Part I, Chapter 1

Lebesgue spaces

The objective of the four chapters composing Part I is to recall (or gently introduce) some elements of functional analysis that will be used throughout the book: Lebesgue integration, weak derivatives, and Sobolev spaces. We focus in this chapter on Lebesgue integration and Lebesgue spaces. Most of the results are stated without proof, but we include various examples. We refer the reader to Adams and Fournier [3], Bartle [16], Brezis [48], Demengel and Demengel [88], Evans [99], Grisvard [110], Malý and Ziemer [138], Rudin [169, Chap. 11], Rudin [170], Sobolev [180], Tartar [189], Yosida [202].

In this book, d is the space dimension, and D denotes a nonempty subset of $\mathbb{R}^d$. Vectors in $\mathbb{R}^d$, $d \geq 2$, and vector-valued functions are denoted in bold font. We abuse the notation by denoting position vectors in $\mathbb{R}^d$ in bold font as well. Moreover, $\|\cdot\|_{\ell^2(\mathbb{R}^d)}$ denotes the Euclidean norm in $\mathbb{R}^d$ (we write $\|\cdot\|_{\ell^2}$ when the context is unambiguous), and $\boldsymbol{a}{\cdot}\boldsymbol{b}$ denotes the Euclidean inner product between two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$. For every pair of integers $m \leq n$, we use the notation $\{m{:}n\} := \{p \in \mathbb{N} \mid m \leq p \leq n\}$.

1.1 Heuristic motivation

If one restricts oneself to computational considerations, the Riemann integral is the only notion of integration that is needed in numerical analysis, since the objects that one manipulates in practice are piecewise smooth functions (e.g., polynomials) defined on meshes. However, the Riemann integral becomes useless when one starts to investigate questions like passage to the limit. For instance, assume that one has an interval $I := (a, b)$, a sequence of finite partitions of this interval, say $(I_h)_{h\in\mathcal{H}}$, and a sequence of real-valued functions $(v_h)_{h\in\mathcal{H}}$ defined on I such that v_h is smooth on each subinterval of I_h for all $h \in \mathcal{H}$. Here, $\mathcal{H}$ is a countable set with 0 as unique accumulation point. In the context of finite elements, the index h refers to the size of the mesh that is used to construct the function v_h. Assume also that one can a priori prove that the sequence $(v_h)_{h\in\mathcal{H}}$ is Cauchy in the following sense: for every $\epsilon > 0$, there is $h(\epsilon)$ such that $\int_a^b |v_{h_1}(x)-v_{h_2}(x)|\,\mathrm{d}x \le \epsilon$ for all $h_1, h_2 \in \mathcal{H}\cap\big(0, h(\epsilon)\big)$. One may then wonder whether v_h converges to some object with interesting properties when $h \to 0$. The answer to this question becomes very intricate if one restricts oneself to the Riemann integral, but it becomes simple if one adopts Lebesgue's point of view. Since the above question arises constantly in this book, we now take some time to recall the key ingredients of Lebesgue's theory.

1.2 Lebesgue measure

To define the Lebesgue integral of a function defined on a subset D of $\mathbb{R}^d$, one needs to measure the volume of sets in $\mathbb{R}^d$. For every bounded rectangular parallelepiped $R := [a_1, b_1] \times \cdots \times [a_d, b_d]$, with $a_i \le b_i$ for all $i \in \{1{:}d\}$, we define the Lebesgue (outer) measure of R to be its volume, i.e., we set $|R| := \prod_{i\in\{1:d\}}(b_i - a_i)$.

Definition 1.1 (Lebesgue's outer measure). *Let $\mathcal{R}(\mathbb{R}^d)$ be the set of all the rectangular parallelepipeds in $\mathbb{R}^d$. Let E be a set in $\mathbb{R}^d$. The* Lebesgue's outer measure *of E is defined as*

$$|E|^* := \inf\Big\{ \sum_{i\in\mathbb{N}} |R_i| \mid E \subset \bigcup_{i\in\mathbb{N}} R_i,\ R_i \in \mathcal{R}(\mathbb{R}^d)\Big\}. \tag{1.1}$$

We expect $|E|^*$ to be a reasonable estimate of the volume of E if E is a reasonable set. The outer Lebesgue measure has the following properties: (i) $|\emptyset|^* = 0$; (ii) If $E \subset F$, then $|E|^* \le |F|^*$; (iii) If $\{E_i\}_{i\in\mathbb{N}}$ is a countable collection of subsets of $\mathbb{R}^d$, then $|\bigcup_{i\in\mathbb{N}} E_i|^* \le \sum_{i\in\mathbb{N}} |E_i|^*$ (countable subadditivity property; see [169, Thm. 11.8]).

Example 1.2 (Countable sets). The outer Lebesgue measure of a countable set $A := \bigcup_{k\in\mathbb{N}}\{\boldsymbol{x}_k\}$ is zero. Let indeed $\epsilon > 0$. We have $\{\boldsymbol{x}_k\} \subset R(\boldsymbol{x}_k, \epsilon^{\frac{1}{d}})$, where $R(\boldsymbol{z}, r)$ is the cube of side r centered at $\boldsymbol{z}$. Hence, $|\{\boldsymbol{x}_k\}|^* \le \epsilon$, i.e., $|\{\boldsymbol{x}_k\}|^* = 0$ since $\epsilon > 0$ is arbitrary. Invoking subadditivity yields $|A|^* = 0$. For example, this implies that the outer measure of the set of the rational numbers is zero, i.e., $|\mathbb{Q}|^* = 0$. □

Definition 1.3 (Lebesgue's measure of a set). *A set $E \subset \mathbb{R}^d$ is said to be* Lebesgue-measurable *if $|S|^* = |S\cap E|^* + |S\cap E^c|^*$ for every subset $S \subset \mathbb{R}^d$, where E^c is the complement of E in $\mathbb{R}^d$.*

It turns out that not all the sets of $\mathbb{R}^d$ are Lebesgue-measurable, but the class of Lebesgue-measurable sets (in short, measurable sets) of $\mathbb{R}^d$, say $\mathcal{L}(\mathbb{R}^d)$, is sufficiently vast that we will only encounter measurable sets in this book. In particular, (i) If E is measurable, then E^c is also measurable; (ii) Open sets of $\mathbb{R}^d$ and closed sets of $\mathbb{R}^d$ are measurable (so that all the usual geometric objects, e.g., parallelepipeds or balls, are measurable); (iii) Countable unions and countable intersections of measurable sets are measurable.

Henceforth, the map $|\cdot| : \mathcal{L}(\mathbb{R}^d) \to [0,\infty]$ such that $|E| := |E|^*$ for all $E \in \mathcal{L}(\mathbb{R}^d)$ is called (d-dimensional) *Lebesgue measure*. Since the action of the Lebesgue measure on measurable sets is simply the outer Lebesgue measure, we infer that (i) $|\emptyset| = 0$; (ii) If $A, B \in \mathcal{L}(\mathbb{R}^d)$ and $A \subset B$, then $|A| \leq |B|$; (iii) The countable subadditivity property holds true on countable collections of measurable sets. By restricting our attention to measurable sets, the property we have gained is that $|A_1 \cup A_2| = |A_1| + |A_2|$ for disjoint measurable sets (since $(A_1 \cup A_2) \cap A_1 = A_1$ and $(A_1 \cup A_2) \cap A_1^c = A_2$). Moreover, if $\{A_k\}_{k\in\mathbb{N}}$ is a countable family of measurable disjoint sets, the union $\bigcup_{k\in\mathbb{N}} A_k$ is measurable and $|\bigcup_{k\in\mathbb{N}} A_k| = \sum_{k\in\mathbb{N}} |A_k|$; see [169, Thm. 11.10].

Example 1.4 (Null sets). Let $A \subset \mathbb{R}^d$. If $|A|^* = 0$, then A is measurable. Let indeed $S \subset \mathbb{R}^d$. Then $|A \cap S|^* \leq |A|^* = 0$, i.e., $|A \cap S|^* = 0$. Moreover, $|S|^* \geq |S \cap A^c|^* = |S \cap A^c|^* + |S \cap A|^*$, and the subadditivity property implies that $|S|^* \leq |S \cap A|^* + |S \cap A^c|^*$, whence the result. □

Example 1.5 (Cantor set). To define the Cantor ternary set, one starts with the interval $[0,1]$, then one deletes the open middle third from $[0,1]$, leaving two line segments: $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Next the open middle third of each of the two remaining segments is deleted, leaving four line segments: $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. This process is continued ad infinitum. Setting $C_0 := [0,1]$ and $C_n := \frac{1}{3}C_{n-1} \cup (\frac{2}{3} + \frac{1}{3}C_{n-1})$, the Cantor ternary set is defined by $C_\infty := \{x \in [0,1] \mid x \in C_k, \forall k \in \mathbb{N}\}$. Then C_∞ is measurable (as the complement of a countable union of measurable sets), $|C_\infty| \leq |C_k|$ for all $k \in \mathbb{N}$, so that $|C_\infty| = 0$, but it can be shown that C_∞ is not countable. □

Definition 1.6 (Equality a.e.). *Let $D \subset \mathbb{R}^d$ be a measurable set, i.e., $D \in \mathcal{L}(\mathbb{R}^d)$. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions. We say that f and g are* equal almost everywhere *if $|\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \neq g(\boldsymbol{x})\}| = 0$. Henceforth, we write $f(\boldsymbol{x}) = g(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in D$, or $f = g$ a.e. in D.*

Definition 1.7 ($\operatorname{ess\,sup}$, $\operatorname{ess\,inf}$). *Let $D \subset \mathbb{R}^d$ be a measurable set and let $f : D \to \mathbb{R}$ be a function. We define*

$$\operatorname*{ess\,sup}_{\boldsymbol{x}\in D} f(\boldsymbol{x}) := \inf\{M \in \mathbb{R} \mid f(\boldsymbol{x}) \leq M \textit{ for a.e. } \boldsymbol{x} \in D\}, \tag{1.2a}$$

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} f(\boldsymbol{x}) := \sup\{m \in \mathbb{R} \mid f(\boldsymbol{x}) \geq m \textit{ for a.e. } \boldsymbol{x} \in D\}. \tag{1.2b}$$

Definition 1.8 (Measurable function). *Let $D \subset \mathbb{R}^d$ be a measurable set. A function $f : D \to \mathbb{R}$ is said to be* measurable *if $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) > r\}$ is measurable for all $r \in \mathbb{R}$.*

The meaning of the above definition is that a function is measurable if all its upper level sets are (Lebesgue) measurable; see also [169, Def. 11.13].

Lemma 1.9 (Characterization). *Let $D \subset \mathbb{R}^d$ be a measurable set. Let $f : D \to \mathbb{R}$. The function f is measurable iff any of the following statements holds true:*

(i) *For all $r \in \mathbb{R}$, the set $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) > r\}$ is measurable.*
(ii) *For all $r \in \mathbb{R}$, the set $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \geq r\}$ is measurable.*
(iii) *For all $r \in \mathbb{R}$, the set $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) < r\}$ is measurable.*
(iv) *For all $r \in \mathbb{R}$, the set $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \leq r\}$ is measurable.*

Proof. Item (i) is the definition of the measurability of f. The identity $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \geq r\} = \bigcap_{n\in\mathbb{N}}\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) > r - \frac{1}{n+1}\}$ proves that (i) implies (ii). $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) < r\} = D \cap \{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \geq r\}^c$ proves that (ii) implies (iii). $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \leq r\} = \bigcap_{n\in\mathbb{N}}\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) < r - \frac{1}{n+1}\}$ proves that (iii) implies (iv), and $\{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) > r\} = D \cap \{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \leq r\}^c$ proves that (iv) implies the measurability of f. (See also [169, Thm. 11.15].) □

For every subset $A \subset \mathbb{R}$, let us denote by $f^{-1}(A) := \{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \in A\}$ the inverse image of A by f. Since every open set in $\mathbb{R}$ is a countable union of open intervals, the above result shows that f is measurable if and only if $f^{-1}(U) = \{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) \in U\}$ is measurable for every open set U of $\mathbb{R}$.

Example 1.10 (Measurable functions). Functions that are piecewise continuous and more generally all the functions that are integrable in the Riemann sense are measurable. □

Corollary 1.11 (Measurability and equality a.e.). *Let $D \subset \mathbb{R}^d$ be a measurable set. Let $f : D \to \mathbb{R}$ be a measurable function. Let $g : D \to \mathbb{R}$ be a function. If $f = g$ a.e. in D, then g is measurable.*

Proof. See Exercise 1.2. □

Theorem 1.12 (Pointwise limit of measurable functions). *Let D be a measurable set in $\mathbb{R}^d$. Let $f_n : D \to \mathbb{R}$ for all $n \in \mathbb{N}$ be real-valued measurable functions. Then*

(i) $\limsup_{n\in\mathbb{N}} f_n$ *and* $\liminf_{n\in\mathbb{N}} f_n$ *are both measurable.*
(ii) *Let $f : D \to \mathbb{R}$. Assume that $f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in D$. Then f is measurable.*

Proof. See Exercise 1.5. □

Example 1.13 (Measurability). Let $D := (0, 1)$. Let $f : D \to \mathbb{R}$ be defined by $f(x) := x$. Let C_∞ be the Cantor set (see Example 1.5). Let $g : D \to \mathbb{R}$ be defined by $g(x) := -2x$ if $x \in C_\infty$, and $g(x) := x$ if $x \notin C_\infty$. The function f is measurable since it is continuous. Recalling that $|C_\infty| = 0$, g is also measurable by virtue of Corollary 1.11 since $f = g$ a.e. in D. □

Theorem 1.14 (Composite functions). *Let D be a measurable set in $\mathbb{R}^d$. Let $g : D \to \mathbb{R}$ be a measurable function. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $f \circ g : D \to \mathbb{R}$ is measurable.*

Proof. For every subset $A \subset \mathbb{R}$, we have $(f \circ g)^{-1}(A) = g^{-1}(f^{-1}(A))$. Let U be an open set in $\mathbb{R}$. Then $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$. But $f^{-1}(U)$ is an open set since f is continuous. Hence, $g^{-1}(f^{-1}(U))$ is measurable since $f^{-1}(U)$ is open and g is measurable. As a result, $(f \circ g)^{-1}(U)$ is measurable. □

Example 1.15 (Composite functions). Let $g : D \to \mathbb{R}$ be a measurable function. Then by virtue of Theorem 1.14, the functions $|g|$, $g + |g|$, $g - |g|$, $|g|^p$ for every $p > 0$, e^g, $\cos(g)$, $\sin(g)$ are also measurable. □

Theorem 1.16 (Operations on measurable functions). *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two measurable functions and let $\lambda \in \mathbb{R}$. Then the functions λf, $f + g$, $|f|$ and fg are measurable.*

Proof. See Exercise 1.6. □

1.3 Lebesgue integral

We say that $g : D \to \mathbb{R}$ is a simple nonnegative function if there exist $m \in \mathbb{N}$, a collection of disjoint measurable sets $\{A_k\}_{k\in\{1:m\}}$ in D, and a collection of nonnegative numbers $\{v_k\}_{k\in\{1:m\}}$ such that $g = \sum_{k\in\{1:m\}} v_k \mathbb{1}_{A_k}$ (where $\mathbb{1}_{A_k}(\boldsymbol{x}) := 1$ if $\boldsymbol{x} \in A_k$ and $\mathbb{1}_{A_k}(\boldsymbol{x}) := 0$ otherwise). The Lebesgue integral of g over D is defined by $\int_D g(\boldsymbol{x})\,\mathrm{d}x := \sum_{k\in\{1:m\}} v_k |A_k|$.

Theorem 1.17 (Simple functions). *Let $D \in \mathcal{L}(\mathbb{R}^d)$. Let $f : D \to [0,\infty]$ be a nonnegative measurable function. Then there exist simple functions $\{g_k\}_{k\in\mathbb{N}}$ s.t. $0 \le g_1 \le g_2 \ldots \le f$ and $\lim_{k\to\infty} g_k(x) = f(x)$ for all $x \in D$.*

Proof. See [170, Thm. 1.17]. □

Definition 1.18 (Lebesgue integral). *Let f be a nonnegative measurable function. The* Lebesgue integral *of f over D is defined in $[0,\infty]$ as follows:*

$$\int_D f(\boldsymbol{x})\,\mathrm{d}x := \sup\Big\{ \int_D g(\boldsymbol{x})\,\mathrm{d}x \mid g \text{ is simple nonnegative and } g \le f \Big\}.$$

Let f be measurable but not necessarily nonnegative. If either $\int_D f^+(\boldsymbol{x})\,\mathrm{d}x$ or $\int_D f^-(\boldsymbol{x})\,\mathrm{d}x$ is finite, where $f^\pm := \max(\pm f, 0)$, the Lebesgue integral of f is defined by

$$\int_D f(\boldsymbol{x})\,\mathrm{d}x := \int_D f^+(\boldsymbol{x})\,\mathrm{d}x - \int_D f^-(\boldsymbol{x})\,\mathrm{d}x. \tag{1.3}$$

We say that f is (Lebesgue-)integrable on D if both terms in (1.3) *are finite.*

This definition agrees with the Riemann integral of f if f is Riemann-integrable. Moreover, since $\int_D |f(\boldsymbol{x})|\,\mathrm{d}x = \int_D f^+(\boldsymbol{x})\,\mathrm{d}x + \int_D f^-(\boldsymbol{x})\,\mathrm{d}x$, we have by construction $\left|\int_D f(\boldsymbol{x})\,\mathrm{d}x\right| \le \int_D |f(\boldsymbol{x})|\,\mathrm{d}x$.

An important property of the Lebesgue integral is that if f is integrable on D, then $\int_D |f(\boldsymbol{x})| \,\mathrm{d}x = 0$ if and only if f vanishes everywhere on D up to a set of zero measure. This leads us to introduce a notion of equivalence classes. Two functions are said to belong to the same class if they coincide *almost everywhere* (henceforth, a.e.), i.e., everywhere but on a set of zero Lebesgue measure. Elements of Lebesgue spaces are, strictly speaking, equivalence classes, although we refer to them simply as functions that are defined almost everywhere. For instance, the function $\phi : (0,1) \to \{0,1\}$ that is 1 on the rational numbers and is zero otherwise is in the same equivalence class as the zero function. Hence, $\phi = 0$ a.e. on $(0,1)$. Integrals are always understood in the Lebesgue sense throughout this book. Whenever the context is unambiguous, we simply write $\int_D f \,\mathrm{d}x$ instead of $\int_D f(\boldsymbol{x}) \,\mathrm{d}x$. We refer the reader to [170, Chap. 1] for more elaborate notions on the measure theory.

Example 1.19 (Cantor set). Let $f : [0,1] \to \mathbb{R}$ be such that $f(x) := 1$ if x is in C_∞ (see Example 1.5) and $f(x) := 0$ otherwise. Then f is measurable (see Corollary 1.11) and $\int_0^1 f(x) \,\mathrm{d}x = 0$. □

Remark 1.20 (Literature). It is reported in Denjoy et al. [89, p. 15] that Lebesgue explained his approach to integration as follows: "I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket, I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral." To get a clearer connection with the integration process, one could say that Lebesgue went to a grocery store every day in a month, bought items, and asked for credit until the end of the month. His debt at the end of a 30-day month is $\int_0^{30} f(t) \,\mathrm{d}t$, where $f(t)$ is the amount of money he owes per day. What Lebesgue has described above are two different ways to compute $\int_0^{30} f(t) \,\mathrm{d}t$. □

1.4 Lebesgue spaces

This section introduces the Lebesgue spaces and reviews their key properties.

1.4.1 Lebesgue space $L^1(D)$

Definition 1.21 (Space L^1). *Let D be an open set in $\mathbb{R}^d$. $L^1(D)$ is the vector space composed of all the real-valued measurable functions that are Lebesgue-integrable on D, and we equip $L^1(D)$ with the norm $\|f\|_{L^1(D)} := \int_D |f| \,\mathrm{d}x$ to make it a normed space.*

Theorem 1.22 (Monotone convergence, Beppo Levi). *Let D be an open set in $\mathbb{R}^d$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(D)$ such that*

$0 \le f_0 \le f_1 \le \ldots \le f_n \le f_{n+1} \le \ldots$ a.e. on D and $\sup_{n\in\mathbb{N}} \int_D f_n \,\mathrm{d}x < \infty$. Then $f_n(x)$ converges to a finite limit for a.e. x in D. Denoting by $f(x)$ the limit in question, f is in $L^1(D)$ and $\lim_{n\in\mathbb{N}} \|f_n - f\|_{L^1(D)} = 0$.

Proof. See [48, Thm. 4.1] or [170, Thm. 1.26]. □

Theorem 1.23 (Lebesgue's dominated convergence). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(D)$ such that:*

(i) *$f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ a.e. in D.*
(ii) *There is $g \in L^1(D)$ such that $|f_n(\boldsymbol{x})| \le g(\boldsymbol{x})$ a.e. in D for all $n \in \mathbb{N}$.*

Then $f \in L^1(D)$ and $f_n \to f$ in $L^1(D)$.

Proof. See [16, p. 123], [48, Thm. 4.2], [170, Thm. 1.34]. □

Example 1.24 (Application). Let $f_n : D := (0, 1) \to \mathbb{R}$, $n \in \mathbb{N}$, with $f_n(x) := 1$ if $x < \frac{1}{n}$ and $f_n(x) := x$ otherwise. We have $f_n(x) \to x$ a.e. in D and $f_n(x) \le g := 1$ a.e. in D. Hence, $f_n \to x$ in $L^1(D)$. □

Theorem 1.25 (Fischer–Riesz). *$L^1(D)$ equipped with the L^1-norm from Definition 1.33 is a Banach space.*

Proof. See [3, Thm. 2.16], [16, p. 142], [48, Thm. 4.8], [170, Thm. 3.11]. □

Remark 1.26 (Lebesgue vs. Riemann). The two key results the notion of Lebesgue integration gave us that were missing in the Riemann integration are Lebesgue's dominated convergence theorem and the fact that $L^1(D)$ is now complete, i.e., it is a Banach space. This answers the question raised in §1.1. □

Theorem 1.27 (Pointwise convergence). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(D)$ and assume that $f \in L^1(D)$ is such that $\|f_n - f\|_{L^1(D)} \to 0$. Then there exist a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a function $g \in L^1(D)$ such that $f_{n_k}(\boldsymbol{x}) \to f(\boldsymbol{x})$ a.e. in D and $|f_{n_k}(\boldsymbol{x})| \le g(\boldsymbol{x})$ a.e. in D for all $k \in \mathbb{N}$.*

Proof. See [48, Thm. 4.9], [170, Thm. 3.12]. □

Example 1.28 (Dirac mass). The assumption that there exists some $g \in L^1(D)$ s.t. $|f_n(\boldsymbol{x})| \le g(\boldsymbol{x})$ a.e. in D for all $n \in \mathbb{N}$, is crucial to apply Lebesgue's dominated convergence theorem. For instance, consider the sequence of functions in $L^1(\mathbb{R})$ s.t. $f_n(x) := 0$ if $|x| > \frac{1}{n}$ and $f_n(x) := \frac{n}{2}$ otherwise. We have $f_n(x) \to 0$ for a.e. x in $\mathbb{R}$ and $\int_{\mathbb{R}} |f_n(x)| \,\mathrm{d}x = 1$, but f_n does not converge in $L^1(\mathbb{R})$. Reasoning by contradiction, let us assume that $f_n \to f$ in $L^1(\mathbb{R})$. Theorem 1.27 implies that there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ s.t. $f_{n_k}(x) \to f(x)$ for a.e. x in $\mathbb{R}$. For all $x \ne 0$, we have $f_{n_k}(x) = 0$ for all n_k such that $n_k > \frac{1}{|x|}$. This implies that $f(x) = 0$ for a.e. x in $\mathbb{R}$. This argument shows that $\int_{\mathbb{R}} |f(x)| \,\mathrm{d}x = 0$, but since we assumed that $f_n \to f$ in $L^1(\mathbb{R})$, we also have $\int_{\mathbb{R}} |f(x)| \,\mathrm{d}x = 1$, which is a contradiction. Actually $(f_n)_{\in\mathbb{N}}$ converges to the Dirac mass at 0 in the distribution sense; see Example 4.3. □

Definition 1.29 (Space $L^1_{\text{loc}}(D)$). *Let D be an open set in $\mathbb{R}^d$. The elements of the following space are called* locally integrable *functions:*

$$L^1_{\text{loc}}(D) := \{v \text{ measurable} \mid \forall \text{ compact } K \subset D,\ v_{|K} \in L^1(K)\}. \tag{1.4}$$

Definition 1.30 (Support). *Let D be a measurable set in $\mathbb{R}^d$. The* support *in D of a function $\varphi : D \to \mathbb{R}$, henceforth denoted by* $\text{supp}(\varphi)$*, is defined to be the closure in D of the subset $\{\boldsymbol{x} \in D \mid \varphi(\boldsymbol{x}) \neq 0\}$.*

Definition 1.31 (Space $C_0^\infty(D)$). *We denote by $C_0^\infty(D)$ the space composed of the functions from D to $\mathbb{R}$ that are C^∞ and whose support in D is compact. The members of $C_0^\infty(D)$ are called* test functions.

Theorem 1.32 (Vanishing integral). *Let D be an open set in $\mathbb{R}^d$. Let $v \in L^1_{\text{loc}}(D)$. Then $\int_D v\varphi \,\mathrm{d}x = 0$ for all $\varphi \in C_0^\infty(D)$ iff $v = 0$ a.e. in D.*

Proof. See [48, Cor. 4.24], [138, p. 6]. □

1.4.2 Lebesgue spaces $L^p(D)$ and $L^\infty(D)$

Definition 1.33 (L^p spaces). *Let D be an open set in $\mathbb{R}^d$. For all $p \in [1, \infty]$, let $L^p(D) := \{f \text{ measurable } \mid \|f\|_{L^p(D)} < \infty\}$, where*

$$\|f\|_{L^p(D)} := \left(\int_D |f|^p \,\mathrm{d}x\right)^{\frac{1}{p}}, \quad \textit{if } p \in [1, \infty), \tag{1.5a}$$

$$\|f\|_{L^\infty(D)} := \operatorname*{ess\,sup}_{\boldsymbol{x}\in D} |f(\boldsymbol{x})| := \inf\{M \in \mathbb{R} \mid |f(\boldsymbol{x})| \leq M \text{ a.e. } \boldsymbol{x} \in D\}. \tag{1.5b}$$

We write $L^p(D; \mathbb{R}^q)$, $q \geq 1$, for the space composed of $\mathbb{R}^q$-valued functions whose components are all in $L^p(D)$, and we use the Euclidean norm in $\mathbb{R}^q$, $\|f\|_{\ell^2(\mathbb{R}^q)}$, instead of $|f|$, to evaluate the norms in (1.5)*. When $q = d$, we write $\boldsymbol{L}^p(D) := L^p(D; \mathbb{R}^d)$.*

Lebesgue's dominated convergence theorem extends to all the L^p spaces, $p \in [1, \infty)$, i.e., if the dominating function g is in $L^p(D)$, the convergence of f_n to f occurs in $L^p(D)$.

Theorem 1.34 (Pointwise convergence). *Let $p \in [1, \infty]$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(D)$ and let $f \in L^p(D)$ such that $\|f_n - f\|_{L^p(D)} \to 0$. Then there exist a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ and a function $g \in L^p(D)$ such that $f_{n_k}(\boldsymbol{x}) \to f(\boldsymbol{x})$ a.e. in D and $|f_{n_k}(\boldsymbol{x})| \leq g(\boldsymbol{x})$ a.e. in D for all $k \in \mathbb{N}$.*

Proof. See [48, Thm. 4.9], [170, Thm. 3.12]. □

Theorem 1.35 (Fischer–Riesz). *For all $p \in [1, \infty]$, $L^p(D)$ equipped with the L^p-norm from Definition* 1.33 *is a Banach space.*

Proof. See [3, Thm. 2.16], [16, p. 142], [48, Thm. 4.8], [170, Thm. 3.11]. □

Among all the Lebesgue spaces, $L^2(D)$ plays a particular role owing to the following important consequence of the Fischer–Riesz theorem.

Theorem 1.36 (L^2 space). *$L^2(D;\mathbb{R})$ is a Hilbert space when equipped with the inner product $(f,g)_{L^2(D)} := \int_D fg\,\mathrm{d}x$. Similarly, $L^2(D;\mathbb{C})$ is a Hilbert space when equipped with the inner product $(f,g)_{L^2(D)} := \int_D f\overline{g}\,\mathrm{d}x$.*

Remark 1.37 (Continuous embedding on bounded sets). Assume that D is bounded. For all $p,q \in [1,\infty]$ with $p \le q$, Hölder's inequality implies that

$$\|f\|_{L^p(D)} \le |D|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^q(D)}, \qquad \forall f \in L^q(D), \tag{1.6}$$

meaning that $L^q(D) \hookrightarrow L^p(D)$ (this notation means that $L^q(D)$ is continuously embedded into $L^p(D)$). One can show that $\lim_{p\to\infty} \|f\|_{L^p(D)} = \|f\|_{L^\infty(D)}$ for all $f \in L^\infty(D)$. Moreover, if $f \in L^p(D)$ for all $p \in [1,\infty)$ and if there is c, uniform w.r.t. p, s.t. $\|f\|_{L^p(D)} \le c$, then $f \in L^\infty(D)$ and $\|f\|_{L^\infty(D)} \le c$; see [3, Thm. 2.14]. □

Theorem 1.38 (Density of $C_0^\infty(D)$). *Let D be an open set in $\mathbb{R}^d$. Then $C_0^\infty(D)$ is dense in $L^p(D)$ for all $p \in [1,\infty)$.*

Proof. See [170, Thm. 3.14]. □

Remark 1.39 (The case of $L^\infty(D)$). $C_0^\infty(D)$ is not dense in $L^\infty(D)$. If D is bounded, the completion of $C^\infty(D)$ in $L^\infty(D)$ is $C^0(D)$, and the completion of $C_0^\infty(D)$ is $\{v \in C^0(D) \mid v_{|\partial D} = 0\}$. □

1.4.3 Duality

Lemma 1.40 (Conjugate, Hölder's inequality). *Let $p \in [1,\infty]$ be a real number. The real number $p' \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{p'} = 1$, with the convention that $p' := 1$ if $p = \infty$ and $p' := \infty$ if $p = 1$, is called* conjugate *of p. Let $f \in L^p(D)$ and $g \in L^{p'}(D)$. Then $fg \in L^1(D)$ and*

$$\int_D |fg|\,\mathrm{d}x \le \|f\|_{L^p(D)}\|g\|_{L^{p'}(D)}. \tag{1.7}$$

Proof. See [3, Thm. 2.4], [16, p. 404], [48, Thm. 4.6], [170, Thm. 3.8]. □

For $p = p' = 2$, Hölder's inequality becomes $\int_D |fg|\,\mathrm{d}x \le \|f\|_{L^2(D)}\|g\|_{L^2(D)}$ for all $f,g \in L^2(D)$, which is nothing but the *Cauchy–Schwarz inequality* in $L^2(D)$. This inequality is useful to bound $|(f,g)_{L^2(D)}|$ since $|(f,g)_{L^2(D)}| \le \int_D |fg|\,\mathrm{d}x$.

Theorem 1.41 (Riesz–Fréchet). *Let $p \in [1,\infty)$. The dual space of $L^p(D)$ can be identified with $L^{p'}(D)$.*

Proof. See [3, pp. 45–49], [48, Thm. 4.11&4.14], [170, Thm. 6.16]. □

Remark 1.42 ($L^\infty(D)$). Theorem 1.41 fails for $p = \infty$. Indeed, the dual of $L^\infty(D)$ strictly contains $L^1(D)$ (see [48, p. 102]). □

Corollary 1.43 (Interpolation inequality). *Let $p, q \in [1, \infty]$ with $p \le q$. For all $r \in [p, q]$, letting $\theta \in [0, 1]$ be s.t. $\frac{1}{r} := \frac{\theta}{p} + \frac{1-\theta}{q}$, we have*

$$\|f\|_{L^r(D)} \le \|f\|^{\theta}_{L^p(D)} \|f\|^{1-\theta}_{L^q(D)}, \qquad \forall f \in L^p(D) \cap L^q(D). \tag{1.8}$$

Recall from §A.2 that for two Banach spaces V and W, $\mathcal{L}(V;W)$ is composed of the linear operators that map V boundedly to W, and that the norm $\|\cdot\|_{\mathcal{L}(V;W)}$ is defined in (A.2).

Theorem 1.44 (Riesz–Thorin). *Let p_0, p_1, q_0, q_1 be four real numbers such that $1 \le p_0 \le p_1 \le \infty$, $1 \le q_0 \le q_1 \le \infty$. Let $T : L^{p_0}(D) + L^{p_1}(D) \longrightarrow L^{q_0}(D) + L^{q_1}(D)$ be a linear operator that maps $L^{p_0}(D)$ and $L^{p_1}(D)$ boundedly to $L^{q_0}(D)$ and $L^{q_1}(D)$, respectively. Then the operator T maps $L^{p_\theta}(D)$ boundedly to $L^{q_\theta}(D)$ for all $\theta \in (0, 1)$, where p_θ and q_θ are defined by $\frac{1}{p_\theta} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Moreover, $\|T\|_{\mathcal{L}(L^{p_\theta};L^{q_\theta})} \le \|T\|^{\theta}_{\mathcal{L}(L^{p_0};L^{q_0})} \|T\|^{1-\theta}_{\mathcal{L}(L^{p_1};L^{q_1})}$.*

Proof. See [189, Thm. 21.2], Bergh and Löfström [18, Chap. 1]. □

Remark 1.45 (Interpolation). Corollary 1.43 and Theorem 1.44 are related to the interpolation theory between Banach spaces (see §A.5). For instance, $L^p(D)$ can be defined for all $p \in (1, \infty)$, up to equivalent norm, by interpolating between $L^1(D)$ and $L^\infty(D)$, i.e., $L^p(D) = [L^1(D), L^\infty(D)]_{\frac{1}{p'},p}$; see Tartar [189, p. 111]. □

1.4.4 Multivariate functions

The following results on multivariate functions are useful in many situations.

Theorem 1.46 (Tonelli). *Let $f : D_1 \times D_2 \to \mathbb{R}$ be a measurable function such that the function $D_1 \ni \boldsymbol{x}_1 \mapsto \int_{D_2} |f(\boldsymbol{x}_1, \boldsymbol{x}_2)| \,\mathrm{d}x_2$ is finite a.e. in D_1 and is in $L^1(D_1)$. Then $f \in L^1(D_1 \times D_2)$.*

Proof. See [48, Thm. 4.4]. □

Theorem 1.47 (Fubini). *Let $f \in L^1(D_1 \times D_2)$. Then the function $D_2 \ni \boldsymbol{x}_2 \mapsto f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is in $L^1(D_2)$ for a.e. $\boldsymbol{x}_1 \in D_1$, and the function $D_1 \ni \boldsymbol{x}_1 \mapsto \int_{D_2} f(\boldsymbol{x}_1, \boldsymbol{x}_2) \,\mathrm{d}x_2$ is in $L^1(D_1)$. Similarly, the function $D_1 \ni \boldsymbol{x}_1 \mapsto f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is in $L^1(D_1)$ for a.e. $\boldsymbol{x}_2 \in D_2$, and the function $D_2 \ni \boldsymbol{x}_2 \mapsto \int_{D_1} f(\boldsymbol{x}_1, \boldsymbol{x}_2) \,\mathrm{d}x_1$ is in $L^1(D_2)$. Moreover, we have*

$$\int_{D_1} \left(\int_{D_2} f(\boldsymbol{x}_1, \boldsymbol{x}_2) \,\mathrm{d}x_2 \right) \mathrm{d}x_1 = \int_{D_2} \left(\int_{D_1} f(\boldsymbol{x}_1, \boldsymbol{x}_2) \,\mathrm{d}x_1 \right) \mathrm{d}x_2, \tag{1.9}$$

and both quantities are equal to $\int_{D_1 \times D_2} f(\boldsymbol{x}_1, \boldsymbol{x}_2) \,\mathrm{d}x_1 \,\mathrm{d}x_2$, where $\mathrm{d}x_1 \,\mathrm{d}x_2$ is the product measure on the Cartesian product $D_1 \times D_2$.

Exercises

Exercise 1.1 (Measurability). Let W be a nonmeasurable subset of $D := (0,1)$. Let $f : W \to \mathbb{R}$ be defined by $f(x) := 1$ if $x \in D\backslash W$ and $f(x) := 0$ if $x \in W$. (i) Is f measurable? (ii) Assume that there is a measurable subset $V \subset W$ s.t. $|V| > 0$. Compute $\sup_{x\in D} f(x)$, $\operatorname{ess\,sup}_{x\in D} f(x)$, $\inf_{x\in D} f(x)$, $\operatorname{ess\,inf}_{x\in D} f(x)$. (iii) Is f a member of $L^\infty(D)$? (iv) Assume now that W has zero measure (hence, W is measurable). Compute $\inf_{x\in D} f(x)$ and $\operatorname{ess\,inf}_{x\in D} f(x)$.

Exercise 1.2 (Measurability and equality a.e.). Prove Corollary 1.11. (*Hint*: consider the sets $A_r := \{\boldsymbol{x} \in D \mid f(\boldsymbol{x}) > r\}$ and $B_r := \{\boldsymbol{x} \in D \mid g(\boldsymbol{x}) > r\}$ for all $r \in \mathbb{R}$, and show that $B_r = (A_r \cap (A_r\backslash B_r)^c) \cup (B_r\backslash A_r)$.)

Exercise 1.3 (Lebesgue's theorem). Let $D := (-1,1)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(D)$ and let $g \in L^1(D)$. Assume that $f_n \to f$ a.e. in D. Propose a counterexample to show that the assumption "$|f_n| \le g$ a.e. for all $n \in \mathbb{N}$" cannot be replaced by "$f_n \le g$ a.e. for all $n \in \mathbb{N}$" in Lebesgue's dominated convergence theorem.

Exercise 1.4 (Compact support). Let $D := (0,1)$ and $f(x) := 1$ for all $x \in D$. What is the support of f in D? Is the support compact?

Exercise 1.5 (Pointwise limit of measurable functions). Let D be a measurable set in $\mathbb{R}^d$. Let $f_n : D \to \mathbb{R}$ for all $n \in \mathbb{N}$ be real-valued measurable functions. (i) Show that $\limsup_{n\in\mathbb{N}} f_n$ and $\liminf_{n\in\mathbb{N}} f_n$ are both measurable. (*Hint*: recall that $\limsup_{n\in\mathbb{N}} f_n(\boldsymbol{x}) := \inf_{n\in\mathbb{N}} \sup_{k\ge n} f_k(\boldsymbol{x})$ and $\liminf_{n\in\mathbb{N}} f_n(\boldsymbol{x}) := \sup_{n\in\mathbb{N}} \inf_{k\ge n} f_k(\boldsymbol{x})$ for all $\boldsymbol{x} \in D$). (ii) Let $f : D \to \mathbb{R}$. Assume that $f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ for every $\boldsymbol{x} \in D$. Show that f is measurable. (iii) Let $f : D \to \mathbb{R}$. Assume that $f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in D$. Show that f is measurable.

Exercise 1.6 (Operations on measurable functions). The objective of this exercise is to prove Theorem 1.6. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two measurable functions and let $\lambda \in \mathbb{R}$. (i) Show that λf is measurable. (*Hint*: use Lemma 1.9). (ii) Idem for $|f|$. (iii) Idem for $f + g$. (iv) Idem for fg. (*Hint*: observe that $fg = \frac{1}{2}(f+g)^2 - \frac{1}{2}(f-g)^2$.)