Appendix A

Banach and Hilbert spaces

The goal of this appendix is to recall basic results on Banach and Hilbert spaces. To stay general, we consider complex vector spaces, i.e., vector spaces over the field $\mathbb{C}$ of complex numbers. The case of real vector spaces is recovered by replacing the field $\mathbb{C}$ by $\mathbb{R}$, by removing the real part symbol $\Re(\cdot)$ and the complex conjugate symbol $\bar{\cdot}$, and by interpreting the symbol $|\cdot|$ as the absolute value instead of the modulus.

A.1 Banach spaces

Let V be a complex vector space.

Definition A.1 (Norm). *A* norm *on V is a map $\|\cdot\|_V : V \to \mathbb{R}_+ := [0, \infty)$ satisfying the following three properties:*

(i) *Definiteness:* $[\,\|v\|_V = 0\,] \iff [\,v = 0\,]$.
(ii) 1*-homogeneity:* $\|\lambda v\|_V = |\lambda|\,\|v\|_V$ *for all* $\lambda \in \mathbb{C}$ *and all* $v \in V$.
(iii) *Triangle inequality:* $\|v + w\|_V \le \|v\|_V + \|w\|_V$ *for all* $v, w \in V$.

For every norm $\|\cdot\|_V : V \to \mathbb{R}_+ := [0, \infty)$, the function $d(x, y) := \|x - y\|_V$, for all $x, y \in V$, is a metric (or distance).

Remark A.2 (Definiteness). Item (i) can be slightly relaxed by requiring only that $[\,\|v\|_V = 0\,] \implies [\,v = 0\,]$, since the 1-homogeneity implies that $[\,v = 0\,] \implies [\,\|v\|_V = 0\,]$. □

Definition A.3 (Seminorm). *A* seminorm *on V is a map $|\cdot|_V : V \to \mathbb{R}_+$ satisfying only the statements* (ii) *and* (iii) *above, i.e.,* 1*-homogeneity and the triangle inequality.*

Definition A.4 (Banach space). *A vector space V equipped with a norm $\|\cdot\|_V$ is called* Banach space *if every Cauchy sequence in V has a limit in V.*

Definition A.5 (Equivalent norms). *Two norms $\|\cdot\|_{V,1}$ and $\|\cdot\|_{V,2}$ are said to be* equivalent *on V if there exists a positive real number c such that*

$$c\,\|v\|_{V,2} \le \|v\|_{V,1} \le c^{-1}\,\|v\|_{V,2}, \qquad \forall v \in V. \tag{A.1}$$

Whenever (A.1) *holds true, V is a Banach space for the norm $\|\cdot\|_{V,1}$ if and only if it is a Banach space for the norm $\|\cdot\|_{V,2}$.*

Remark A.6 (Finite dimension). If V is finite-dimensional, all the norms in V are equivalent. This result is false in infinite-dimensional vector spaces. Actually, the unit ball in V is a compact set (for the norm topology) if and only if V is finite-dimensional; see Brezis [48, Thm. 6.5], Lax [131, §5.2]. □

A.2 Bounded linear maps and duality

Definition A.7 (Linear, antilinear map). *Let V, W be complex vector spaces. A map $A : V \to W$ is said to be* linear *if $A(v_1 + v_2) = A(v_1) + A(v_2)$ for all $v_1, v_2 \in V$ and $A(\lambda v) = \lambda A(v)$ for all $\lambda \in \mathbb{C}$ and all $v \in V$, and it is said to be* antilinear *if $A(v_1 + v_2) = A(v_1) + A(v_2)$ for all $v_1, v_2 \in V$ and $A(\lambda v) = \overline{\lambda} A(v)$ for all $\lambda \in \mathbb{C}$ and all $v \in V$.*

Definition A.8 (Bounded (anti)linear map). *Assume that V and W are equipped with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. The (anti)linear map $A : V \to W$ is said to be* bounded *or* continuous *if*

$$\|A\|_{\mathcal{L}(V;W)} := \sup_{v \in V} \frac{\|A(v)\|_W}{\|v\|_V} < \infty. \tag{A.2}$$

In this book, we systematicaly abuse the notation by implicitly assuming that the argument in this type of supremum is nonzero. Bounded (anti)linear maps in Banach spaces are called operators*.*

The complex vector space composed of the bounded linear maps from V to W is denoted by $\mathcal{L}(V;W)$. One readily verifies that the map $\|\cdot\|_{\mathcal{L}(V;W)}$ defined in (A.2) is indeed a norm on $\mathcal{L}(V;W)$.

Proposition A.9 (Banach space). *Assume that W is a Banach space. Then $\mathcal{L}(V;W)$ equipped with the norm* (A.2) *is also a Banach space. The same statement holds true for the complex vector space composed of all the bounded antilinear maps from V to W.*

Proof. See Rudin [170, p. 87], Yosida [202, p. 111]. □

Example A.10 (Continuous embedding). Assume that $V \subset W$ and that there is a real number c such that $\|v\|_W \le c\|v\|_V$ for all $v \in V$. This means that the embedding of V into W is continuous. We say that V is continuously embedded into W, and we write $V \hookrightarrow W$. □

The dual of a real Banach space V is composed of the bounded linear maps from V to $\mathbb{R}$. The same definition can be adopted if V is a complex space, but to stay consistent with the formalism considered in the weak formulation of complex-valued PDEs, we define the dual space as being composed of bounded antilinear maps from V to $\mathbb{C}$.

Definition A.11 (Dual space). *Let V be a complex vector space. The* dual space *of V is denoted by V' and is composed of the bounded antilinear maps from V to $\mathbb{C}$. An element $A \in V'$ is called* bounded antilinear form, *and its action on an element $v \in V$ is denoted either by $A(v)$ or $\langle A, v\rangle_{V',V}$.*

Owing to Proposition A.9, V' is a Banach space with the norm

$$\|A\|_{V'} = \sup_{v\in V} \frac{|A(v)|}{\|v\|_V} = \sup_{v\in V} \frac{|\langle A, v\rangle_{V',V}|}{\|v\|_V}, \qquad \forall A \in V'. \tag{A.3}$$

Remark A.12 (Linear vs. antilinear form). If $A : V \to \mathbb{C}$ is an antilinear form, then $\overline{A}$ (defined by $\overline{A}(v) := \overline{A(v)} \in \mathbb{C}$ for all $v \in V$) is a linear form. □

A.3 Hilbert spaces

Let V be a complex vector space.

Definition A.13 (Inner product). *An* inner product *on V is a map $(\cdot,\cdot)_V : V \times V \to \mathbb{C}$ satisfying the following three properties:* (i) *Sesquilinearity (the prefix sesqui means one and a half): $(\cdot, w)_V$ is a linear map for all fixed $w \in V$, whereas $(v, \cdot)_V$ is an antilinear map for all fixed $v \in V$. If V is a real vector space, the inner product is a bilinear map (i.e., it is linear in both of its arguments).* (ii) *Hermitian symmetry: $(v, w)_V = \overline{(w, v)_V}$ for all $v, w \in V$.* (iii) *Positive definiteness: $(v, v)_V \geq 0$ for all $v \in V$ and $[\,(v, v)_V = 0\,] \iff [\,v = 0\,]$. (Notice that $(v, v)_V$ is always real owing to the Hermitian symmetry and that $(0, \cdot)_V = (\cdot, 0)_V = 0$ owing to sesquilinearity.)*

Proposition A.14 (Cauchy–Schwarz). *Let $(\cdot,\cdot)_V$ be an inner product on V. By setting*

$$\|v\|_V := (v, v)_V^{\frac{1}{2}}, \qquad \forall v \in V, \tag{A.4}$$

one defines a norm on V. This norm is said to be induced by the inner product. Moreover, we have the Cauchy–Schwarz inequality

$$|(v, w)_V| \leq \|v\|_V \|w\|_V, \qquad \forall v, w \in V. \tag{A.5}$$

Definition A.15 (Hilbert space). *A* Hilbert space *V is an inner product space that is complete with respect to the induced norm (and is therefore a Banach space).*

Theorem A.16 (Riesz–Fréchet). *Let V be a complex Hilbert space. For all $A \in V'$, there exists a unique $v \in V$ s.t. $(v, w)_V = \langle A, w\rangle_{V',V}$ for all $w \in V$, and we have $\|v\|_V = \|A\|_{V'}$.*

Proof. See Brezis [48, Thm. 5.5], Lax [131, p. 56], Yosida [202, p. 90]. □

A.4 Compact operators

Definition A.17 (Compact operator). *Let V, W be two complex Banach spaces. The operator $T \in \mathcal{L}(V; W)$ is said to be* compact *if from every bounded sequence $(v_n)_{n\in\mathbb{N}}$ in V, one can extract a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that the sequence $(T(v_{n_k}))_{k\in\mathbb{N}}$ converges in W. Equivalently T is said to be compact if T maps the unit ball in V into a relatively compact set in W (that is, a set whose closure in W is compact).*

Example A.18 (Compact embedding). Assume that $V \subset W$ and that the embedding of V into W is compact. Then from every bounded sequence $(v_n)_{n\in\mathbb{N}}$ in V, one can extract a subsequence that converges in W. □

Proposition A.19 (Composition). *Let W, X, Y, Z be four Banach spaces and let $A \in \mathcal{L}(Z; Y)$, $K \in \mathcal{L}(Y; X)$, $B \in \mathcal{L}(X; W)$ be three operators. Assume that K is compact. Then the operator $B \circ K \circ A$ is compact.*

The following compactness result is used at several instances in this book. The reader is referred to Tartar [189, Lem. 11.1] and Girault and Raviart [107, Thm. 2.1, p. 18] for a slightly more general statement and references.

Lemma A.20 (Peetre–Tartar). *Let X, Y, Z be three Banach spaces. Let $A \in \mathcal{L}(X; Y)$ be an injective operator and let $T \in \mathcal{L}(X; Z)$ be a compact operator. Assume that there is $c > 0$ such that $c\|x\|_X \le \|A(x)\|_Y + \|T(x)\|_Z$ for all $x \in X$. Then there is $\alpha > 0$ such that*

$$\alpha\|x\|_X \le \|A(x)\|_Y, \qquad \forall x \in X. \tag{A.6}$$

Proof. We prove (A.6) by contradiction. Assume that there is a sequence $(x_n)_{n\in\mathbb{N}}$ of X s.t. $\|x_n\|_X = 1$ and $\|A(x_n)\|_Y$ converges to zero as $n \to \infty$. Since T is compact and the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ s.t. $(T(x_{n_k}))_{k\in\mathbb{N}}$ is a Cauchy sequence in Z. Owing to the inequality

$$\alpha\|x_{n_k} - x_{m_k}\|_X \le \|A(x_{n_k}) - A(x_{m_k})\|_Y + \|T(x_{n_k}) - T(x_{m_k})\|_Z,$$

$(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Let x be its limit, so that $\|x\|_X = 1$. The boundedness of A implies $A(x_{n_k}) \to A(x)$, and $A(x) = 0$ since $A(x_{n_k}) \to 0$. Since A is injective, $x = 0$, which contradicts $\|x\|_X = 1$. □

We finish this section with a striking property of compact operators.

Theorem A.21 (Approximability and compactness). *Let V, W be Banach spaces. If there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{L}(V;W)$ of finite rank (i.e., $\dim(\mathrm{im}(T_n)) < \infty$ for all $n \in \mathbb{N}$) such that $\lim_{n\to\infty} \|T - T_n\|_{\mathcal{L}(V;W)} = 0$, then T is compact. Conversely, if W is a Hilbert space and $T \in \mathcal{L}(V;W)$ is a compact operator, then there exists a sequence of operators in $\mathcal{L}(V;W)$ of finite rank, $(T_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \|T - T_n\|_{\mathcal{L}(V;W)} = 0$.*

Proof. See Brezis [48, pp. 157-158]. □

A.5 Interpolation between Banach spaces

Interpolation between Banach spaces is often used to combine known results to derive new results that could be difficult to obtain directly. An important application is the derivation of functional inequalities in fractional-order Sobolev spaces (see §2.2.2). There are many interpolation methods; see, e.g., Bergh and Löfström [18], Tartar [189], and the references therein. For simplicity we focus on the real interpolation K-method; see [18, §3.1] and [189, Chap. 22].

Let V_0 and V_1 be two normed vector spaces that are continuously embedded into a common topological vector space $\mathcal{V}$. Then V_0+V_1 is a normed vector space with the (canonical) norm $\|v\|_{V_0+V_1} := \inf_{v=v_0+v_1}(\|v_0\|_{V_0}+\|v_1\|_{V_1})$. Moreover, if V_0 and V_1 are Banach spaces, then V_0+V_1 is also a Banach space; see [18, Lem. 2.3.1]. For all $v \in V_0 + V_1$ and all $t > 0$, we define

$$K(t, v) := \inf_{v=v_0+v_1} (\|v_0\|_{V_0} + t\|v_1\|_{V_1}). \tag{A.7}$$

For all $t > 0$, $v \mapsto K(t, v)$ defines a norm on $V_0 + V_1$ that is equivalent to the canonical norm. One can verify that the function $t \mapsto K(t, v)$ is nondecreasing and concave (and therefore continuous) and that the function $t \mapsto \frac{1}{t}K(t, v)$ is increasing.

Definition A.22 (Interpolated space). *Let $\theta \in (0, 1)$ and let $p \in [1,\infty]$. The interpolated space $[V_0, V_1]_{\theta,p}$ is defined to be the vector space*

$$[V_0, V_1]_{\theta,p} := \{v \in V_0 + V_1 \mid \|t^{-\theta}K(t, v)\|_{L^p(\mathbb{R}_+;\frac{\mathrm{d}t}{t})} < \infty\}, \tag{A.8}$$

where $\|\varphi\|_{L^p(\mathbb{R}_+;\frac{\mathrm{d}t}{t})} := \left(\int_0^\infty |\varphi(t)|^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}$ for all $p \in [1,\infty)$ and $\|\varphi\|_{L^\infty(\mathbb{R}_+;\frac{\mathrm{d}t}{t})} := \sup_{0<t<\infty} |\varphi(t)|$. This space is equipped with the norm

$$\|v\|_{[V_0,V_1]_{\theta,p}} := \|t^{-\theta}K(t, v)\|_{L^p(\mathbb{R}_+;\frac{\mathrm{d}t}{t})}. \tag{A.9}$$

If V_0 and V_1 are Banach spaces, so is $[V_0, V_1]_{\theta,p}$.

Remark A.23 (Value for θ). Since $K(t,v) \geq \min(1,t)\|v\|_{V_0+V_1}$, the space $[V_0,V_1]_{\theta,p}$ reduces to $\{0\}$ if $t^{-\theta}\min(1,t) \notin L^p(\mathbb{R}_+;\frac{\mathrm{d}t}{t})$. In particular, $[V_0,V_1]_{\theta,p}$ is trivial if $\theta \in \{0,1\}$ and $p < \infty$. □

Remark A.24 (Gagliardo set). The function $t \mapsto K(t,v)$ has a simple geometric interpretation. Introducing the Gagliardo set $G(v) := \{(x_0,x_1) \in \mathbb{R}^2 \mid v = v_0 + v_1 \text{ with } \|v_0\|_{V_0} \leq x_0,\ \|v_1\|_{V_1} \leq x_1\}$, one can verify that $G(v)$ is convex and that $K(t,v) = \inf_{v\in\partial G(v)}(x_0 + tx_1)$, so that the map $t \mapsto K(t,v)$ is one way to explore the boundary of $G(v)$; see [18, p. 39]. □

Remark A.25 (Intersection). The vector space $V_0 \cap V_1$ can be equipped with the (canonical) norm $\|v\|_{V_0\cap V_1} := \max(\|v\|_{V_0}, \|v\|_{V_1})$. One can verify that $K(t,v) \leq \min(1,t)\|v\|_{V_0\cap V_1}$ for all $v \in V_0 \cap V_1$, which implies the boundedness of the embedding $V_0 \cap V_1 \hookrightarrow [V_0,V_1]_{\theta,p}$ for all $\theta \in (0,1)$ and all $p \in [1,\infty]$. Hence, if $V_0 \subset V_1$, then $V_0 \hookrightarrow [V_0,V_1]_{\theta,p}$. □

Lemma A.26 (Continuous embedding). *Let $\theta \in (0,1)$ and $p,q \in [1,\infty]$ with $p \leq q$. Then we have $[V_0,V_1]_{\theta,p} \hookrightarrow [V_0,V_1]_{\theta,q}$.*

Theorem A.27 (Riesz–Thorin, interpolation of operators). *Let $A : V_0 + V_1 \to W_0 + W_1$ be a linear operator that maps V_0 and V_1 boundedly to W_0 and W_1, respectively. Then for all $\theta \in (0,1)$ and all $p \in [1,\infty]$, A maps $[V_0,V_1]_{\theta,p}$ boundedly to $[W_0,W_1]_{\theta,p}$. Moreover, we have*

$$\|A\|_{\mathcal{L}([V_0,V_1]_{\theta,p};[W_0,W_1]_{\theta,p})} \leq \|A\|^{1-\theta}_{\mathcal{L}(V_0;W_0)} \|A\|^{\theta}_{\mathcal{L}(V_1;W_1)}. \tag{A.10}$$

Proof. See [189, Lem. 22.3]. □

Theorem A.28 (Lions–Peetre, reiteration). *Let $\theta_0, \theta_1 \in (0,1)$ with $\theta_0 \neq \theta_1$. Assume that $[V_0,V_1]_{\theta_0,1} \hookrightarrow W_0 \hookrightarrow [V_0,V_1]_{\theta_0,\infty}$ and $[V_0,V_1]_{\theta_1,1} \hookrightarrow W_1 \hookrightarrow [V_0,V_1]_{\theta_1,\infty}$. Then for all $\theta \in (0,1)$ and all $p \in [1,\infty]$, $[W_0,W_1]_{\theta,p} = [V_0,V_1]_{\eta,p}$ with equivalent norms, where $\eta := (1-\theta)\theta_0 + \theta\theta_1$.*

Proof. See Tartar [189, Thm. 26.2]. □

Theorem A.29 (Lions–Peetre, extension). *Let V_0, V_1, F be three Banach spaces. Let $A \in \mathcal{L}(V_0 \cap V_1; F)$. Then A extends into a linear continuous map from $[V_0,V_1]_{\theta,1;J}$ to F iff*

$$\exists c < \infty, \qquad \|A(v)\|_F \leq c\|v\|^{1-\theta}_{V_0}\|v\|^{\theta}_{V_1}, \qquad \forall v \in V_0 \cap V_1. \tag{A.11}$$

Proof. See [189, Lem. 25.3]. □

Theorem A.30 (Interpolation of dual spaces). *Let $\theta \in (0,1)$ and $p \in [1,\infty)$. Then $[V_0,V_1]'_{\theta,p} = [V_1',V_0']_{1-\theta,p'}$ where $p' := \frac{p}{p-1}$ (with the convention that $p' := \infty$ if $p = 1$).*

Proof. See [189, Lem. 41.3] or Bergh and Löfström [18, Thm. 3.7.1]. □

Appendix B

Differential calculus

This appendix briefly overviews some basic facts of differential calculus concerning Fréchet derivatives and their link to the notions of gradient, Jacobian matrix, and Hessian matrix.

B.1 Fréchet derivative

Let V, W be Banach spaces and let U be an open set in V. The space $C^0(U;W)$ consists of those functions $f : U \to W$ that are continuous in U.

Definition B.1 (Fréchet derivative). *Let $f \in C^0(U;W)$. We say that f is Fréchet differentiable (or differentiable) at $x \in U$ if there is a bounded linear operator $Df(x) \in \mathcal{L}(V;W)$ such that*

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Df(x)(h)\|_W}{\|h\|_V} = 0. \tag{B.1}$$

The operator $Df(x)$ is called Fréchet derivative *of f at x. If the map $Df : U \to \mathcal{L}(V;W)$ is continuous, we say that f is of class C^1 in U, and we write $f \in C^1(U;W)$.*

The above process can be repeated to define $D(Df)(x)$. For an integer $n \geq 2$, let us denote by $\mathcal{M}_n(V, \dots, V; W)$ the space spanned by the multilinear maps from $V \times \dots \times V$ (n times) to W. Upon identifying $\mathcal{L}(V;\mathcal{L}(V;W))$ with $\mathcal{M}_2(V, V; W)$ and setting $D^2f(x) := D(Df)(x)$, we have $D^2f(x) \in \mathcal{M}_2(V, V; W)$. The n-th Fréchet derivative of f at x is defined recursively as being the Fréchet derivative of $D^{n-1}f$ at x for all $n \geq 2$, that is,

$$D^n f(x) \in \mathcal{M}_n(\underbrace{V, \dots, V}_{n \text{ times}}; W).$$

If $D^n f : U \to \mathcal{M}_n(V, \dots, V; W)$ is continuous, we write $f \in C^n(U;W)$.

Let us restate some elementary properties of the Fréchet derivative (for the chain rule, the reader is referred, e.g., to Cartan [64, pp. 28-96], Ciarlet and Raviart [78, p. 227]). For an integer $n \geq 1$, $\mathcal{S}_n$ denotes the set of permutations of the integer set $\{1{:}n\} := \{1, \ldots, n\}$.

Lemma B.2 (Leibniz product rule). *Let $f \in C^n(U; W_1)$, $g \in C^n(U; W_2)$, $n \geq 1$, and let $b : W_1 \times W_2 \to W_3$ be a bilinear map, where U is an open set in V and V, W_1, W_2 are Banach spaces. The following holds true for all $x \in U$:*

$$D^n b(f(x), g(x)) = \sum_{l \in \{0:n\}} \binom{n}{l} b(D^{n-l} f(x), D^l g(x)), \qquad \forall x \in U. \tag{B.2}$$

Theorem B.3 (Symmetry). *Let V, W be Banach spaces. Let $n \geq 2$ and let $\mathcal{S}_n$ be the set of the permutations of $\{1{:}n\}$. Let $f \in C^n(U; W)$ where U is an open set in V. Then $D^n f$ is symmetric, i.e.,*

$$D^n f(x)(h_1, \ldots, h_n) = D^n f(x)(h_{\sigma(1)}, \ldots, h_{\sigma(n)}), \qquad \forall x \in U, \tag{B.3}$$

for all $\sigma \in \mathcal{S}_n$ and all $h_1, \ldots, h_n \in V$.

Theorem B.3 with $n := 2$ is often called *Clairaut or Schwarz theorem* in the literature.

Lemma B.4 (Chain rule). *Let $f \in C^n(U; W_1)$ and $g \in C^n(W_1; W_2)$, $n \geq 1$, where V, W_1, W_2 are Banach spaces and let U be an open set in V. Then we have*

$$D^n(g \circ f)(x)(h_1, \ldots, h_n) = \sum_{\sigma \in \mathcal{S}_n} \sum_{l \in \{1:n\}} \sum_{1 \leq r_1 + \ldots + r_l = n} \frac{1}{l! r_1! \ldots r_l!} \times \tag{B.4}$$
$$D^l g(f(x))(D^{r_1} f(x)(h_{\sigma(1)}, \ldots, h_{\sigma(s_1)}), \ldots, D^{r_l} f(x)(h_{\sigma(s_{l-1}+1)}, \ldots, h_{\sigma(n)})).$$

with $s_0 := 0$, $s_1 := r_1$, $s_2 := r_1 + r_2$, $\ldots$, $s_{l-1} := r_1 + \ldots + r_{l-1}$.

The identity (B.4) is often called *Faà di Bruno's formula* in the literature.

Example B.5. For $n = 1$, (B.4) yields

$$D(f \circ g)(x)(h) = Dg(f(x))(Df(x)(h)),$$

i.e., $D(f \circ g)(x) = Dg(f(x)) \circ Df(x)$. □

B.2 Vector and matrix representation

Assume that $V = \mathbb{R}^d$ and let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ be the canonical Cartesian basis of $\mathbb{R}^d$. (We use boldface notation for elements in V). Let U be an open set of $\mathbb{R}^d$. We say that f is differentiable in the direction $\boldsymbol{e}_i$ at $\boldsymbol{x} \in U$ if there is an element in W, say $\partial_i f(\boldsymbol{x}) \in W$, such that $\lim_{t\to 0} |t|^{-1}(f(\boldsymbol{x}+t\boldsymbol{e}_i) - f(\boldsymbol{x}) - t\partial_i f(\boldsymbol{x})) = 0$. If f is Fréchet differentiable at $\boldsymbol{x}$, it is differentiable along any direction $\boldsymbol{e}_i$ for $i \in \{1{:}d\}$ (the converse is not necessarily true), and we have

$$\partial_i f(\boldsymbol{x}) = Df(\boldsymbol{x})(\boldsymbol{e}_i). \tag{B.5}$$

More generally, let $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ be a multi-index. The number $|\alpha| := \alpha_1 + \ldots + \alpha_d$ is called the *length* of α. For all $f \in C^n(U;W)$ and every multi-index α s.t. $|\alpha| = n$, we write

$$\partial^\alpha f(\boldsymbol{x}) := \underbrace{\partial_1 \ldots \partial_1}_{\alpha_1 \text{ times}} \ldots \underbrace{\partial_d \ldots \partial_d}_{\alpha_d \text{ times}} f(\boldsymbol{x}) = D^n f(\boldsymbol{x})(\underbrace{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_1}_{\alpha_1 \text{ times}}, \ldots, \underbrace{\boldsymbol{e}_d, \ldots, \boldsymbol{e}_d}_{\alpha_d \text{ times}}), \tag{B.6}$$

and the order of the partial derivatives is irrelevant owing to Theorem B.3.

Let us finally assume that W is also finite-dimensional, e.g., $W := \mathbb{R}^m$ or $W := \mathbb{C}^m$. For $m = 1$, we adopt the convention that the *gradient* of f at $\boldsymbol{x}$, say $\nabla f(\boldsymbol{x})$, is the column vector with components

$$(\nabla f(\boldsymbol{x}))_i := \partial_i f(\boldsymbol{x}), \qquad \forall i \in \{1{:}d\}. \tag{B.7}$$

Identifying $\boldsymbol{h}$ with a column vector in $\mathbb{R}^d$, the action of $Df(\boldsymbol{x})$ is such that the following identities hold true for all $\boldsymbol{h} = \sum_{i\in\{1:d\}} h_i \boldsymbol{e}_i \in \mathbb{R}^d$:

$$Df(\boldsymbol{x})(\boldsymbol{h}) = \sum_{i\in\{1:d\}} \partial_i f(\boldsymbol{x}) h_i = (\nabla f(\boldsymbol{x}), \boldsymbol{h})_{\ell^2(\mathbb{R}^d)}, \tag{B.8}$$

where $(\cdot,\cdot)_{\ell^2(\mathbb{R}^d)}$ denotes the Euclidean product in $\mathbb{R}^d$. Assuming that $m \geq 2$, consider a basis of $\mathbb{R}^m$ and define the $m{\times}d$ *Jacobian matrix* of f at $\boldsymbol{x}$, say $\mathbb{J}_f(\boldsymbol{x})$, by its entries

$$(\mathbb{J}_f(\boldsymbol{x}))_{ij} := \partial_j f_i(\boldsymbol{x}), \qquad \forall i, j \in \{1{:}d\}, \tag{B.9}$$

where f_i is the i-th component of f in the chosen basis. Then we have

$$Df(\boldsymbol{x})(\boldsymbol{h}) = \mathbb{J}_f(\boldsymbol{x})\boldsymbol{h}, \qquad \forall \boldsymbol{h} \in \mathbb{R}^d. \tag{B.10}$$

Note that when $m = 1$, $\mathbb{J}_f(\boldsymbol{x})$ is the transpose of the gradient of f at $\boldsymbol{x}$, i.e., $\mathbb{J}_f(\boldsymbol{x}) = (\nabla f(\boldsymbol{x}))^\mathsf{T}$. For a scalar-valued function f, one can introduce the (symmetric) $d{\times}d$ *Hessian matrix* at $\boldsymbol{x}$, say $H_f(\boldsymbol{x})$, with entries

$$(H_f)_{ij} := \partial_{ij} f(\boldsymbol{x}), \qquad \forall i, j \in \{1{:}d\}, \tag{B.11}$$

leading to the following representation:

$$D^2 f(\boldsymbol{x})(\boldsymbol{h}_1, \boldsymbol{h}_2) = \boldsymbol{h}_1^\mathsf{T} H_f(\boldsymbol{x})\boldsymbol{h}_2 = \boldsymbol{h}_2^\mathsf{T} H_f(\boldsymbol{x})\boldsymbol{h}_1, \qquad \forall \boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathbb{R}^d. \tag{B.12}$$

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