# Part I, Chapter 3

# Traces and Poincaré inequalities

This chapter reviews two types of results on the Sobolev spaces $W^{s,p}(D)$ introduced in the previous chapter. The first one concerns the notion of trace (i.e., loosely speaking, the boundary values) of functions in $W^{s,p}(D)$. The second one is about functional inequalities (due to Poincaré and Steklov) essentially bounding the $L^p$-norm of a function by that of its gradient. The validity of these results relies on some smoothness properties on the boundary of the set $D$. In this book, we mainly focus on Lipschitz sets. For any subset $S \subset \mathbb{R}^d$, $d \geq 1$, $\text{int}(S)$ denotes the interior of $S$ and $\overline{S}$ its closure.

## 3.1 Lipschitz sets and domains

**Definition 3.1 (Domain).** *Let $D$ be a nonempty subset of $\mathbb{R}^d$. In this book, $D$ is called* domain *if it is open, bounded, and connected.*

For instance, a domain in $\mathbb{R}$ is an open and bounded interval. At many instances in this book we will need to say something on the smoothness of the boundary $\partial D$ of a domain $D \subset \mathbb{R}^d$, $d \geq 2$. To stay simple, we are going to focus our attention on the class of Lipschitz domains. In simple words, a Lipschitz domain $D$ in $\mathbb{R}^d$, $d \geq 2$, is such that at every point $\boldsymbol{x} \in \partial D$, the boundary can be represented in a neighborhood of $\boldsymbol{x}$ as the graph of a Lipschitz function. Equivalently there exists a cone with nonzero aperture angle that can be moved in the neighborhood of $\boldsymbol{x}$ without changing direction and without exiting $D$. Let us now give some precise definitions.

**Definition 3.2 (Lipschitz set and domain).** *An open set $D$ in $\mathbb{R}^d$, $d \geq 2$, is said to be* Lipschitz *if for all $\boldsymbol{x} \in \partial D$, there exists a neighborhood $V_{\boldsymbol{x}}$ of $\boldsymbol{x}$ in $\mathbb{R}^d$, a rotation $\boldsymbol{R}_{\boldsymbol{x}} : \mathbb{R}^d \to \mathbb{R}^d$, and two real numbers $\alpha > 0$, $\beta > 0$ ($\alpha$ and $\beta$ may depend on $\boldsymbol{x}$) s.t. the following holds true:*

(i) $V_{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{R}_{\boldsymbol{x}}(B_\alpha \times I_\beta)$ *with* $B_\alpha := B_{\mathbb{R}^{d-1}}(\boldsymbol{0}, \alpha)$, $I_\beta := (-\beta, \beta)$.

(ii) *There exists a Lipschitz function* $\phi_{\boldsymbol{x}} : B_\alpha \to \mathbb{R}$ *such that* $\phi_{\boldsymbol{x}}(\mathbf{0}) = 0$, $\|\phi_{\boldsymbol{x}}\|_{L^\infty(B_\alpha)} \le \frac{1}{2}\beta$ *and (see Figure 3.1)*

$$D \cap V_{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{R}_{\boldsymbol{x}}(\{(\boldsymbol{y}', y_d) \in B_\alpha \times I_\beta \mid y_d < \phi_{\boldsymbol{x}}(\boldsymbol{y}')\}), \tag{3.1a}$$

$$\partial D \cap V_{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{R}_{\boldsymbol{x}}(\{(\boldsymbol{y}', y_d) \in B_\alpha \times I_\beta \mid y_d = \phi_{\boldsymbol{x}}(\boldsymbol{y}')\}). \tag{3.1b}$$

*We say that* $D$ *is a* Lipschitz domain *if it is a domain and a Lipschitz set.*

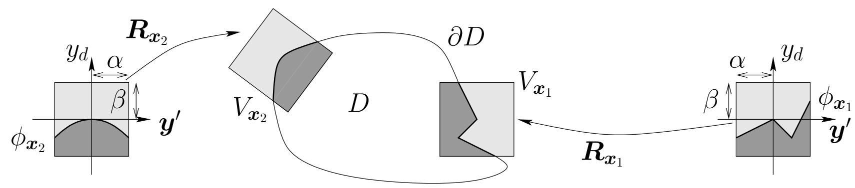


**Fig. 3.1** Lipschitz domain and mappings $(\boldsymbol{R}_{\boldsymbol{x}_1}, \phi_{\boldsymbol{x}_1})$, $(\boldsymbol{R}_{\boldsymbol{x}_2}, \phi_{\boldsymbol{x}_2})$.

**Definition 3.3 (Cone property).** *Let* $D$ *be an open set in* $\mathbb{R}^d$, $d \ge 2$. *We say that* $D$ *has the* uniform cone property *if for all* $\boldsymbol{x} \in \partial D$, *there exists a neighborhood* $V_{\boldsymbol{x}}$ *of* $\boldsymbol{x}$ *in* $\mathbb{R}^d$, *a rotation* $\boldsymbol{R}_{\boldsymbol{x}} : \mathbb{R}^d \to \mathbb{R}^d$, *positive real numbers* $\alpha$, $\beta$, $h$, $\theta \in (0, \frac{\pi}{2}]$ *(which may depend on* $\boldsymbol{x}$*) s.t. the following holds true:*

(i) $V_{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{R}_{\boldsymbol{x}}(B_\alpha \times I_\beta)$ *with* $B_\alpha := B_{\mathbb{R}^{d-1}}(\mathbf{0}, \alpha)$, $I_\beta := (-\beta, \beta)$.
(ii) *For all* $\boldsymbol{y} \in (\overline{D} \cap V_{\boldsymbol{x}})$, *we have* $\boldsymbol{y} + \boldsymbol{R}_{\boldsymbol{x}}(C) \subset D$ *with the cone* $C := \{(\boldsymbol{y}', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid -h < y_d < -\cot(\theta)\|\boldsymbol{y}'\|_{\ell^2(\mathbb{R}^{d-1})}\}$.

**Lemma 3.4 (Lipschitz domain and cone property).** *A domain in* $\mathbb{R}^d$, $d \ge 2$, *has the (uniform) cone property iff it is Lipschitz.*

*Proof.* See Grisvard [110, Thm. 1.2.2.2]. □

**Remark 3.5 (Finite covering).** Let $D$ be a domain in $\mathbb{R}^d$. Since $\partial D$ is compact, there is a finite set $\mathcal{L} \subset \mathbb{N}$ and a finite covering $\bigcup_{i \in \mathcal{L}} V_{\boldsymbol{x}_i}$ of $\partial D$ with $\boldsymbol{x}_i \in \partial D$ for all $i \in \mathcal{L}$. Definition 3.2 and Definition 3.3 can be equivalently reformulated for the finite set $\{\boldsymbol{x}_i\}_{i \in \mathcal{L}}$ with coefficients $\{\alpha_i, \beta_i, \theta_i, h_i\}_{i \in \mathcal{L}}$ that are bounded from below away from zero (the change of coordinates described by the rotation $\boldsymbol{R}_{\boldsymbol{x}_i}$ being fixed in each $V_{\boldsymbol{x}_i}$). □

**Remark 3.6 (Terminology).** In the literature, the term "domain" is sometimes defined without requiring $D$ to be bounded. We have incorporated this requirement in our definition since we mostly consider bounded sets in this book. Domains that are Lipschitz in the sense of Definition 3.2 are sometimes called *strongly Lipschitz*. It is also possible to weaken this definition. For instance, some authors say that a domain $D$ in $\mathbb{R}^d$ is *weakly Lipschitz* if for every $\boldsymbol{x} \in \partial D$, there exists a neighborhood $V_{\boldsymbol{x}} \ni \boldsymbol{x}$ in $\mathbb{R}^d$ and a global bilipschitz mapping $\boldsymbol{M}_{\boldsymbol{x}} : \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d$ such that $D \cap V_{\boldsymbol{x}} = \boldsymbol{M}_{\boldsymbol{x}}(\mathbb{R}^{d-1} \times \mathbb{R}_-) \cap V_{\boldsymbol{x}}$

and $\partial D \cap V_{\boldsymbol{x}} = \boldsymbol{M}_{\boldsymbol{x}}(\mathbb{R}^{d-1}\times\{0\}) \cap V_{\boldsymbol{x}}$. A strongly Lipschitz domain is weakly Lipschitz (using the notation of Definition 3.2, it suffices to set $\boldsymbol{M}_{\boldsymbol{x}}(\boldsymbol{y}', y_d) = \boldsymbol{x} + \boldsymbol{R}_{\boldsymbol{x}}(\boldsymbol{y}', y_d + \phi_{\boldsymbol{x}}(\boldsymbol{y}')))$, but a weakly Lipschitz domain may not be strongly Lipschitz. For instance, the two-brick domain (see Example 3.7) and the logarithmic spiral $\{re^{\mathrm{i}\theta} \mid r > 0, \theta \in \mathbb{R}, a_1 e^{-\theta} < r < a_2 e^{-\theta}\} \subsetneq \mathbb{R}^2$ (with positive real numbers $a_1, a_2$ s.t. $e^{-2\pi} a_2 < a_1 < a_2$ and $\mathrm{i}^2 = -1$) are weakly Lipschitz but are not strongly Lipschitz; see Axelsson and McIntosh [13]. These examples show that the image of a strongly Lipschitz domain by a bilipschitz mapping is not necessarily strongly Lipschitz. A weakly Lipschitz domain is strongly Lipschitz if the mapping $\boldsymbol{M}_{\boldsymbol{x}}$ is continuously differentiable. The source of the difficulty is that the implicit function theorem does not hold true for Lipschitz functions; see [110, pp. 7–10] for more details. In this book, we only consider strongly Lipschitz domains and, unless explicitly stated otherwise, when we say "let $D$ be a Lipschitz domain" we mean that $D$ is strongly Lipschitz in the sense of Definition 3.2. □

Lipschitz domains have many important properties:

(i) Outward normal: the outward-pointing unit normal $\boldsymbol{n}$ is well defined a.e. on the boundary of a Lipschitz domain (this follows from Rademacher's theorem (Theorem 2.7)). For an interval in $\mathbb{R}$, the outward unit normal is conventionally set to be $-1$ at the left endpoint and $+1$ at the right endpoint (in coherence with the conventional orientation of $\mathbb{R}$ from left to right).

(ii) One-sided property: a Lipschitz domain is always located on one side of its boundary, i.e., there cannot be slits or cuts; see Costabel and Dauge [82], Grisvard [110, §1.7] for discussions on domains with cuts.

(iii) Convexity: any Lipschitz domain is quasiconvex (see Remark 2.12). Conversely every convex domain is Lipschitz (see [110, Cor. 1.2.2.3]).

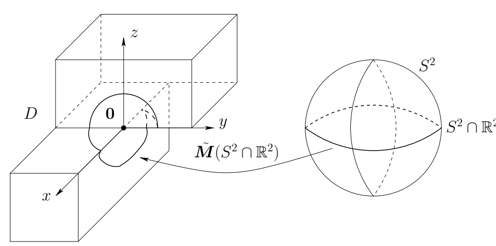


**Fig. 3.2** (Surprising) example of non-Lipschitz domain: the two-brick assembly.

**Example 3.7 (Two-brick domain).** Consider the bricks $B_1 := (-2,0) \times (-2,2) \times (0,2)$ and $B_2 := (-2,2) \times (-2,0) \times (-2,0)$, and the two-brick assembly $D := \mathrm{int}(\overline{B_1} \cup \overline{B_2})$ illustrated in Figure 3.2. Let us show that $D$ is not a Lipschitz domain by using the uniform cone property. For any $\epsilon \in (0,1)$, let $V_{\mathbf{0}}$ be the ball of radius $3\epsilon$ centered at $\mathbf{0}$. The points $\boldsymbol{a} := (\epsilon, -\epsilon, 0)$ and $\boldsymbol{a}' :=$

$(-\epsilon, \epsilon, 0)$ are both in $V_{\mathbf{0}} \cap \overline{D}$. Let us assume that the uniform cone property holds, and let $\boldsymbol{\zeta} := (\zeta_x, \zeta_y, \zeta_z)^\mathsf{T} := R_{\mathbf{0}}((0,0,-1)^\mathsf{T})$. Item (ii) in Definition 3.3 requires that $\boldsymbol{a} + \frac{1}{2}h\boldsymbol{\zeta} \in D$, which in turn implies that $\zeta_z < 0$. But also we must have $\boldsymbol{a}' + \frac{1}{2}h\boldsymbol{\zeta} \in D$, which implies that $\zeta_z > 0$. This contradiction implies that Item (ii) from Definition 3.3 cannot hold true for any neighborhood of $\mathbf{0}$. In other words, one cannot find a coordinate system such that the boundary of $D$ is described by the graph of a Lipschitz function in the neighborhood of the origin. Incidentally, one can show that $D$ is a weakly Lipschitz domain. Letting $\boldsymbol{\psi} : S^2 \to S^2$ be a bilipschitz homeomorphism of the unit sphere in $\mathbb{R}^3$ that maps the circle $S^2 \cap \mathbb{R}^2$ to the curve shown in bold in Figure 3.2, a mapping $\boldsymbol{M_0} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the definition from Remark 3.6 can be defined by $\boldsymbol{M_0}(\boldsymbol{x}) := \|\boldsymbol{x}\|_{\ell^2}\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_{\ell^2}})$ if $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{M_0}(\mathbf{0}) := \mathbf{0}$. That $\boldsymbol{M_0}$ is a bilipschitz mapping results from the identity $\|r_1\boldsymbol{\omega}_1 - r_2\boldsymbol{\omega}_2\|_{\ell^2}^2 = |r_1 - r_2|^2 + r_1 r_2 \|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\|_{\ell^2}^2$ with the notation $r_i := \|\boldsymbol{x}_i\|_{\ell^2}$, $\omega_i := \frac{\boldsymbol{x}_i}{\|\boldsymbol{x}_i\|_{\ell^2}}$. □

**Remark 3.8 (Stronger smoothness).** $D$ is said to be of *class* $C^m$ or *piecewise of class* $C^m$, $m \geq 1$, if all the local mappings $\phi_{\boldsymbol{x}}$ in the Definition 3.2 are of class $C^m$ or piecewise of class $C^m$, respectively. In this case, the outward unit normal is well defined for all $\boldsymbol{x} \in \partial D$ and is of class $C^{m-1}$. □

## 3.2 Traces as functions at the boundary

Boundary values of functions in $L^p(D)$, $p \in [1, \infty)$, are in general not well defined. For instance, let $D := (0,1)^2$ and $v(x_1, x_2) := x_1^{-\frac{\alpha}{p}}$ with $\alpha \in (0,1)$. Then $v \in L^p(D)$ but $v|_{x_1=0} = \infty$. The main idea of this section is to show that it is possible to define the boundary value of a function $v \in W^{s,p}(D)$ if $s$ is large enough. But how large? A first possibility is to invoke Morrey's theorem (see (2.10)): if $sp > d$, one can consider the continuous representative of $v$ to define the boundary value of $v$. The purpose of the trace theory is to give a meaning to boundary values under the weaker assumption $sp > 1$ (and $s \geq 1$ if $p = 1$) in every space dimension. In what follows, we consider Sobolev spaces defined on $\partial D$ by using the local mappings $\phi_{\boldsymbol{x}}$ from the Definition 3.2: letting $\boldsymbol{\psi}_{\boldsymbol{x}}(\boldsymbol{\xi}) := (\boldsymbol{\xi}, \phi_{\boldsymbol{x}}(\boldsymbol{\xi}))$ for all $\boldsymbol{\xi}$ in the open ball $B(\mathbf{0}, \alpha)$ in $\mathbb{R}^{d-1}$, we say that $v$ is in $W^{s,p}(\partial D)$ if $v \circ \boldsymbol{\psi}_{\boldsymbol{x}} \in W^{s,p}(B(\mathbf{0}, \alpha))$ for all $\boldsymbol{x} \in \partial D$. When $D$ is Lipschitz (resp., of class $C^{1,1}$), this approach defines $W^{s,p}(\partial D)$ up to $s = 1$ (resp., $s = 2$). We refer to Grisvard [110, §1.3.3] for more details.

### 3.2.1 The spaces $W_0^{s,p}(D)$, $W^{s,p}(D)$ and their traces

**Definition 3.9 ($W_0^{s,p}(D)$).** *Let $s > 0$ and $p \in [1, \infty)$. Let $D$ be an open set in $\mathbb{R}^d$. We define*

$$W_0^{s,p}(D) := \overline{C_0^\infty(D)}^{W^{s,p}(D)}, \tag{3.2}$$

*i.e.,, $W_0^{s,p}(D)$ is the closure of the subspace $C_0^\infty(D)$ in $W^{s,p}(D)$. For $p=2$, we write $H_0^s(D) := W_0^{s,2}(D)$.*

We will see below in Theorem 3.19 that $W^{s,p}(D) = W_0^{s,p}(D)$ if $sp \le 1$ whereas $W_0^{s,p}(D)$ is a proper subspace of $W^{s,p}(D)$ if $sp > 1$ and $D$ is bounded.

**Theorem 3.10 (Trace).** *Let $p \in [1,\infty)$. Let $s > \frac{1}{p}$ if $p > 1$ or $s \ge 1$ if $p = 1$. Let $D$ be a Lipschitz domain in $\mathbb{R}^d$. There is a bounded linear map $\gamma^{\mathrm{g}} : W^{s,p}(D) \to L^p(\partial D)$ such that:*

(i) *$\gamma^{\mathrm{g}}(v) = v_{|\partial D}$ whenever $v$ is smooth, e.g., $v \in C^0(\overline{D})$.*
(ii) *The kernel of $\gamma^{\mathrm{g}}$ is $W_0^{s,p}(D)$.*
(iii) *If $s = 1$ and $p = 1$, or if $s \in (\frac{1}{2}, \frac{3}{2})$ and $p = 2$, or if $s \in (\frac{1}{p}, 1]$ and $p \notin \{1,2\}$, then $\gamma^{\mathrm{g}} : W^{s,p}(D) \to W^{s-\frac{1}{p},p}(\partial D)$ is bounded and surjective, that is, there exists $C_{\gamma^{\mathrm{g}}}$ s.t. for every function $g \in W^{s-\frac{1}{p},p}(\partial D)$, one can find a function $u_g \in W^{s,p}(D)$, called a* lifting *of $g$, s.t.*

$$\gamma^{\mathrm{g}}(u_g) = g \qquad \text{and} \qquad \|u_g\|_{W^{s,p}(D)} \le C_{\gamma^{\mathrm{g}}} \ell_D^{\frac{1}{p}} \|g\|_{W^{s-\frac{1}{p},p}(\partial D)}, \tag{3.3}$$

*where $\ell_D$ is a characteristic length of $D$, e.g., $\ell_D := \mathrm{diam}(D)$.*

*Proof.* See Brezis [48, p. 315] ($s = 1$, $p \in [1,\infty)$), Grisvard [110, Thm. 1.5.1.2 & Cor. 1.5.1.6], McLean [141, Thm. 3.38] ($s \in (\frac{1}{2}, \frac{3}{2})$, $p = 2$); see Gagliardo [104] for the original work with $s = 1$, $p \in [1,\infty)$. □

**Remark 3.11 (Notation).** The superscript g stands for “gradient” since $\gamma^{\mathrm{g}}(v)$ is meaningful for $v \in W^{1,1}(D)$, i.e., $\gamma^{\mathrm{g}}(v)$ makes sense if the weak gradient of $v$ is integrable. □

**Example 3.12 (Elliptic PDEs).** Theorem 3.10 (with $s = 1$ and $p = 2$) is crucial in the analysis of elliptic PDEs, where a natural functional setting for the solution is the space $H^1(D)$. Whenever a homogeneous Dirichlet condition is enforced (prescribing to zero the value of the solution at the boundary), Item (ii) shows that the solution is in $H_0^1(D)$. When the boundary condition prescribes a nonzero value, the surjectivity of $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ is invoked to identify a proper functional setting (see Chapter 31). □

**Remark 3.13 ($W^{1,\infty}(D)$).** The trace theory in $W^{1,\infty}(D)$ is not trivial since $C^\infty(D)$ is not dense in $L^\infty(D)$; see Remark 1.39. The situation simplifies if $D$ is quasiconvex since $W^{1,\infty}(D) = C^{0,1}(D)$ in this case (see Remark 2.12). □

**Remark 3.14 (Trace of gradient).** If $v \in W^{s,p}(D)$ with $p \in [1,\infty)$ and $s > 1 + \frac{1}{p}$ if $p > 1$ or $s \ge 2$ if $p = 1$, then $\nabla v \in \boldsymbol{W}^{s-1,p}(D)$, and we can apply Theorem 3.10 componentwise, i.e., $\gamma^{\mathrm{g}}(\nabla v) \in \boldsymbol{W}^{s-1-\frac{1}{p}}(\partial D)$. □

Repeated applications of Theorem 3.10 lead to the following important result to define the domain of various finite element interpolation operators (for simplicity we only consider integrability on the manifold).

**Theorem 3.15 (Trace on low-dimensional manifolds).** *Let $p \in [1,\infty)$ and let $D$ be a Lipschitz domain in $\mathbb{R}^d$. Let $M$ be a smooth, or polyhedral, manifold of dimension $r$ in $\overline{D}$, $r \in \{0{:}d\}$. Then there is a bounded trace operator from $W^{s,p}(D)$ to $L^p(M)$ provided $sp > d-r$ (or $s \geq d-r$ if $p=1$).*

When solving boundary value problems, one sometimes has to enforce a Neumann boundary condition which consists of prescribing the value of the normal derivative $\partial_n u := \boldsymbol{n}{\cdot}\nabla u$ at the boundary; see Chapter 31. Enforcing such a boundary condition is ambiguous if $\boldsymbol{n}$ is discontinuous. For instance, irrespective of the smoothness of the function in question, the normal derivative on polygons and polyhedra cannot be continuous. Let us start to address this problem by considering the simpler case where the boundary enjoys some additional smoothness property.

**Theorem 3.16 (Normal derivative).** *Let $p \in (1,\infty)$ and $s - \frac{1}{p} \in (1,2)$. Let $D$ be a domain in $\mathbb{R}^d$ with a boundary of class $C^{k,1}$, with $k := 1$ if $s \leq 2$ and $k := 2$ otherwise. There is a bounded linear map $\gamma^{\partial_n} : W^{s,p}(D) \to W^{s-1-\frac{1}{p},p}(\partial D)$ so that $\gamma^{\partial_n}(v) = (\boldsymbol{n}{\cdot}\nabla v)_{|\partial D}$ for all $v \in C^1(\overline{D})$, and letting $\gamma_1 := (\gamma^{\mathrm{g}}, \gamma^{\partial_n}) : W^{s,p}(D) \to W^{s-\frac{1}{p},p}(\partial D) \times W^{s-1-\frac{1}{p},p}(\partial D)$,*

(i) *The map $\gamma_1$ is bounded and surjective.*
(ii) *The kernel of $\gamma_1$ is $W_0^{s,p}(D)$.*

*Proof.* See Grisvard [110, Thm. 1.5.1.2] for the statement (i) and [110, Cor. 1.5.1.6] for the statement (ii). □

The above theorem can be extended to polygons ($d = 2$) as detailed in [110, Thm. 1.5.2.1]. The situation is more subtle when $D$ is only Lipschitz. An extension of the notion of the normal derivative in this case is introduced in §4.3, and we refer the reader to Example 4.16 where $\boldsymbol{n}{\cdot}\nabla u$ is defined by duality.

### 3.2.2 The spaces $\widetilde{W}^{s,p}(D)$

We have seen that a function $v \in W^{s,p}(D)$ has a trace at the boundary $\partial D$ if $s$ is large enough. Another closely related question is whether the *zero-extension* of $v$ to the whole space $\mathbb{R}^d$ belongs to $W^{s,p}(\mathbb{R}^d)$. For instance, the zero-extension to $\mathbb{R}^d$ of a test function $\varphi \in C_0^\infty(D)$ is in $C_0^\infty(\mathbb{R}^d)$. For every function $v \in L^1(D)$, we denote by $\widetilde{v}$ the extension by zero of $v$ to $\mathbb{R}^d$, i.e., $\widetilde{v}(\boldsymbol{x}) := v(\boldsymbol{x})$ if $\boldsymbol{x} \in D$ and $\widetilde{v}(\boldsymbol{x}) := 0$ otherwise.

**Definition 3.17 ($\widetilde{W}^{s,p}(D)$).** *Let $s > 0$ and $p \in [1,\infty]$. Let $D$ be an open subset of $\mathbb{R}^d$. We define*

$$\widetilde{W}^{s,p}(D) := \{v \in W^{s,p}(D) \mid \widetilde{v} \in W^{s,p}(\mathbb{R}^d)\}. \tag{3.4}$$

*For $p = 2$, we write $\widetilde{H}^s(D) := \widetilde{W}^{s,2}(D)$.*

**Theorem 3.18 (Completion).** *$\widetilde{W}^{s,p}(D)$ is a Banach space equipped with the norm $\|v\|_{\widetilde{W}^{s,p}(D)} := \|\widetilde{v}\|_{W^{s,p}(\mathbb{R}^d)}$. Moreover, $\|v\|_{\widetilde{W}^{s,p}(D)} = \|v\|_{W^{s,p}(D)}$ if $s \in \mathbb{N}$. If $s \notin \mathbb{N}$ and $D$ is a Lipschitz domain in $\mathbb{R}^d$, $\|v\|_{\widetilde{W}^{s,p}(D)}$ is equivalent to the norm $(\|v\|^p_{W^{s,p}(D)} + \ell_D^{sp} \sum_{|\alpha|=m} \int_D (\rho(\boldsymbol{x}))^{-\sigma p} |\partial^\alpha v|^p \,\mathrm{d}x)^{\frac{1}{p}}$, where $m := \lfloor s \rfloor$, $\sigma := s - m$, and $\rho$ is the distance to $\partial D$, i.e., $\rho(\boldsymbol{x}) := \inf_{\boldsymbol{y} \in \partial D} \|\boldsymbol{x} - \boldsymbol{y}\|_{\ell^2}$.*

*Proof.* See Grisvard [110, Lem. 1.3.2.6], Tartar [189, Lem. 37.1]. □

**Theorem 3.19 ($W^{s,p}(D)$, $W_0^{s,p}(D)$, $\widetilde{W}^{s,p}(D)$).** *Let $s > 0$ and $p \in (1, \infty)$. Let $D$ be a Lipschitz domain in $\mathbb{R}^d$. The following holds true:*

$$W^{s,p}(D) = W_0^{s,p}(D) = \widetilde{W}^{s,p}(D) \qquad (sp < 1), \tag{3.5a}$$

$$W^{s,p}(D) = W_0^{s,p}(D) \neq \widetilde{W}^{s,p}(D) \qquad (sp = 1), \tag{3.5b}$$

$$W^{s,p}(D) \neq W_0^{s,p}(D) = \widetilde{W}^{s,p}(D) \qquad (sp > 1, s - \frac{1}{p} \notin \mathbb{N}). \tag{3.5c}$$

*For all $sp > 1$, $W_0^{s,p}(D)$ is a proper subspace of $W^{s,p}(D)$. (The above equalities mean that the sets coincide and the associated norms are equivalent, i.e., the topologies are identical.)*

*Proof.* See Grisvard [110, Thm. 1.4.2.4, Cor. 1.4.4.5], Tartar [189, Chap. 33], Lions and Magenes [135, Thm. 11.1]; see also Exercise 3.4 for a proof of the fact that $\widetilde{W}^{1,p}(D) \hookrightarrow W_0^{1,p}(D)$. □

**Remark 3.20 ($D = \mathbb{R}^d$).** We have $W_0^{s,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d) = \widetilde{W}^{s,p}(\mathbb{R}^d)$ for all $s > 0$ and all $p \in [1, \infty)$; see [110, p. 24], [189, Lem. 6.5]. □

**Remark 3.21 (Embedding of $\widetilde{W}^{s,p}(D)$).** The same conclusions as in Theorems 2.31 and 2.35 hold true for $\widetilde{W}^{s,p}(D)$ since the $(s,p)$-extension property is available. □

**Remark 3.22 (Density).** Let $D$ be a Lipschitz domain in $\mathbb{R}^d$, $s > 0$, $p \in (1, \infty)$. Then $C_0^\infty(D)$ is dense in $\widetilde{W}^{s,p}(D)$; see [110, Thm. 1.4.2.2]. □

**Remark 3.23 (Interpolation).** Let $p \in [1, \infty)$, $s \in (0, 1)$. We have $W^{s,p}(D) = [L^p(D), W^{1,p}(D)]_{s,p}$ with equivalent norms; see Remark 2.20 and [189, Lem. 36.1]. Let us now define

$$W_{00}^{s,p}(D) := [L^p(D; \mathbb{R}^q), W_0^{1,p}(D)]_{s,p}. \tag{3.6}$$

It is established in Chandler-Wilde et al. [65, Cor. 4.10] that for $p = 2$,

$$\widetilde{H}^s(D) = H_{00}^s(D). \tag{3.7}$$

(More generally, we conjecture that $\widetilde{W}^{s,p}(D) = W_{00}^{s,p}(D)$.) The equality (3.7) together with Theorem 3.19 implies that $H_{00}^s(D) = H_0^s(D)$ if $s \neq \frac{1}{2}$. □

## 3.3 Poincaré–Steklov inequalities

We list here a series of functional inequalities that will be used repeatedly in the book; see Remark 3.32 for some historical background and some comments on the terminology.

**Lemma 3.24 (Poincaré–Steklov).** *Let $D$ be a Lipschitz domain in $\mathbb{R}^d$. Let $\ell_D := \operatorname{diam}(D)$. Let $p \in [1, \infty]$. There is $C_{\mathrm{PS},p}$ (the subscript $p$ is omitted when $p = 2$) s.t.*

$$C_{\mathrm{PS},p} \|v - \underline{v}_D\|_{L^p(D)} \le \ell_D |v|_{W^{1,p}(D)}, \qquad \forall v \in W^{1,p}(D), \tag{3.8}$$

*where $\underline{v}_D := \frac{1}{|D|}\int_D v \,\mathrm{d}x$. The following holds true when $D$ is convex:*

$$C_{\mathrm{PS},1} = 2, \quad C_{\mathrm{PS}} := C_{\mathrm{PS},2} = \pi, \quad C_{\mathrm{PS},p} \ge \frac{1}{2}\left(\frac{2}{p}\right)^{\frac{1}{p}}, \ p > 1. \tag{3.9}$$

**Remark 3.25 (Best constant).** The values in (3.9) are proved in Acosta and Durán [2] for $p = 1$, in Bebendorf [17] for $p = 2$ (see also Payne and Weinberger [157] for the general idea), and in Chua and Wheeden [72, Thm. 1.2] for general $p$. The constants given in (3.9) for $p \in \{1, 2\}$ are the best possible. Uniform bounds on the Poincaré–Steklov constant for possibly nonconvex sets are a delicate issue; see Exercise 22.3 and Veeser and Verfürth [194]. □

**Lemma 3.26 (Fractional Poincaré–Steklov).** *Let $p \in [1, \infty)$ and $s \in (0, 1)$. Let $D$ be a Lipschitz domain in $\mathbb{R}^d$. Let $\ell_D := \operatorname{diam}(D)$. Let us set $\underline{v}_D := \frac{1}{|D|}\int_D v \,\mathrm{d}x$. The following holds true:*

$$\|v - \underline{v}_D\|_{L^p(D)} \le \ell_D^s \left(\frac{\ell_D^d}{|D|}\right)^{\frac{1}{p}} |v|_{W^{s,p}(D)}. \tag{3.10}$$

*We also have $|v - \underline{v}_D|_{W^{r,p}(D)} = |v|_{W^{r,p}(D)} \le \ell_D^{s-r} |v|_{W^{s,p}(D)}$ for all $r \in (0, s]$.*

*Proof.* A direct proof is proposed in Exercise 3.3 following [97, Lem. 7.1]. See also Dupont and Scott [92, Prop. 6.1] and Heuer [116]. The factor $\frac{\ell_D^d}{|D|}$ is often called eccentricity of $D$. □

**Lemma 3.27 (Poincaré–Steklov).** *Let $p \in [1, \infty]$ and let $D$ be a Lipschitz domain. Let $\ell_D := \operatorname{diam}(D)$. There is $C_{\mathrm{PS},p} > 0$ (the subscript $p$ is omitted when $p = 2$) such that*

$$C_{\mathrm{PS},p} \|v\|_{L^p(D)} \le \ell_D \|\nabla v\|_{\boldsymbol{L}^p(D)}, \qquad \forall v \in W_0^{1,p}(D). \tag{3.11}$$

*Proof.* See Brezis [48, Cor. 9.19], Evans [99, Thm. 3, §5.6]. □

**Remark 3.28 (Unit).** The Poincaré–Steklov constant $C_{\mathrm{PS},p}$ is a dimensionless number. Its value remains unchanged if $D$ is translated or rotated.

Moreover, assuming $\mathbf{0} \in D$, if $\tilde{D} = \lambda^{-1}D$ with $\lambda > 0$, the two domains $D$ and $\tilde{D}$ have the same Poincaré–Steklov constant. □

**Remark 3.29 (Norm equivalence).** The Poincaré–Steklov inequality implies that the seminorm $|\cdot|_{W^{1,p}(D)}$ is a norm equivalent to $\|\cdot\|_{W^{1,p}(D)}$ in $W_0^{1,p}(D)$. For instance, for the $H^1$-norm $\|v\|^2_{H^1(D)} = \|v\|^2_{L^2(D)} + \ell_D^2|v|^2_{H^1(D)}$ (recall that $|v|_{H^1(D)} = \|\nabla v\|_{\boldsymbol{L}^2(D)}$), we obtain

$$\frac{C_{\text{PS}}}{(1+C_{\text{PS}}^2)^{\frac{1}{2}}}\|v\|_{H^1(D)} \leq \ell_D|v|_{H^1(D)} \leq \|v\|_{H^1(D)}, \qquad \forall v \in H_0^1(D).$$ □

**Lemma 3.30 (Extended Poincaré–Steklov).** *Let $p \in [1,\infty)$ and let $D$ be a Lipschitz domain in $\mathbb{R}^d$. Let $\ell_D := \text{diam}(D)$. Let $f$ be a bounded linear form on $W^{1,p}(D)$ whose restriction on constant functions is not zero. There is $\check{C}_{\text{PS},p} > 0$ (the subscript $p$ is omitted when $p = 2$) such that*

$$\check{C}_{\text{PS},p}\|v\|_{L^p(D)} \leq \ell_D\|\nabla v\|_{\boldsymbol{L}^p(D)} + |f(v)|, \qquad \forall v \in W^{1,p}(D). \tag{3.12}$$

*In particular, letting* $\ker(f) := \{v \in W^{1,p}(D) \mid f(v) = 0\}$, *we have*

$$\check{C}_{\text{PS},p}\|v\|_{L^p(D)} \leq \ell_D\|\nabla v\|_{\boldsymbol{L}^p(D)}, \qquad \forall v \in \ker(f). \tag{3.13}$$

*Moreover, if $f(\mathbb{1}_D) = 1$ (where $\mathbb{1}_D$ is the indicator function of $D$), we have*

$$\check{C}_{\text{PS},p}\|v - f(v)\mathbb{1}_D\|_{L^p(D)} \leq \ell_D\|\nabla v\|_{\boldsymbol{L}^p(D)}, \qquad \forall v \in W^{1,p}(D). \tag{3.14}$$

*Proof.* We use the Peetre–Tartar lemma (Lemma A.20) to prove (3.12). Let $X := W^{1,p}(D)$, $Y := \boldsymbol{L}^p(D) \times \mathbb{R}$, $Z := L^p(D)$, and $A : X \ni v \mapsto (\nabla v, f(v)) \in Y$. Owing to Lemma 2.11 and the hypotheses on $f$, $A$ is continuous and injective. Moreover, the embedding $X \hookrightarrow Z$ is compact owing to Theorem 2.35. This proves (3.12), and (3.13) is a direct consequence of (3.12). To prove (3.14), we apply (3.12) to the function $\tilde{v} := v - f(v)\mathbb{1}_D$. This function is in $\ker(f)$ since $f(\mathbb{1}_D) = 1$ and it satisfies $\nabla\tilde{v} = \nabla v$. □

**Example 3.31 (Zero mean-value).** Lemma 3.30 can be applied with $f(v) := |U|^{-1}\int_U v\,\mathrm{d}x$, where $U$ is a subset of $D$ of nonzero measure (the boundedness of $f$ follows from $|f(v)| \leq |U|^{-\frac{1}{p}}\|v\|_{L^p(D)}$ by Hölder's inequality). One can also apply Lemma 3.30 with $f(v) := |\partial D_1|^{-1}\int_{\partial D_1} v\,\mathrm{d}s$, where $\partial D_1$ is a subset of $\partial D$ of nonzero $(d-1)$-measure (the boundedness of $f$ is a consequence of Theorem 3.10). □

**Remark 3.32 (Terminology).** The inequality (3.8) is often called *Poincaré inequality* in the literature, and it is sometimes associated with other names like Wirtinger or Friedrichs. It turns out that Poincaré proved (3.8) for a convex domain in 1890 in [158] (the problem is formulated at the bottom of page 252, and the theorem is given at the bottom of page 258). Poincaré refined his estimates of $C_{\mathrm{PS}}$ in 1894 in [159, p. 76] and gave $C_{\mathrm{PS}} \geq \frac{16}{9}$ for a three-dimensional convex domain. Without invoking the convexity assumption, he has also showed in [158] that the best constant $C_{\mathrm{PS}}^2$ in the inequality

$$C_{\mathrm{PS}}^2 \|v\|_{L^2(D)}^2 \leq \ell_D^2 (\alpha \|v\|_{L^2(\partial D)}^2 + |v|_{H^1(D)}^2), \quad \forall v \in H^1(D), \tag{3.15}$$

is the smallest eigenvalue of the Laplacian supplemented with the Robin boundary condition $(\alpha v + \partial_n v)_{|\partial D} = 0$ (cf. statement in the middle of page 240: "and we must conclude that $k_1$ is the minimum of the ratio $B/A$" (in French)). The simplest form of (3.8) on an interval with $p = 2$ can be traced to the work of Steklov (see [184, Lem. 2, p. 156] for the Russian version published in 1897 with $C_{\mathrm{PS}} \geq \sqrt{2}$ and [182, pp. 294–295] for the 1901 French version with $C_{\mathrm{PS}} = \pi$ for functions that are either zero at both ends of the interval or are of zero mean). Steklov makes ample references to the work of Poincaré in each paper. He revisited the work of Poincaré on the spectrum of the Laplacian in [183, 185]. He proved in [183, Thm. VII, p. 66] and in [185, Thm. XIV, p. 107] that $C_{\mathrm{PS}}^2$ in (3.11) is the smallest eigenvalue of the Laplacian supplemented with homogeneous Dirichlet boundary conditions. He reproved that $C_{\mathrm{PS}}^2$ in (3.8) is the smallest eigenvalue of the Laplacian supplemented with homogeneous Neumann boundary conditions in [185, Thm. XV, p. 110]. A detailed survey of the literature on the best constant in (3.8) can be found in Kuznetsov and Nazarov [130]. Note that [183] is cited in [130] for the work of Steklov on the Laplacian with Neumann boundary condition, whereas the paper in question only deals with Dirichlet boundary conditions. For mysterious reasons, the paper by Friedrichs, Eine invariante Formulierung des Newtonschen Gravititationsgesetzes und des Grenzüberganges vom Einsteinschen zum Newtonschen Gesetz. *Math. Ann.* 98 (1927), 566–575, is sometimes cited in the literature in relation to Poincaré's inequalities, including in [130], but the topic of this paper is not even remotely related to the Poincaré inequality. One early work of Friedrichs related to Poincaré's inequalities is a semi-discrete version of (3.15) published in Courant et al. [85, Eq. (13)]. Finally, it seems that the name of Wirtinger has been attached for the first time in 1916 to the inequality $\|f\|_{L^2(0,2\pi)} \leq |f|_{H^1(0,2\pi)}$ for periodic functions by Blaschke in his book [24, p. 105] without any specific reference. A little bit at odd with the rest of the literature, we henceforth adopt the Poincaré–Steklov terminology to refer to inequalities like (3.8) and (3.11). □

## Exercises

**Exercise 3.1 (Scaling).** Let $D \subset \mathbb{R}^d$ be a Lipschitz domain. Let $\lambda > 0$ and $\tilde{D} := \lambda^{-1} D$. (i) Show that $D$ and $\tilde{D}$ have the same Poincaré–Steklov constant in (3.8). (ii) Same question for (3.11).

**Exercise 3.2 (Poincaré–Steklov, 1D).** Let $D := (0,1)$ and $u \in C^1(D;\mathbb{R})$. Prove the following bounds: (i) $\|u\|^2_{L^2(D)} \le \frac{1}{2}\|u'\|^2_{L^2(D)}$ if $u(0) = 0$. (*Hint*: $u(x) = \int_0^x u'(t)\,\mathrm{d}t$.) (ii) $\|u\|^2_{L^2(D)} \le \frac{1}{\sqrt{8}}\|u'\|^2_{L^2(D)}$ if $u(0) = u(1) = 0$. (*Hint*: as above, but distinguish whether $x \in (0, \frac{1}{2})$ or $x \in (\frac{1}{2}, 1)$.) (iii) $\|u\|^2_{L^2(D)} \le \frac{1}{6}\|u'\|^2_{L^2(D)} + \underline{u}^2$ with $\underline{u} := \int_0^1 u\,\mathrm{d}x$. (*Hint*: square the identity $u(x) - u(y) = \int_x^y u'(t)\,\mathrm{d}t$.) (iv) $\max_{x\in\overline{D}} |u(x)|^2 \le 2u(1)^2 + 2\|u'\|^2_{L^2(D)}$. (*Hint*: square $u(x) = u(1) + \int_1^x u'(t)\,\mathrm{d}t$.) (v) $\max_{x\in\overline{D}} |u(x)|^2 \le 2(\|u\|^2_{L^2(D)} + \|u'\|^2_{L^2(D)})$. (*Hint*: prove that $u(x)^2 \le 2u(y)^2 + 2\|u'\|^2_{L^2(D)}$ and integrate over $y \in D$.)

**Exercise 3.3 (Fractional Poincaré–Steklov).** (i) Prove (3.10). (*Hint*: write $\int_D |v(\boldsymbol{x}) - \underline{v}_D|^p \mathrm{d}x = \int_D |D|^{-p} \left|\int_D (v(\boldsymbol{x}) - v(\boldsymbol{y}))\,\mathrm{d}y\right|^p \mathrm{d}x$.) (ii) Prove that $|v - \underline{v}_D|_{W^{r,p}(D)} \le \ell_D^{s-r}|v|_{W^{s,p}(D)}$ for all $r \in (0, s]$ and all $s \in (0,1)$.

**Exercise 3.4 (Zero-extension in $W_0^{1,p}(D)$).** Let $p \in [1,\infty)$. Let $D$ be an open set in $\mathbb{R}^d$. Show that $W_0^{1,p}(D) \hookrightarrow \widetilde{W}^{1,p}(D)$ and $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^d)} \le \|u\|_{W^{1,p}(D)}$ for all $u \in W_0^{1,p}(D)$.

**Exercise 3.5 (Integral representation).** Let $v : [0,\infty) \longrightarrow \mathbb{R}$ be a continuous function with bounded derivative, and let $w : [0,\infty) \longrightarrow \mathbb{R}$ be such that $w(x) := \frac{1}{x}\int_0^x (v(t) - v(x))\,\mathrm{d}t$. (i) Show that $|w(x)| \le \frac{Mx}{2}$ where $M := \sup_{x\in[0,\infty)} |\partial_x v(x)|$. (ii) Estimate $w(0)$. (iii) Show that $\partial_t(tw(t)) = -t\partial_t v(t)$. (iv) Prove that $v(x)-v(0) = -w(x) - \int_0^x \frac{w(t)}{t}\,\mathrm{d}t$. (*Hint*: observe that $v(x) - v(0) = \int_0^x \frac{1}{t}\,(t\partial_t v(t))\,\mathrm{d}t$, use (iii), and integrate by parts.) (v) Prove the following integral representation formula (see Grisvard [110, pp. 29-30]):

$$v(0) = v(x) + \frac{1}{x}\int_0^x (v(t) - v(x))\,\mathrm{d}t + \int_0^x \frac{1}{y^2}\int_0^y (v(t) - v(y))\,\mathrm{d}t\,\mathrm{d}y.$$

**Exercise 3.6 (Trace inequality in $W^{s,p}$, $sp > 1$).** Let $s \in (0,1)$, $p \in [1,\infty)$, and $sp > 1$. Let $a > 0$ and $F$ be an open bounded subset of $\mathbb{R}^{d-1}$. Let $D := F\times(0,a)$. Let $v \in C^1(D) \cap C^0(\overline{D})$. (i) Let $\boldsymbol{y} \in F$. Using the integral representation from Exercise 3.5, show that there are $c_1(s,p)$ and $c_2(s,p)$ such that

$$|v(\boldsymbol{y},0)| \le a^{-\frac{1}{p}}\|v(\boldsymbol{y},.)\|_{L^p(0,a)} + (c_1(s,p) + c_2(s,p))a^{s-\frac{1}{p}}|v(\boldsymbol{y},.)|_{W^{s,p}(0,a)}.$$

(ii) Accept as a fact that there is $c$ (depending on $s$ and $p$) such that

$$\int_F\int_0^a\int_0^a \frac{|v(\boldsymbol{x}_{d-1},x_d) - v(\boldsymbol{x}_{d-1},y_d)|^p}{|x_d - y_d|^{sp+1}}\,\mathrm{d}x_1 \ldots \mathrm{d}x_{d-1}\,\mathrm{d}x_d\,\mathrm{d}y_d \le c\,|v|_{W^{s,p}(D)}.$$

Prove that $\|v(.,0)\|_{L^p(F)} \le c'\,(a^{-\frac{1}{p}}\|v\|_{L^p(D)} + a^{s-\frac{1}{p}}|v|_{W^{s,p}(D)})$. *Note*: this shows that the trace operator $\gamma^{\mathrm{g}} : C^1(D) \cap C^0(\overline{D}) \to L^p(F)$ is bounded uniformly w.r.t. the norm of $W^{s,p}(D)$ when $sp > 1$. This means that $\gamma^{\mathrm{g}}$ can be extended to $W^{s,p}(D)$ since $C^1(D) \cap C^0(\overline{D})$ is dense in $W^{s,p}(D)$.