{k+1,\infty,\Omega}.$$ □

1.5.2 Interpolation in $H(\mathrm{div};\Omega)$

We analyze in this section the interpolation properties of the Raviart–Thomas finite element introduced in §1.2.7.

We assume that the mapping $T_K : \widehat{K} \to K$ is linear, i.e., $T_K(\widehat{x}) = J_K\widehat{x}+b_K$ with $J_K \in \mathbb{R}^{d,d}$ and $b_K \in \mathbb{R}^d$. For a vector-field $v \in [W^{s,p}(K)]^d$, set $\widehat{v}(\widehat{x}) = \det(J_K)J_K^{-1}v(x)$, i.e., $\widehat{v} = \psi_K(v)$ where ψ_K is the Piola transformation defined by (1.81).

Lemma 1.113. *Let $s \geq 0$ and $1 \leq p \leq \infty$ (with $x^{\pm\frac{1}{p}} = 1$ for all $x > 0$ if $p = \infty$). Then, there is c such that, for all K and $w \in [W^{s,p}(K)]^d$ with $\nabla\cdot w \in W^{s,p}(K)$,*

$$|w|_{s,p,K} \leq c\,\|J_K^{-1}\|_d^s\,\|J_K\|_d\,|\det(J_K)|^{-\frac{1}{p'}}\,|\widehat{w}|_{s,p,\widehat{K}}, \tag{1.104}$$

$$|\nabla\cdot w|_{s,p,K} \leq c\,\|J_K^{-1}\|_d^s\,|\det(J_K)|^{-\frac{1}{p'}}\,|\nabla\cdot\widehat{w}|_{s,p,\widehat{K}}. \tag{1.105}$$

Proof. The proof is similar to that of Lemma 1.101; note however the different factors appearing in (1.94) and (1.104) resulting from the fact that a different mapping ψ_K has been used. □

Let $\{K, \mathbb{RT}_0, \Sigma\}$ be the Raviart–Thomas finite element and let $\mathcal{I}_K^{\mathrm{RT}}$ be the associated local interpolation operator defined in (1.42).

Theorem 1.114. *Let $p > \frac{2d}{d+2}$. There is c such that, for all $v \in [W^{1,p}(K)]^d$ with $\nabla\cdot v \in W^{1,p}(K)$,*

$$\|\mathcal{I}_K^{\mathrm{RT}}v - v\|_{0,p,K} \leq c\,\sigma_K h_K |v|_{1,p,K},$$
$$\|\nabla\cdot(\mathcal{I}_K^{\mathrm{RT}}v - v)\|_{0,p,K} \leq c\,h_K|\nabla\cdot v|_{1,p,K}.$$

Proof. Set $V(\widehat{K}) = [W^{1,p}(\widehat{K})]^d$ with $p > \frac{2d}{d+2}$. The operator

$$\mathcal{F} : [W^{1,p}(\widehat{K})]^d \ni \widehat{w} \longmapsto \widehat{w} - \mathcal{I}_{\widehat{K}}^{\mathrm{RT}}\widehat{w} \in [L^p(\widehat{K})]^d,$$

is continuous. Since $[\mathbb{P}_0]^d \subset \mathbb{RT}_0$ and $\mathcal{F}$ vanishes on $[\mathbb{P}_0]^d$, it is clear that, for all $\widehat{w} \in V(\widehat{K})$,

$$\begin{aligned}
\|\widehat{w} - \mathcal{I}_{\widehat{K}}^{\mathrm{RT}}\widehat{w}\|_{0,p,\widehat{K}} &= \|\mathcal{F}(\widehat{w})\|_{0,p,\widehat{K}} = \inf_{\widehat{p}\in[\mathbb{P}_0]^d}\|\mathcal{F}(\widehat{w}+\widehat{p})\|_{0,p,\widehat{K}} \\
&\leq \|\mathcal{F}\|_{[W^{1,p}(\widehat{K})]^d,[L^p(\widehat{K})]^d}\inf_{\widehat{p}\in[\mathbb{P}_0]^d}\|\widehat{w}+\widehat{p}\|_{1,p,\widehat{K}} \\
&\leq c\inf_{\widehat{p}\in[\mathbb{P}_0]^d}\|\widehat{w}+\widehat{p}\|_{1,p,\widehat{K}} \leq c\,|\widehat{w}|_{1,p,\widehat{K}},
\end{aligned}$$

the last estimate resulting from the Deny–Lions Lemma applied componentwise. Let $v \in [W^{1,p}(K)]^d$ and set $\widehat{v} = \psi_K(v)$. Lemma 1.113 implies

$$\begin{aligned}
\|v - \mathcal{I}_K^{\mathrm{RT}}v\|_{0,p,K} &\leq c\,\|J_K\|_d\,|\det(J_K)|^{-\frac{1}{p'}}\,\|\widehat{v} - \mathcal{I}_{\widehat{K}}^{\mathrm{RT}}\widehat{v}\|_{0,p,\widehat{K}} \\
&\leq c\,\|J_K\|_d\,|\det(J_K)|^{-\frac{1}{p'}}\,|\widehat{v}|_{1,p,\widehat{K}} \\
&\leq c\,\|J_K\|_d^2\,\|J_K^{-1}\|_d\,|v|_{1,p,K} \\
&\leq c\,(\|J_K\|_d\,\|J_K^{-1}\|_d)\,\|J_K\|_d\,|v|_{1,p,K}.
\end{aligned}$$

The estimate on $\|\mathcal{I}_K^{\mathrm{RT}}v - v\|_{0,p,K}$ then results from (1.92). To prove the estimate on the divergence of the interpolation error, use Lemma 1.41, yielding

$$\|\nabla\cdot(\mathcal{I}_K^{\mathrm{RT}}v) - \nabla\cdot v\|_{0,p,K} = \|\pi_K^0[\nabla\cdot v] - \nabla\cdot v\|_{0,p,K} \le c\, h_K |\nabla\cdot v|_{1,p,K}.$$

Since $\nabla\cdot v$ is scalar-valued, the technique to prove the last inequality is identical to that used in the proof of Theorem 1.103. □

Corollary 1.115. *Let the assumptions of Theorem* 1.114 *hold. Let Ω be a polyhedron and let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine meshes of Ω. Let $\mathcal{I}_h^{\mathrm{RT}}$ be the global Raviart–Thomas interpolation operator defined in* (1.85)*. Let $p > \frac{2d}{d+2}$. Then, there is c such that, for all h and $v \in [W^{1,p}(\Omega)]^d$ with $\nabla\cdot v \in W^{1,p}(\Omega)$,*

$$\|v - \mathcal{I}_h^{\mathrm{RT}}v\|_{0,p,\Omega} + \|\nabla\cdot(v - \mathcal{I}_h^{\mathrm{RT}}v)\|_{0,p,\Omega} \le c\, h(\|v\|_{1,p,\Omega} + \|\nabla\cdot v\|_{1,p,\Omega}). \quad (1.106)$$

1.5.3 Interpolation in $H(\mathrm{curl};\Omega)$

The purpose of this section is to analyze the interpolation properties of the Nédélec finite element introduced in §1.2.8.

The space dimension is $d = 2$ or 3. The results are stated for $d = 3$, those for $d = 2$ being similar. As in the previous section, we assume that the mapping $T_K : \widehat{K} \to K$ is linear, i.e., $T_K(\widehat{x}) = J_K\widehat{x} + b_K$ with $J_K \in \mathbb{R}^{d,d}$ and $b_K \in \mathbb{R}^d$. For a vector-field $v \in [W^{s,p}(K)]^3$ with $s \ge 0$ and $p \ge 1$, we set $\widehat{v}(\widehat{x}) = J_K v(T_K(x))$, i.e., $\widehat{v} = \psi_K(v)$ where ψ_K is the transformation defined in (1.86). Denote by $\|\cdot\|_{\mathbb{R}^3}$ the Euclidean vector norm in $\mathbb{R}^3$ and by $\|\cdot\|_{\mathbb{R}^{3,3}}$ the associated matrix norm.

Lemma 1.116. *Let $s \ge 0$ and $1 \le p \le \infty$ (with $x^{\pm\frac{1}{p}} = 1$ for all $x > 0$ if $p = \infty$). There is c such that, for all K and $w \in [W^{s,p}(K)]^3$ with $\nabla\times w \in [W^{s,p}(K)]^3$,*

$$|w|_{s,p,K} \le c\,\|J_K^{-1}\|_{\mathbb{R}^{3,3}}^{s+1}|\det(J_K)|^{\frac{1}{p}}\,|\widehat{w}|_{s,p,\widehat{K}},$$
$$|\nabla\times w|_{s,p,K} \le c\,\|J_K^{-1}\|_{\mathbb{R}^{3,3}}^{s+2}|\det(J_K)|^{\frac{1}{p}}\,|\nabla\times\widehat{w}|_{s,p,\widehat{K}}.$$

Proof. The proof is similar to that of Lemma 1.101 and uses Lemma 1.89. Let us prove the second inequality with $s = 1$. Observe that

$$\|\partial_{x_i}\nabla\times v\|_{[L^p(K)]^3}^p = \|\nabla\times(\partial_{x_i}v)\|_{[L^p(K)]^3}^p = \|\mathcal{C}(\partial_{x_i}v)\|_{[L^p(K)]^{3,3}}^p$$
$$= |\det(J_K)|\int_{\widehat{K}}\left\|\sum_{j=1}^3 \partial_{x_i}\widehat{x}_j\,(J_K^{-1})^T\mathcal{C}(\partial_{\widehat{x}_j}\widehat{v})\,(J_K^{-1})\right\|_{\mathbb{R}^{3,3}}^p$$
$$\le |\det(J_K)|\|(J_K^{-1})^T\|_{\mathbb{R}^{3,3}}^p\|J_K^{-1}\|_{\mathbb{R}^{3,3}}^p\left(\sum_{j=1}^3|\partial_{x_i}\widehat{x}_j|^2\right)^{\frac{p}{2}}\int_{\widehat{K}}\left(\sum_{j=1}^3\|\partial_{\widehat{x}_j}\nabla\times\widehat{v}\|_{\mathbb{R}^3}^2\right)^{\frac{p}{2}}.$$

Then, since $\|(J_K^{-1})^T\|_{\mathbb{R}^{3,3}} = \|J_K^{-1}\|_{\mathbb{R}^{3,3}}$ and $\sum_{j=1}^3|\partial_{x_i}\widehat{x}_j|^2 \le \|J_K^{-1}\|_{\mathbb{R}^{3,3}}^2$, the desired result is obtained. □

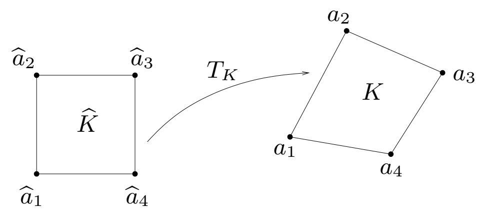


Fig. 1.23. Non-affine transformation mapping the unit square to a quadrangle.

Let $\{K, \mathbb{N}_0, \Sigma\}$ be the Nédélec finite element and let $\mathcal{I}_K^{\mathrm{N}}$ be the associated local interpolation operator defined in (1.48).

Theorem 1.117. *Let $p > 2$. There is c such that, for all $v \in [W^{1,p}(K)]^3$ with $\nabla\times v \in [W^{1,p}(K)]^3$,*

$$\|\mathcal{I}_K^{\mathrm{N}} v - v\|_{0,p,K} \le c\, \sigma_K h_K |v|_{1,p,K},$$
$$\|\nabla\times(\mathcal{I}_K^{\mathrm{N}} v - v)\|_{0,p,K} \le c\, h_K |\nabla\times v|_{1,p,K}.$$

Proof. The proof is similar to that of Theorem 1.114. □

Corollary 1.118. *Let the assumptions of Theorem* 1.117 *hold. Let Ω be a polyhedron and let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine meshes of Ω. Let $\mathcal{I}_h^{\mathrm{N}}$ be the global Nédélec interpolation operator defined in* (1.90). *Let $p > 2$. Then, there is c such that, for all h and $v \in [W^{1,p}(\Omega)]^3$ with $\nabla\times v \in [W^{1,p}(\Omega)]^3$,*

$$\|v - \mathcal{I}_h^{\mathrm{N}} v\|_{0,p,\Omega} + \|\nabla\times(v - \mathcal{I}_h^{\mathrm{N}} v)\|_{0,p,\Omega} \le c\, h(\|v\|_{1,p,\Omega} + \|\nabla\times v\|_{1,p,\Omega}). \quad (1.107)$$

1.5.4 Interpolation in $W^{s,p}(\Omega)$ on non-affine meshes

Interpolation on general quadrangles. This section contains a brief introduction to error estimates applicable to finite elements on quadrangles. For the sake of simplicity, we assume that Ω is a polygonal domain in $\mathbb{R}^2$ and that the reference cell $\widehat{K}$ is the unit square. For proofs and further insight, see, e.g., [GiR86, p. 104].

Let K be a non-degenerate, convex quadrangle in $\mathbb{R}^2$. We readily see that there exists a unique bijective transformation $T_K \in [\mathbb{Q}_1(\widehat{K})]^2$ such that $T_K(\widehat{K}) = K$ (see Figure 1.23); T_K maps the edges of $\widehat{K}$ to the edges of K, but unless K is a parallelogram T_K is not affine.

In this section, we assume again that $\psi_K(v) = v \circ T_K$.

Lemma 1.119. *Let K be a convex quadrangle in $\mathbb{R}^2$ and let T_K be the unique bijective transformation in $[\mathbb{Q}_1(\widehat{K})]^2$ mapping the unit square $\widehat{K}$ to K. Let J_K be the Jacobian matrix of T_K. Then, there exists c such that*

$$\|\det(J_K)\|_{L^\infty(\widehat{K})} \le c\, h_K^2, \qquad \|\det(J_K^{-1})\|_{L^\infty(K)} \le c\, \tfrac{1}{\rho_K^2},$$
$$\|J_K\|_{[L^\infty(\widehat{K})]^{2,2}} \le c\, h_K, \qquad \|J_K^{-1}\|_{[L^\infty(K)]^{2,2}} \le c\, \tfrac{h_K}{\rho_K^2},$$

where $h_K = \operatorname{diam}(K)$ *and* $\rho_K = \min_{1\le i\le 4} \rho_i$, ρ_i *being the diameter of the circle inscribed in the triangle formed by the three vertices* $(a_j)_{j\ne i}$ *of* K.

Theorem 1.120 (Local interpolation). *Let* $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ *be the reference finite element with* $\widehat{K} = [0,1]^2$ *and associated normed vector space* $V(\widehat{K})$. *Assume that there exists an integer* k *such that* $\mathbb{Q}_k \subset \widehat{P}$ *and* $H^{k+1}(\widehat{K}) \subset V(\widehat{K})$. *Let* K *be a quadrangle in* $\mathbb{R}^2$ *and let* $\mathcal{I}_K^k$ *be the local interpolation operator in* K *defined in* (1.59). *Then, setting* $\sigma_K = \frac{h_K}{\rho_K}$, *there exists* c *such that, for all* $m \in \{0, \ldots, k+1\}$ *and* $v \in H^{k+1}(K)$,

$$\begin{cases} \|v - \mathcal{I}_K^k v\|_{0,K} \le c\, \sigma_K h_K^{k+1} |v|_{k+1,K}, \\ |v - \mathcal{I}_K^k v|_{m,K} \le c\, \sigma_K^{4m-1} h_K^{k+1-m} |v|_{k+1,K}. \end{cases} \tag{1.108}$$

Definition 1.121 (Shape-regularity). *Let* ρ_K *be as in Lemma* 1.119. *A family* $\{\mathcal{T}_h\}_{h>0}$ *of quadrangular meshes is said to be* shape-regular *if there exists* σ_0 *such that*

$$\forall h,\ \forall K \in \mathcal{T}_h, \quad \sigma_K = \tfrac{h_K}{\rho_K} \le \sigma_0.$$

Corollary 1.122 (Global interpolation). *Let the assumptions of Theorem* 1.120 *hold. Let* Ω *be a polygonal domain in* $\mathbb{R}^2$. *Let* $\{\mathcal{T}_h\}_{h>0}$ *be a family of quadrangular meshes of* Ω *and assume that* $\{\mathcal{T}_h\}_{h>0}$ *is shape-regular according to Definition* 1.121. *Denote by* V_h^k *the approximation space based on* $\mathcal{T}_h$ *and* $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$. *Let* $\mathcal{I}_h^k$ *be the corresponding interpolation operator. Then, there exists* c *such that, for all* h *and* $v \in H^{k+1}(\Omega)$,

$$\|v - \mathcal{I}_h^k v\|_{0,\Omega} + \sum_{m=1}^{k+1} h^m \left(\sum_{K\in\mathcal{T}_h} \|v - \mathcal{I}_h^k v\|_{m,K}^2 \right)^{\frac{1}{2}} \le c\, h^{k+1} |v|_{k+1,\Omega}.$$

In particular, if V_h^k *is* H^1*-conforming,*

$$\forall h,\ \forall v \in H^{k+1}(\Omega), \quad |v - \mathcal{I}_h^k v|_{1,\Omega} \le c\, h^k |v|_{k+1,\Omega}.$$

Remark 1.123.

(i) In Theorem 1.120, the exponent on σ_K is larger than that obtained in (1.96) for affine meshes.

(ii) We deduce from Lemma 1.119 that for a shape-regular family of quadrangular meshes, the condition number of J_K is controlled uniformly with respect to h and $K \in \mathcal{T}_h$. □

Interpolation on domains with curved boundary. The goal of this section is to highlight an important practical result, namely that using a high-order reference finite element on a domain with curved boundary only pays off if the boundary is accurately represented. In particular, if a domain with curved boundary is approximated geometrically with affine meshes, using finite elements of degree larger than one is not asymptotically more accurate than using first-order finite elements.

For the sake of simplicity, we restrict the discussion to Lagrange geometric finite elements on simplices (see §1.3.2), and we consider *isoparametric* interpolation. For proofs and further insight, see [Ber89, BrS94, Cia91, CiR72^b, Len86, Zlá73, Zlá74].

Let $\{\widetilde{\mathcal{T}}_h\}_{h>0}$ be a family of *affine* meshes of Ω and set $\widetilde{\Omega}_h = \bigcup_{\widetilde{K}\in\widetilde{\mathcal{T}}_h} \widetilde{K}$. Let $k_{\text{geo}} \geq 2$ and let $F_h : \tilde{\Omega}_h \to \Omega_h = F_h(\widetilde{\Omega}_h)$ be a mapping such that $\forall \widetilde{K} \in \widetilde{\mathcal{T}}_h$, $F_{h|\widetilde{K}} \in [\mathbb{P}_{k_{\text{geo}}}]^d$. Using the mapping F_h, a new triangulation is constructed from $\widetilde{\mathcal{T}}_h$ by setting $\mathcal{T}_h = \{F_h(\widetilde{K})\}_{\widetilde{K}\in\widetilde{\mathcal{T}}_h}$. The concept of shape-regular family of meshes can be extended as follows:

Definition 1.124. *The family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be* shape-regular *if the affine family $(\widetilde{\mathcal{T}}_h)_h$ is shape-regular according to Definition* 1.107 *and if the mappings $\{F_h\}_{h>0}$ satisfy the following properties:*

(i) *F_h is the identity away from $\partial\Omega_h$; that is, $F_{h|\widetilde{K}} = \mathcal{I}$ if $\partial\widetilde{K} \cap \partial\widetilde{\Omega}_h = \emptyset$.*

(ii) $\sup_{x\in\partial\Omega} \operatorname{dist}(x, \partial\Omega_h) \leq c\, h^{k_{\text{geo}}+1}$ *with c independent of h.*

(iii) *The norm of the Jacobian matrix of F_h and the norm of its inverse are bounded uniformly in $[W^{k_{\text{geo}},\infty}(\Omega_h)]^{d,d}$ with respect to h.*

Theorem 1.125. *Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a Lagrange finite element of degree k with $k+1 > \frac{d}{2}$. Let Ω be a domain in $\mathbb{R}^d$ and let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of meshes according to Definition* 1.124 *with $k_{\text{geo}} = k$. Let*

$$V_h^k = \{v \in \mathcal{C}^0(\overline{\Omega}_h);\, v \circ F_h \in \widetilde{V}_h^k\},$$

where $\widetilde{V}_h^k$ is the approximation space based on the mesh $\widetilde{\mathcal{T}}_h$ and the reference finite element $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$. Let $\mathcal{I}_h^k$ be the interpolation operator on V_h^k. Then, there exists c such that

$$\forall h,\ \forall v \in H^{k+1}(\Omega_h), \quad \|v - \mathcal{I}_h^k v\|_{0,\Omega_h} + h\,|v - \mathcal{I}_h^k v|_{1,\Omega_h} \leq c\, h^{k+1} |v|_{k+1,\Omega_h}.$$

Moreover,

$$\forall v \in H^1(\Omega_h), \quad \lim_{h\to 0} \left(\inf_{v_h \in V_h^k} |v - v_h|_{1,\Omega_h} \right) = 0.$$

Proof. See, e.g., [BrS94, p. 117]. □

Remark 1.126. A different approach to extend the concept of shape-regularity is presented in [Cia91, p. 227]. Assume, for instance, that the geometric finite element is the Lagrange finite element $\mathbb{P}_2$. Let $\widehat{a}_i$, $0 \le i \le d$, be the vertices of $\widehat{K}$ and let $\widehat{a}_l$, $d+1 \le l \le \frac{1}{2}d(d+3)$, be the other nodes. Consider a similar notation for the nodes a_i and a_l of K. Let K° be the convex hull of the $(d+1)$ vertices of K and denote by a_l°, $d+1 \le l \le \frac{1}{2}d(d+3)$, the nodes located at the midpoints of the edges of K°. The shape-regularity criterion considered in [Cia91, p. 241] involves two conditions:

(i) The family of meshes formed by the simplices K° is shape-regular according to Definition 1.107.
(ii) There exists c such that, for all $l \in \{d+1, \dots, \frac{1}{2}d(d+3)\}$,

$$\forall h,\ \forall K, \quad \|a_l^\circ - a_l\|_d \le c\, h^2.$$

This definition can be extended to the Lagrange finite element $\mathbb{P}_3$ [Cia91, p. 247]. A general theory is presented in [CiR72^b]. □

1.6 Interpolation of Non-Smooth Functions

This section is concerned with the problem of interpolating non-smooth functions, e.g., functions that are too rough to be in the domain of the Lagrange interpolation operator. This situation occurs, for instance, when interpolating discontinuous functions, e.g., in $L^2(\Omega)$ or in $H^1(\Omega)$ in dimension $d \ge 2$. Throughout this section, Ω is a polyhedron and $\{\mathcal{T}_h\}_{h>0}$ is a shape-regular family of affine, simplicial, geometrically conforming meshes.

1.6.1 Clément interpolation

An interpolation technique to handle functions in L^1 using H^1-conforming Lagrange finite elements was first analyzed by Clément [Clé75]. The main ingredient is a regularization operator based on *macroelements* consisting of element patches. Let $P_{c,h}^k$ be the H^1-conforming approximation space based on the $\mathbb{P}_k$ Lagrange finite element; see (1.76). Let $\{a_1, \dots, a_N\}$ be the Lagrange nodes and let $\{\varphi_1, \dots, \varphi_N\}$ be the global shape functions in $P_{c,h}^k$. Associate with each node a_i the macroelement A_i consisting of the simplices containing a_i. Examples of macroelements are shown in Figure 1.24. Clearly, the macroelements can only assume a finite number of configurations, say n_{cf}. Denote by $\{\widehat{A}_n\}_{1 \le n \le n_{\mathrm{cf}}}$ the list of reference configurations. Define the application $j : \{1, \dots, N\} \to \{1, \dots, n_{\mathrm{cf}}\}$ such that $j(i)$ is the index of the reference configuration associated with the macroelement A_i. Define a $\mathcal{C}^0$-diffeomorphism F_{A_i} from $\widehat{A}_{j(i)}$ to A_i such that $\forall \widehat{K} \in \widehat{A}_{j(i)}$, $F_{A_i|\widehat{K}}$ is affine. The Clément interpolation operator $\mathcal{C}_h$ is then defined by local L^2-projections onto the macroelements. More precisely, for a reference macroelement $\widehat{A}_n$ and

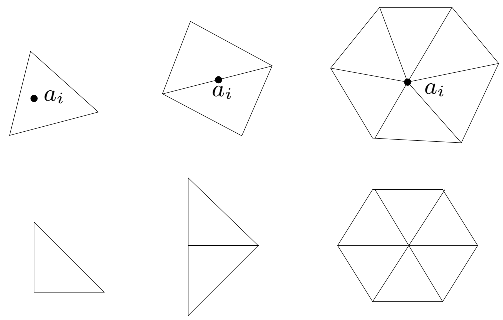


Fig. 1.24. Examples of macroelements A_i (top) and reference configuration $\widehat{A}_i$ (bottom) associated with a node a_i.

a function $\widehat{v} \in L^1(\widehat{A}_n)$, let $\widehat{\mathcal{C}}_n\widehat{v}$ be the unique polynomial in $\mathbb{P}_k$ such that $\int_{\widehat{A}_n}(\widehat{\mathcal{C}}_n\widehat{v} - \widehat{v})p = 0$ for all $p \in \mathbb{P}_k$. Then, the Clément interpolation operator is defined as follows:

$$\mathcal{C}_h : L^1(\Omega) \ni v \longmapsto \mathcal{C}_h v = \sum_{i=1}^{N} \widehat{\mathcal{C}}_{j(i)}(v \circ F_{A_i})(F_{A_i}^{-1}(a_i))\,\varphi_i \in P_{\mathrm{c},h}^k. \tag{1.109}$$

The stability and interpolation properties of the Clément operator are stated in the following:

Lemma 1.127 (Clément). *Under the above assumptions, the following properties hold:*

(i) *Stability: Let $1 \le p < +\infty$ and $0 \le m \le 1$. There is c such that*

$$\forall h,\ \forall v \in W^{m,p}(\Omega), \quad \|\mathcal{C}_h v\|_{W^{m,p}(\Omega)} \le c\,\|v\|_{W^{m,p}(\Omega)}. \tag{1.110}$$

(ii) *Approximation: For $K \in \mathcal{T}_h$, denote by Δ_K the set of elements in $\mathcal{T}_h$ sharing at least one vertex with K. Let F be an interface between two elements of $\mathcal{T}_h$, and denote by Δ_F the set of elements in $\mathcal{T}_h$ sharing at least one vertex with F; see Figure 1.25. Let l, m, and p satisfy $1 \le p < +\infty$ and $0 \le m \le l \le k+1$. Then, there is c such that*

$$\forall h,\ \forall K \in \mathcal{T}_h,\ \forall v \in W^{l,p}(\Delta_K), \quad \|v - \mathcal{C}_h v\|_{m,p,K} \le c\,h_K^{l-m}\|v\|_{l,p,\Delta_K}.$$

Similarly, if $m + \frac{1}{p} \le l \le k+1$,

$$\forall h,\ \forall K \in \mathcal{T}_h,\ \forall v \in W^{l,p}(\Delta_F), \quad \|v - \mathcal{C}_h v\|_{m,p,F} \le c\,h_F^{l-m-\frac{1}{p}}\|v\|_{l,p,\Delta_F}.$$

Proof. See [Clé75, Ber89, BeG98]. □

An easy consequence of this result is the following:

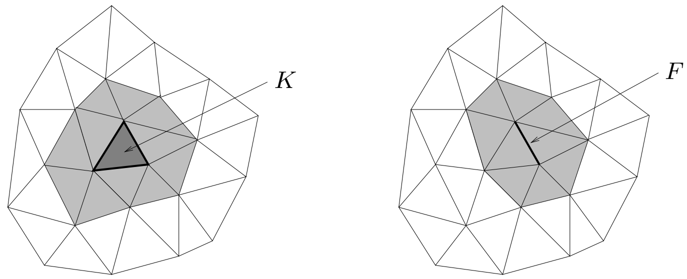


Fig. 1.25. Left: the shaded zone illustrates the set Δ_K of simplices sharing at least one vertex with the simplex K. Right: the shaded zone illustrates the set Δ_F of simplices sharing at least one vertex with the interface F.

Corollary 1.128. *Let the assumptions of Lemma* 1.127 *hold, let* $0 \le l \le k+1$, *and let* $0 \le m \le \min(1, l)$. *Then, there is* c *such that*

$$\forall h,\ \forall v \in W^{l,p}(\Omega), \qquad \inf_{v_h \in P^k_{\mathrm{c},h}} \|v - v_h\|_{m,p,\Omega} \le c\, h^{l-m} \|v\|_{l,p,\Omega}. \tag{1.111}$$

Remark 1.129.

(i) One difficulty with the Clément interpolation operator is that it does not preserve homogeneous boundary conditions, i.e., if v vanishes at the boundary, this is generally not the case for $\mathcal{C}_h v$. This problem is usually solved by setting boundary nodal values to zero. It can be shown that the Clément interpolant thus modified satisfies the estimates of Lemma 1.127.

(ii) The technique presented above can be generalized to other finite elements and to domains with curved boundaries; see, e.g., [Ber89, BeG98]. □

1.6.2 Scott–Zhang interpolation

Besides the fact that the Clément operator does not preserve boundary conditions, another difficulty is that it is not a projection. In [ScZ90], Scott and Zhang have addressed these two issues and defined an alternative interpolation operator.

Consider the notation and assumptions of the previous section. With each node a_i in the approximation space $P^k_{\mathrm{c},h}$ we associate either a d-simplex or a $(d-1)$-simplex, say Ξ_i, as follows: If a_i is in the interior of a d-simplex, say K, we simply set $\Xi_i = K$. If a_i is on a face, i.e., a $(d-1)$-simplex, say F, we set $\Xi_i = F$. Whenever a_i is at the boundary and in the intersection of many faces, it is important to pick the one face such that $F \subset \partial\Omega$. Let n_i be the number of nodes belonging to Ξ_i and denote by $\{\varphi_{i,q}\}_{1\le q\le n_i}$ the restrictions to Ξ_i of the local shape functions associated with the nodes lying in Ξ_i; see Figure 1.26. Conventionally set $\varphi_{i,1} = \varphi_i$. We now construct a family $\{\gamma_{i,q}\}_{1\le q\le n_i}$ as follows: For an integer q, $1 \le q \le n_i$, define $\gamma_{i,q} \in \operatorname{span}\{\varphi_{i,1}, \ldots, \varphi_{i,n_i}\}$ to be the unique function such that

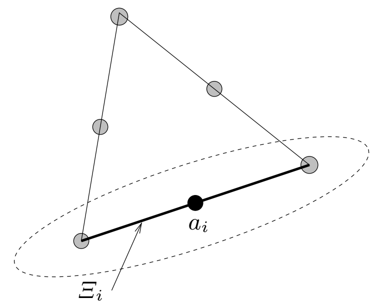


Fig. 1.26. Example of a node a_i with associated $(d-1)$-simplex Ξ_i ($d = 2$) containing $n_i = 3$ nodes.

$$\int_{\Xi_i} \gamma_{i,q}\varphi_{i,r} = \delta_{qr}, \quad 1 \le q, r \le n_i. \tag{1.112}$$

Then, the Scott–Zhang interpolation operator is defined as follows:

$$\mathcal{SZ}_h : W^{l,p}(\Omega) \ni v \longmapsto \mathcal{SZ}_h v(x) = \sum_{i=1}^{N} \varphi_i \int_{\Xi_i} \gamma_{i,1} v \in P^k_{c,h}. \tag{1.113}$$

It is clear that $\mathcal{SZ}_h$ preserves homogeneous boundary conditions, i.e., $v_{|\partial\Omega} = 0$ implies $\mathcal{SZ}_h v_{|\partial\Omega} = 0$. Furthermore, (1.112) implies $\mathcal{SZ}_h v_h = v_h$ for all $v_h \in P^k_{c,h}$. The interpolation properties of the Scott–Zhang interpolation operator are stated in the following:

Lemma 1.130 (Scott–Zhang). *Let p and l satisfy $1 \le p < +\infty$ and $l \ge 1$ if $p = 1$, and $l > \frac{1}{p}$ otherwise. Then, there is c such that the following properties hold:*

(i) *Stability: for all $0 \le m \le \min(1, l)$,*

$$\forall h,\ \forall v \in W^{l,p}(\Omega), \quad \|\mathcal{SZ}_h v\|_{m,p,\Omega} \le c\,\|v\|_{l,p,\Omega}. \tag{1.114}$$

(ii) *Approximation: provided $l \le k+1$, for all $0 \le m \le l$,*

$$\forall h,\ \forall K \in \mathcal{T}_h,\ \forall v \in W^{l,p}(\Delta_K), \qquad \|v - \mathcal{SZ}_h v\|_{m,p,K} \le c\, h_K^{l-m} |v|_{l,p,\Delta_K},$$

where Δ_K is defined in Lemma 1.127.

1.6.3 Orthogonal projections

Projection onto H^1-conforming spaces. Let $P^k_{c,h}$ be the H^1-conforming approximation space based on the $\mathbb{P}_k$ Lagrange finite element; see (1.76). The results presented in this section also hold for tensor product finite element spaces, e.g., the approximation space $Q^k_{c,h}$ defined in (1.77). Consider the following orthogonal projection operators:

$$\Pi_{c,h}^{0,k} : L^2(\Omega) \longrightarrow P_{c,h}^k \quad \text{and} \quad \Pi_{c,h}^{1,k} : H^1(\Omega) \longrightarrow P_{c,h}^k,$$

with scalar products $(u,v)_{0,\Omega} = \int_\Omega uv$ and $(u,v)_{1,\Omega} = \int_\Omega uv + \int_\Omega \nabla u \cdot \nabla v$, respectively. Recall that

$$\forall v_h \in P_{c,h}^k, \qquad (\Pi_{c,h}^{0,k}(u), v_h)_{0,\Omega} = (u, v_h)_{0,\Omega},$$
$$\forall v_h \in P_{c,h}^k, \qquad (\Pi_{c,h}^{1,k}(u), v_h)_{1,\Omega} = (u, v_h)_{1,\Omega},$$

and that $\Pi_{c,h}^{0,k} v$ (resp., $\Pi_{c,h}^{1,k} v$) is the closest function to v in $P_{c,h}^k$ for the L^2-norm (resp., H^1-norm). The operator $\Pi_{c,h}^{1,k}$ is often called the *elliptic projector* or the *Riesz projector*.

Lemma 1.131 (Stability). *Let $k \geq 1$. The following estimates hold:*

$$\forall v \in L^2(\Omega), \quad \|\Pi_{c,h}^{0,k} v\|_{0,\Omega} \leq \|v\|_{0,\Omega}, \tag{1.115}$$
$$\forall v \in H^1(\Omega), \quad \|\Pi_{c,h}^{1,k} v\|_{1,\Omega} \leq \|v\|_{1,\Omega}. \tag{1.116}$$

Moreover, if the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, there exists c such that

$$\forall h,\ \forall v \in H^1(\Omega), \quad \|\Pi_{c,h}^{0,k} v\|_{1,\Omega} \leq c\,\|v\|_{1,\Omega}. \tag{1.117}$$

Proof. The stability estimates (1.115)–(1.116) directly follow from the definition of orthogonal projections. Indeed, using the Pythagoras identity yields

$$\forall v \in L^2(\Omega), \quad \|v\|_{0,\Omega} = \|\Pi_{c,h}^{0,k} v\|_{0,\Omega} + \|v - \Pi_{c,h}^{0,k} v\|_{0,\Omega},$$

and a similar identity holds for $\Pi_{c,h}^{1,k}$. The proof of (1.117) is the subject of Exercise 1.17. □

Remark 1.132. Under reasonable assumptions, the stability estimate (1.116) can be substantially improved. In particular, the elliptic projector is stable in $W^{1,p}(\Omega)$; see Theorem 3.21 and [BrS94, p. 170]. □

For $s \geq 1$ and $v \in L^2(\Omega)$, we define the so-called *negative-norm*

$$\|v\|_{-s,\Omega} = \sup_{w \in H^s(\Omega) \cap H_0^1(\Omega)} \frac{(v,w)_{0,\Omega}}{\|w\|_{s,\Omega}}.$$

Note that this is not the norm considered to define the dual space $H^{-s}(\Omega)$, except in the particular case $s = 1$. Here, the norm $\|\cdot\|_{-s,\Omega}$ is simply used as a quantitative measure for functions in $L^2(\Omega)$.

Proposition 1.133. *Let $k \geq 1$ and $1 \leq s \leq k+1$. Then, there is c such that*

$$\forall h,\ \forall v \in L^2(\Omega), \quad \|v - \Pi_{c,h}^{0,k} v\|_{-s,\Omega} \leq c\, h^s \inf_{v_h \in P_{c,h}^k} \|v - v_h\|_{0,\Omega}. \tag{1.118}$$

Proof. Let $v \in L^2(\Omega)$ and $w \in H^s(\Omega) \cap H_0^1(\Omega)$. Since $s \le k+1$, Lemma 1.127 implies

$$\|w - \mathcal{C}_h w\|_{0,\Omega} \le c\, h^s |w|_{s,\Omega}.$$

Furthermore, since $v - \Pi_{\mathrm{c},h}^{0,k} v$ is L^2-orthogonal to $P_{\mathrm{c},h}^k$,

$$(v - \Pi_{\mathrm{c},h}^{0,k} v, w)_{0,\Omega} = (v - \Pi_{\mathrm{c},h}^{0,k} v, w - \mathcal{C}_h w)_{0,\Omega} \le c\, h^s \|v - \Pi_{\mathrm{c},h}^{0,k} v\|_{0,\Omega}\, \|w\|_{s,\Omega}.$$

The result follows easily. □

Finally, we state approximation properties for smooth functions.

Proposition 1.134. *Let $k \ge 1$ and $1 \le l \le k$.*

(i) *There exists c such that, for all h and $v \in H^{l+1}(\Omega)$,*

$$\|v - \Pi_{\mathrm{c},h}^{0,k}(v)\|_{0,\Omega} \le c\, h^{l+1} |v|_{l+1,\Omega}, \tag{1.119}$$

$$\|v - \Pi_{\mathrm{c},h}^{1,k}(v)\|_{1,\Omega} \le c\, h^{l} |v|_{l+1,\Omega}. \tag{1.120}$$

(ii) *If Ω is convex, there exists c such that, for all h and $v \in H^{l+1}(\Omega)$,*

$$\|v - \Pi_{\mathrm{c},h}^{1,k}(v)\|_{0,\Omega} \le c\, h^{l+1} |v|_{l+1,\Omega}. \tag{1.121}$$

(iii) *If the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, there exists c such that, for all h and $v \in H^{l+1}(\Omega)$,*

$$\|v - \Pi_{\mathrm{c},h}^{0,k}(v)\|_{1,\Omega} \le c\, h^{l} |v|_{l+1,\Omega}. \tag{1.122}$$

Proof. See Exercise 1.18. □

Projection onto totally discontinuous spaces. Let $k \ge 0$ and consider the L^2-orthogonal projection $\Pi_{\mathrm{td},h}^{0,k}$ from $L^2(\Omega)$ to the space $P_{\mathrm{td},h}^k$ defined in (1.67). Clearly, for $v \in L^2(\Omega)$ and $K \in \mathcal{T}_h$,

$$\begin{cases} (\Pi_{\mathrm{td},h}^{0,k} v)_{|K} \in \mathbb{P}_k, \\ \displaystyle\int_K (\Pi_{\mathrm{td},h}^{0,k} v - v) q = 0, \quad \forall q \in \mathbb{P}_k. \end{cases} \tag{1.123}$$

In particular, $\Pi_{\mathrm{td},h}^{0,0} v$ is the piecewise constant function equal to $\frac{1}{\mathrm{meas}(K)} \int_K v$ on all cells $K \in \mathcal{T}_h$. The approximation properties of $\Pi_{\mathrm{td},h}^{0,k}$ are stated in the following:

Proposition 1.135. *There exists c, independent of h, such that, for all $0 \le l \le k+1$, $1 \le p \le \infty$, and $v \in W^{l,p}(\Omega)$,*

$$\|v - \Pi_{\mathrm{td},h}^{0,k} v\|_{0,p,\Omega} \le c\, h^{l} |v|_{l,p,\Omega}. \tag{1.124}$$

Proof. Straightforward verification. □

Remark 1.136.

(i) The shape-regularity assumption on the mesh is not required for (1.124).

(ii) An estimate similar to (1.118) holds for $\Pi_{\mathrm{td},h}^{0,k} v$. □

1.6.4 The discrete commutator property

The so-called discrete commutator property is a powerful tool to analyze nonlinear problems; see Bertoluzza [Ber99] and [JoS87]. As a corollary of Lemma 1.130, we infer the following:

Lemma 1.137 (Bertoluzza). *Let the hypotheses of Lemma* 1.130 *hold. Then, there is c such that, for all h, v_h in $P^k_{\mathrm{c},h}$, ϕ in $W^{s+1,\infty}(\Omega)$, and* $0 \le m \le s \le 1$,

$$\|\phi v_h - \mathcal{SZ}_h(\phi v_h)\|_{m,p,\Omega} \le c\, h^{1+s-m} \|v_h\|_{s,p,\Omega} \|\phi\|_{s+1,\infty,\Omega}.$$

Proof. We prove the result locally. Let K be a cell in the mesh $\mathcal{T}_h$. Denote by x_K some point in K, say the barycenter of K. Let ϕ be a function in $W^{s+1,\infty}(\Omega)$. Define $R_K = \phi - \phi(x_K)$. It is clear that $R_K \in W^{1,\infty}(\Omega)$ and

$$\|R_K\|_{0,\infty,\Delta_K} \le c\, h_K \|\phi\|_{1,\infty,\Omega},$$
$$\|R_K\|_{1,\infty,\Delta_K} \le c\, \|\phi\|_{1,\infty,\Omega}.$$

Let $\overline{v}_h$ be the mean value of v_h on Δ_K. Then, one readily verifies that

$$\|\overline{v}_h\|_{0,p,\Delta_K} \le c\, \|v_h\|_{0,p,\Delta_K},$$
$$\|v_h - \overline{v}_h\|_{m,p,\Delta_K} \le c\, h_K^{s-m} \|v_h\|_{s,p,\Delta_K}, \qquad 0 \le m \le s \le 1.$$

Furthermore, observe that

$$\begin{aligned}\|\phi v_h - \mathcal{SZ}_h(\phi v_h)\|_{m,p,K} \le\ & \|(\mathcal{I} - \mathcal{SZ}_h)(\phi \overline{v}_h)\|_{m,p,K} \\ & + \|(\mathcal{I} - \mathcal{SZ}_h)\big(\phi(v_h - \overline{v}_h)\big)\|_{m,p,K},\end{aligned}$$

and denote by R_1 and R_2 the two residuals in the right-hand side. Since $s \ge 0$, $1 + s \ge \frac{1}{p}$ if $p = 1$ and $1 + s > \frac{1}{p}$ if $p > 1$; moreover, $s \le 1 \le k$. As a result, one can use Lemma 1.130 to control R_1 as follows:

$$\begin{aligned}R_1 &\le c\, h_K^{1+s-m} \|\phi \overline{v}_h\|_{s+1,p,\Delta_K} \le c\, h_K^{1+s-m} \|\overline{v}_h\|_{0,p,\Delta_K} \|\phi\|_{s+1,\infty,\Omega} \\ &\le c\, h_K^{1+s-m} \|v_h\|_{0,p,\Delta_K} \|\phi\|_{s+1,\infty,\Omega}.\end{aligned}$$

For the other residual, use the fact that $\mathcal{SZ}_h$ is linear, $P^k_{\mathrm{c},h}$ is invariant under $\mathcal{SZ}_h$, and $\mathcal{SZ}_h(\overline{v}_h) = \overline{v}_h$ on K to obtain

$$(\mathcal{I} - \mathcal{SZ}_h)\big(\phi(v_h - \overline{v}_h)\big) = (\mathcal{I} - \mathcal{SZ}_h)\big((\phi - \phi(x_K))(v_h - \overline{v}_h)\big).$$

As a result,

$$\begin{aligned}R_2 &= \|(\mathcal{I} - \mathcal{SZ}_h)\big(R_K(v_h - \overline{v}_h)\big)\|_{m,p,K} \\ &\le c\, h_K^{1-m} |R_K(v_h - \overline{v}_h)|_{1,p,\Delta_K} \\ &\le c\, h_K^{1-m} (\|R_K\|_{0,\infty,\Delta_K} |v_h - \overline{v}_h|_{1,p,\Delta_K} \\ &\qquad + |R_K|_{1,\infty,\Delta_K} \|v_h - \overline{v}_h\|_{0,p,\Delta_K}) \\ &\le c\, h_K^{1-m} (h_K |v_h - \overline{v}_h|_{1,p,\Delta_K} + \|v_h - \overline{v}_h\|_{0,p,\Delta_K}) \|\phi\|_{1,\infty,\Omega} \\ &\le c\, h_K^{1+s-m} \|v_h\|_{s,p,\Delta_K} \|\phi\|_{1,\infty,\Omega}.\end{aligned}$$

Then, the desired result follows easily from the shape-regularity of the mesh, which implies that $\sup_{K'\in\mathcal{T}_h}(\mathrm{card}\{K \in \mathcal{T}_h;\, K' \subset \Delta_K\})$ is a fixed constant independent of h. $\square$

1.7 Inverse Inequalities

The goal of this section is to compare various functional norms on approximation spaces. Such spaces being finite-dimensional, all the norms therein are equivalent. The purpose of inverse inequalities is to specify how the equivalence constants depend on h. For the sake of simplicity, we restrict ourselves to affine meshes and to finite elements for which $\psi_K(v) = v \circ T_K$.

Lemma 1.138 (Local inverse inequalities). *Let $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ be a finite element. Let $l \geq 0$ be such that $\widehat{P} \subset W^{l,\infty}(\widehat{K})$. Let $\{\mathcal{T}_h\}_{h>0}$ be a shape-regular family of affine meshes in $\mathbb{R}^d$ with $h \leq 1$. Let $0 \leq m \leq l$ and $1 \leq p, q \leq \infty$. Then, there is c, independent of h, K, p, and q, such that, for all $v \in P_K = \{\widehat{p} \circ T_K^{-1};\, \widehat{p} \in \widehat{P}\}$,*

$$\|v\|_{l,p,K} \leq c\, h_K^{m-l+d(\frac{1}{p}-\frac{1}{q})} \|v\|_{m,q,K}. \tag{1.125}$$

Proof. (1) Since all the norms in $\widehat{P} \subset W^{l,\infty}(\widehat{K})$ are equivalent, there exists c, only depending on $\widehat{K}$ and l, such that, for all $\widehat{v} \in \widehat{P}$, $\|\widehat{v}\|_{l,\infty,\widehat{K}} \leq c\|\widehat{v}\|_{0,1,\widehat{K}}$; hence,

$$\forall \widehat{v} \in \widehat{P}, \qquad \|\widehat{v}\|_{l,p,\widehat{K}} \leq c\, \|\widehat{v}\|_{0,q,\widehat{K}}. \tag{1.126}$$

(2) Let $v \in P_K$ and $0 \leq j \leq l$. Using (1.93), (1.94), (1.126), and the shape-regularity of the family $\{\mathcal{T}_h\}_{h>0}$ yields

$$\begin{aligned}|v|_{j,p,K} &\leq c\, h_K^{-j}\, |\det(J_K)|^{\frac{1}{p}}\, \|\widehat{v}\|_{j,p,\widehat{K}} \leq c\, h_K^{-j}\, |\det(J_K)|^{\frac{1}{p}}\, \|\widehat{v}\|_{0,q,\widehat{K}} \\ &\leq c\, h_K^{-j}\, |\det(J_K)|^{\frac{1}{p}-\frac{1}{q}}\, \|v\|_{0,q,K} \leq c\, h_K^{-j+d(\frac{1}{p}-\frac{1}{q})}\, \|v\|_{0,q,K}.\end{aligned}$$

Since $h_K \leq h \leq 1$ by assumption,

$$\forall v \in P_K,\ \forall j \in \{0,\ldots,l\}, \quad \|v\|_{j,p,K} \leq c\, h_K^{-j+d(\frac{1}{p}-\frac{1}{p})} \|v\|_{0,q,K}.$$

Taking $j = l$ yields (1.125) for $m = 0$.
(3) Let $0 \leq m \leq l$. Let α be a multi-index such that $0 \leq |\alpha| \leq l$. If $|\alpha| < l - m$,

$$\|\partial^\alpha v\|_{0,p,K} \leq \|v\|_{l-m,p,K} \leq c h_K^{m-l+d(\frac{1}{p}-\frac{1}{p})} \|v\|_{0,q,K} \leq c h_K^{m-l+d(\frac{1}{p}-\frac{1}{p})} \|v\|_{m,q,K}.$$

If $l - m \leq |\alpha| \leq l$, one can find two multi-indices β and γ such that $\alpha = \beta + \gamma$ and $|\beta| = l - m$. Hence,

$$\begin{aligned}\|\partial^\alpha v\|_{0,p,K} &= \|\partial^\beta(\partial^\gamma v)\|_{0,p,K} \leq \|\partial^\gamma v\|_{l-m,p,K} \\ &\leq c\, h_K^{m-l+d(\frac{1}{p}-\frac{1}{q})}\, \|\partial^\gamma v\|_{0,q,K} \leq c\, h_K^{m-l+d(\frac{1}{p}-\frac{1}{q})}\, \|v\|_{m,q,K},\end{aligned}$$

since $|\gamma| \leq m$. This proves that for all multi-index α such that $0 \leq |\alpha| \leq l$, $\|\partial^\alpha v\|_{0,p,K} \leq c\, h_K^{m-l+d(\frac{1}{p}-\frac{1}{q})} \|v\|_{m,q,K}$. The conclusion follows readily. □

Example 1.139. For $p = q$, $l = 1$ and $m = 0$, Lemma 1.138 yields $\|v\|_{1,p,K} \leq c\, h_K^{-1} \|v\|_{0,p,K}$ for all h, $K \in \mathcal{T}_h$, and $v \in P_K$. □

To obtain global inverse inequalities, the quantity h_K^{-1} must be controlled. This observation leads to the following:

Definition 1.140 (Quasi-uniformity). *A family of meshes $\{\mathcal{T}_h\}_{h>0}$ is said to be* quasi-uniform *if and only if it is shape-regular and there is c such that*

$$\forall h,\ \forall K \in \mathcal{T}_h, \quad h_K \geq c\, h. \tag{1.127}$$

Corollary 1.141 (Global inverse inequalities). *Along with the hypotheses of Lemma* 1.138*, assume that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. Set $W_h = \{v_h;\, \forall K \in \mathcal{T}_h,\ v_h \circ T_K \in \widehat{P}\}$. Then, using the usual convention if $p = \infty$ or $q = \infty$, there is c, independent of h, such that, for all $v_h \in W_h$ and $0 \leq m \leq l$,*

$$\left(\sum_{K \in \mathcal{T}_h} \|v_h\|_{l,p,K}^p \right)^{\frac{1}{p}} \leq c\, h^{m-l+\min(0,\frac{d}{p}-\frac{d}{q})} \left(\sum_{K \in \mathcal{T}_h} \|v_h\|_{m,q,K}^q \right)^{\frac{1}{q}}. \tag{1.128}$$

Proof. Let $v_h \in W_h$. Assume $p \neq \infty$ and $q \neq \infty$ (these two cases are treated similarly).
(1) Assume $p \geq q$. Then, (1.125) implies

$$\sum_{K \in \mathcal{T}_h} \|v_h\|_{l,p,K}^p \leq c\, h^{p(m-l+d(\frac{1}{p}-\frac{1}{q}))} \sum_{K \in \mathcal{T}_h} \|v_h\|_{m,q,K}^p.$$

To conclude, use the inequality $(\sum_{i \in I} a_i^p)^{\frac{1}{p}} \leq (\sum_{i \in I} a_i^q)^{\frac{1}{q}}$ which holds for $p \geq q \geq 0$ and for all finite sequence of non-negative numbers $\{a_i\}_{i \in I}$; see Exercise 1.20. (2) Assume $p \leq q$. Then, (1.125) implies

$$\begin{aligned}
\sum_{K \in \mathcal{T}_h} \|v_h\|_{l,p,K}^p &\leq c\, h^{p(m-l)} \sum_{K \in \mathcal{T}_h} h_K^{dp(\frac{1}{p}-\frac{1}{q})} \|v_h\|_{m,q,K}^p \\
&\leq c\, h^{p(m-l)} \left(\sum_{K \in \mathcal{T}_h} h_K^{dp(\frac{1}{p}-\frac{1}{q})\frac{q}{q-p}} \right)^{\frac{q-p}{q}} \left(\sum_{K \in \mathcal{T}_h} \|v_h\|_{m,q,K}^q \right)^{\frac{p}{q}} \\
&\leq c\, h^{p(m-l)} \operatorname{meas}(\Omega)^{\frac{q-p}{q}} \left(\sum_{K \in \mathcal{T}_h} \|v_h\|_{m,q,K}^q \right)^{\frac{p}{q}}.
\end{aligned}$$ □

The following result is often used when dealing with nonlinear problems:

Lemma 1.142. *Along with the hypotheses of Lemma* 1.138*, assume that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. Set $W_h = \{v_h;\, \forall K \in \mathcal{T}_h,\ v_h \circ T_K \in \widehat{P}\}$. Then, there is c, independent of h, such that, for all $v_h \in W_h \cap H^1(\Omega)$,*

$$\|v_h\|_{L^\infty(\Omega)} \le \begin{cases} c\,(1+|\log h|)\|v_h\|_{1,\Omega} & \textit{in dimension } 2,\\ c\,h^{-\frac{1}{2}}\|v_h\|_{1,\Omega} & \textit{in dimension } 3. \end{cases} \tag{1.129}$$

Proof. See Exercise 1.21 for a proof in dimension 3. □

Remark 1.143.

(i) A simple consequence of Corollary 1.141 is that for all $v_h \in W_h \cap W^{1,p}(\Omega)$, $\|v_h\|_{1,p,\Omega} \le c\,h^{-1}\|v_h\|_{0,p,\Omega}$.

(ii) A necessary and sufficient condition for quasi-uniformity is that there exists τ such that $\rho_K \ge \tau h$ for all h and $K \in \mathcal{T}_h$. Indeed, if $\{\mathcal{T}_h\}_{h>0}$ satisfies the above property, then $\frac{h_K}{\rho_K} \le \tau^{-1}\frac{h_K}{h} \le \tau^{-1}$ for all h and $K \in \mathcal{T}_h$, thus showing that the family $\{\mathcal{T}_h\}_{h>0}$ is shape-regular. Furthermore, $h_K \ge \rho_K \ge \tau\,h$ implies (1.127). Conversely, if $\{\mathcal{T}_h\}_{h>0}$ is a quasi-uniform mesh family, $\rho_K \ge \frac{1}{\sigma}h_K \ge \frac{c}{\sigma}h$ for all $h > 0$ and $K \in \mathcal{T}_h$.

(iii) In two dimensions, one can construct a finer triangulation from an initial triangulation by connecting all the edge midpoints. Repeating this procedure yields a quasi-uniform family of meshes; see [Zha95].

(iv) See, e.g., [GiR86, p. 103] and [BrS94, p. 109] for further insight. □

1.8 Exercises

Exercise 1.1. Let $\mathcal{I}_h^1$ be the one-dimensional $\mathbb{P}_1$ Lagrange interpolation operator defined in (1.6).

(i) Prove that for all h and $v \in \mathcal{C}^0(\overline{\Omega})$, $\|\mathcal{I}_h^1 v\|_{\mathcal{C}^0(\overline{\Omega})} \le \|v\|_{\mathcal{C}^0(\overline{\Omega})}$.
(ii) Prove that for all h and $v \in \mathcal{C}^1(\overline{\Omega})$, $\|v - \mathcal{I}_h^1 v\|_{\mathcal{C}^0(\overline{\Omega})} \le h\|v\|_{\mathcal{C}^1(\overline{\Omega})}$. (*Hint*: Use the mean-value theorem.)

Exercise 1.2 (Hermite finite element).

(i) Prove Lemma 1.82 and Proposition 1.83.
(ii) Prove (1.103) for $v \in H^4(\Omega)$ without using the results of §1.5. (*Hint*: Adapt the proof of Proposition 1.5 by showing that on a mesh interval I_i, $(v - \mathcal{I}_h^{\mathrm{H}} v_{|I_i})'''$ vanishes at least at one point of I_i.)

Exercise 1.3 ($\mathbb{P}_k$ Lagrange finite element).

(i) Let $p \in \mathbb{P}_k$ with $k \ge 1$. Assume that p vanishes on the $\mathbb{R}^d$-hyperplane of equation $\lambda = 0$. Prove that there is $q \in \mathbb{P}_{k-1}$ such that $p = \lambda q$. Then, prove Proposition 1.34. (*Hint*: By induction on k.)
(ii) Prove that if $k \le d$ and $p \in \mathbb{P}_k$ vanishes at all the faces of K, then $p = 0$.
(iii) Prove that the number of nodes of a $\mathbb{P}_k$ Lagrange finite element located on any edge of K is $(k+1)$ in arbitrary dimension $d \ge 2$. Prove that the number of nodes located on any face of K is the dimension of $\mathbb{P}_k$ in dimension $(d-1)$. Justify Remark 1.76.