Part I, Chapter 4

Distributions and duality in Sobolev spaces

The dual space of a Sobolev space is not only composed of functions (defined almost everywhere), but this space also contains more sophisticated objects called distributions, which are defined by their action on smooth functions with compact support. For instance, the function $\frac{1}{x}$ is not in $L^1(0,1)$, but the map $\varphi \mapsto \int_0^1 \frac{1}{x}\varphi(x)\,\mathrm{d}x$ can be given a meaning for every smooth function that vanishes at 0. Dual Sobolev spaces are useful to handle singularities on the right-hand side of PDEs. They are also useful to give a meaning to the tangential and the normal traces of $\mathbb{R}^d$-valued fields that are not in $W^{s,p}(D;\mathbb{R}^d)$ with $sp > 1$. The extension is done in this case by invoking integration by parts formulas involving the curl or the divergence operators.

4.1 Distributions

The notion of distribution is a powerful tool that extends the concept of integrable functions and weak derivatives. In particular, we will see that every distribution is differentiable in some reasonable sense.

Definition 4.1 (Distribution). *Let D be an open set in $\mathbb{R}^d$. A linear map*

$$T : C_0^\infty(D) \ni \varphi \longmapsto \langle T, \varphi\rangle := T(\varphi) \in \mathbb{R} \text{ or } \mathbb{C}, \tag{4.1}$$

is called distribution *in D if for every compact subset K of D, there exist an integer p, called the* order *of T, and a real number c (both can depend on K) s.t. for all $\varphi \in C_0^\infty(D)$ with* $\mathrm{supp}(\varphi) \subset K$*, we have*

$$|\langle T, \varphi\rangle| \le c \max_{|\alpha|\le p} \big(\ell_D^{|\alpha|} \|\partial^\alpha \varphi\|_{L^\infty(K)}\big). \tag{4.2}$$

Let T be distribution of order p. We henceforth abuse the notation by using the symbol T to denote the extension by density of T to $C_0^p(D)$.

Example 4.2 (Locally integrable functions). Every function v in $L^1_{\mathrm{loc}}(D)$ can be identified with the following distribution:

$$T_v : C_0^\infty(D) \ni \varphi \longmapsto \langle T_v, \varphi\rangle := \int_D v\varphi \,\mathrm{d}x.$$

This identification is possible owing to Theorem 1.32, since two functions $v, w \in L^1_{\mathrm{loc}}(D)$ are such that $v = w$ a.e. in D iff $\int_D v\varphi \,\mathrm{d}x = \int_D w\varphi \,\mathrm{d}x$ for all $\varphi \in C_0^\infty(D)$. We will abuse the notation by writing v instead of T_v. Notice that the identification is also compatible with the Riesz–Fréchet theorem (Theorem 1.41) in $L^2(D)$, which allows one to identify $L^2(D)$ with its dual space by means of the L^2-inner product. □

Example 4.3 (Dirac mass or measure). Let $\boldsymbol{a}$ be a point in D. The *Dirac mass* (or Dirac measure) at $\boldsymbol{a}$ is the distribution defined by $\langle \delta_{\boldsymbol{a}}, \varphi\rangle := \varphi(\boldsymbol{a})$ for all $\varphi \in C_0^\infty(D)$. There is no function $f \in L^1_{\mathrm{loc}}(D)$ such that $\delta_{\boldsymbol{a}} = T_f$. Otherwise, one would have $0 = \int_D f\varphi \,\mathrm{d}x$ for all $\varphi \in C_0^\infty(D\backslash\{\boldsymbol{a}\})$, and owing to Theorem 1.32, this would imply that $f = 0$ a.e. in $D\backslash\{\boldsymbol{a}\}$, i.e., $f = 0$ a.e. in D. Hence, $\delta_{\boldsymbol{a}} \notin T(L^1_{\mathrm{loc}}(D))$. This example shows that there are distributions that cannot be identified with functions in $L^1_{\mathrm{loc}}(D)$. □

Definition 4.4 (Distributional derivative). *Let T be a distribution in D and let $i \in \{1{:}d\}$. The* distributional derivative *$\partial_i T$ is the distribution in D such that $\langle \partial_i T, \varphi\rangle := -\langle T, \partial_i\varphi\rangle$ for all $\varphi \in C_0^\infty(D)$. More generally, for a multi-index $\alpha \in \mathbb{N}^d$, the distributional derivative $\partial^\alpha T$ is the distribution in D acting as $\langle \partial^\alpha T, \varphi\rangle := (-1)^{|\alpha|}\langle T, \partial^\alpha\varphi\rangle$. We set conventionally $\partial^0 T := T$, and $\nabla T := (\partial_1 T, \ldots, \partial_d T)^{\mathsf{T}}$.*

Example 4.5 (Weak derivative). The notion of distributional derivative extends the notion of weak derivative. Let $v \in L^1_{\mathrm{loc}}(D)$ and assume that v has a weak α-th partial derivative, say $\partial^\alpha v \in L^1_{\mathrm{loc}}(D)$. Just like in Example 4.2, we can identify v and $\partial^\alpha v$ with the distributions T_v and $T_{\partial^\alpha v}$ such that $\langle T_v, \psi\rangle := \int_D v\psi \,\mathrm{d}x$ and $\langle T_{\partial^\alpha v}, \varphi\rangle := (-1)^{|\alpha|}\int_D v\partial^\alpha\varphi \,\mathrm{d}x$. This implies that $\langle T_{\partial^\alpha v}, \varphi\rangle = (-1)^{|\alpha|}\langle T_v, \partial^\alpha\varphi\rangle$, which according to Definition 4.4 shows that $\partial^\alpha T_v = T_{\partial^\alpha v}$, i.e., the distributional derivative of T_v is equal to the distribution associated with the weak derivative of v. □

Example 4.6 (Step function). Let $D := (-1, 1)$. Let $w \in L^1(D)$ be defined by $w(x) := -1$ if $x < 0$ and $w(x) := 1$ otherwise. For all $\varphi \in C_0^\infty(D)$, we have $-\int_D w\partial_x\varphi \,\mathrm{d}x = \int_{-1}^0 \partial_x\varphi \,\mathrm{d}x - \int_0^1 \partial_x\varphi \,\mathrm{d}x = 2\varphi(0) = 2\langle \delta_0, \varphi\rangle$. This shows that the distributional derivative of w is twice the Dirac mass at 0, i.e., we write $\partial_x w = 2\delta_0$. As established in Example 4.3, δ_0 cannot be identified with any function in $L^1_{\mathrm{loc}}(D)$. Hence, w does not have a weak derivative but w has a distributional derivative. Consider now the function $v(x) := 1 - |x|$ in $L^1(D)$. By proceeding as in Example 2.5, one shows that v has a weak derivative and $\partial_x v(x) = 1$ if $x < 0$, and $\partial_x v(x) = -1$ otherwise. As established in Example 4.5, the distributional derivative of v and its weak derivative coincide. Notice though that the distributional second derivative of v is $\partial_{xx}v = -2\delta_0$ which is not a weak derivative. □

Example 4.7 (Dirac measure on the unit sphere). (i) Let $\boldsymbol{a} \in \mathbb{R}^d$. Definition 4.4 implies that $\langle \partial^\alpha \delta_{\boldsymbol{a}}, \varphi \rangle = (-1)^{|\alpha|}\partial^\alpha \varphi(\boldsymbol{a})$. (ii) Let $u : \mathbb{R}^d \to \mathbb{R}$ be such that $u(\boldsymbol{x}) := 1$ if $\|\boldsymbol{x}\|_{\ell^2} \leq 1$ and $u(\boldsymbol{x}) := 0$ otherwise. Let $B(\boldsymbol{0}, 1)$ and $S(\boldsymbol{0}, 1)$ be the unit ball and unit sphere in $\mathbb{R}^d$. We define the Dirac measure supported in $S(\boldsymbol{0}, 1)$ by $\langle \delta_{S(\boldsymbol{0},1)}, \varphi \rangle := \int_{S(\boldsymbol{0},1)} \varphi \, \mathrm{d}s$. Let $\boldsymbol{e}_i$ be one of the canonical unit vectors of $\mathbb{R}^d$. Then $\langle \partial_i u, \varphi \rangle = -\int_{B(\boldsymbol{0},1)} \partial_i \varphi \, \mathrm{d}x = -\int_{B(\boldsymbol{0},1)} \nabla{\cdot}(\varphi \boldsymbol{e}_i) \, \mathrm{d}x$, which proves that $\langle \partial_i u, \varphi \rangle = -\int_{S(\boldsymbol{0},1)} \boldsymbol{n}{\cdot}\boldsymbol{e}_i \varphi \, \mathrm{d}s$. Hence, $\nabla u = -\delta_{S(\boldsymbol{0},1)}\boldsymbol{n}$. □

Definition 4.8 (Distributional convergence). *Let D be an open set in $\mathbb{R}^d$. We say that a sequence of distributions $\{T_n\}_{n\in\mathbb{N}}$* converges in the distribution sense *if one has $\lim_{n\to\infty}\langle T_n, \varphi \rangle = \langle T, \varphi \rangle$ for all $\varphi \in \mathcal{C}_0^\infty(D)$.*

Example 4.9 (Oscillating functions). Let $D := (0, 1)$ and $f_n(x) := \sin(nx)$ for all $n \geq 1$. This sequence does not converge in $L^1(D)$, but $\langle T_{f_n}, \varphi \rangle = \int_0^1 \sin(nx)\varphi \, \mathrm{d}x = \int_0^1 \frac{1}{n}\cos(nx)\varphi' \, \mathrm{d}x$, so that $\lim_{n\to\infty}\langle T_{f_n}, \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(D)$, i.e., $T_{f_n} \to 0$ in the sense of distributions. Up to an abuse of notation we say that f_n converges to 0 in the sense of distributions. Likewise one can show that $\cos(nx) \to 0$ in the sense of distributions. Let us now consider $g_n(x) := \sin^2(nx)$ for all $n \geq 1$. Using the identity $\sin^2(nx) = \frac{1}{2} - \frac{1}{2}\cos(2nx)$ and the above results, we conclude that $g_n \to \frac{1}{2}$ in the sense of distributions. □

4.2 Negative-order Sobolev spaces

Equipped with the notion of distributions we can now define Sobolev spaces of negative order by duality using $W_0^{s,p}(D)$.

Definition 4.10 ($W^{-s,p}(D)$). *Let $s > 0$ and $p \in (1, \infty)$. Let D be an open set in $\mathbb{R}^d$. We define the space $W^{-s,p}(D) := \big(W_0^{s,p'}(D)\big)'$ with $\frac{1}{p} + \frac{1}{p'} = 1$ (for $p = 2$, we write $H^{-s}(D) := W^{-s,2}(D)$), equipped with the norm*

$$\|T\|_{W^{-s,p}(D)} := \sup_{w\in W_0^{s,p'}(D)} \frac{|\langle T, w \rangle|}{\|w\|_{W^{s,p'}(D)}}. \tag{4.3}$$

Identifying $L^p(D)$ with the dual space of $L^{p'}(D)$ (see Theorem 1.41), we infer that $L^p(D) \hookrightarrow W^{-s,p}(D)$ (and both spaces coincide for $s = 0$ since $W_0^{0,p'}(D) = L^{p'}(D)$ by Theorem 1.38). Moreover, any element $T \in W^{-s,p}(D)$ is a distribution since, assuming $s = m \in \mathbb{N}$, we have

$$|\langle T,\varphi\rangle| \le \|T\|_{W^{-m,p}(D)}|D|^{\frac{1}{p'}}\binom{m+d}{d}^{\frac{1}{p'}}\max_{|\alpha|\le m}\big(\ell_D^{|\alpha|}\|\partial^\alpha\varphi\|_{L^\infty(K)}\big), \qquad (4.4)$$

for all compact subset $K \subsetneq D$ and all $\varphi \in C_0^\infty(D)$ with $\text{supp}(\varphi) \subset K$. The argument can be adapted to the case where $s = m+\sigma$, $\sigma \in (0,1)$.

Example 4.11 (Dirac measure). Some of the objects in $W^{-s,p}(D)$ are not functions but distributions. For instance, the Dirac mass at a point $\boldsymbol{a} \in D$ is in $W^{-s,p}(D)$ if $sp' > d$. □

Theorem 4.12 ($W^{-1,p}(D)$). *Let $p \in (1,\infty)$. Let D be an open, bounded set in $\mathbb{R}^d$. For all $f \in W^{-1,p}(D)$, there are functions $\{g_i\}_{i\in\{0:d\}}$, all in $L^{p'}(D)$, s.t. $\|f\|_{W^{-1,p}(D)} = \max_{i\in\{0:d\}}\|g_i\|_{L^{p'}(D)}$ and*

$$\langle f,v\rangle = \int_D g_0 v\,\mathrm{d}x + \sum_{i\in\{1:d\}}\int_D g_i\partial_i v\,\mathrm{d}x, \qquad \forall v \in W_0^{1,p}(D). \qquad (4.5)$$

More generally, for all $m \in \mathbb{N}$, one has $v \in W^{-m,p}(D)$ if and only if $v = \sum_{|\alpha|\le m}\partial^\alpha g_\alpha$ where $g_\alpha \in L^{p'}(D)$.

Proof. See Brezis [48, Prop. 9.20] for the case $m = 1$ and Adams and Fournier [3, Thm. 3.9]. □

Example 4.13 (Gradient). Let $s \in (0,1)$, $p \in (1,\infty)$, and $sp \neq 1$. If D is a Lipschitz domain in $\mathbb{R}^d$, then the linear operator ∇ maps $W^{s,p}(D)$ boundedly to $\boldsymbol{W}^{s-1,p}(D)$, i.e., we have $\nabla \in \mathcal{L}(W^{s,p}(D);\boldsymbol{W}^{s-1,p}(D))$; see Grisvard [110, Thm. 1.4.4.6]. □

Remark 4.14 (Interpolation). Assuming that D is a Lipschitz domain, an alternative definition of negative-order spaces relies on the interpolation theory between Banach spaces (see §A.5). Let $p \in (1,\infty)$ and $s \in (0,1)$. Recalling the space $W^{-1,p}(D)$ from Definition 4.10, let us set

$$\check{W}^{-s,p}(D) := [W^{-1,p}(D), L^p(D)]_{1-s,p}.$$

Theorem A.30 and the definition (3.6) of $W_{00}^{s,p'}(D)$ imply that

$$\check{W}^{-s,p}(D) = [L^{p'}(D), W_0^{1,p'}(D)]'_{s,p'} = \big(W_{00}^{s,p'}(D)\big)'.$$

The arguments from Remark 3.23 imply that $\check{H}^{-s}(D) = H^{-s}(D)$ if $s \neq \frac{1}{2}$ since $H_{00}^s(D) = H_0^s(D)$ in this case (see (3.7)). (One can also infer that $\check{W}^{-s,p}(D) = W^{-s,p}(D)$ for $sp \neq 1$, if $\widetilde{W}^{s,p}(D) = W_{00}^{s,p}(D)$, as conjectured in Remark 3.23.) □

4.3 Normal and tangential traces

The goal of this section is to give a meaning to the normal or tangential component of $\mathbb{R}^d$-valued fields for which we only have integrability properties on the divergence or the curl, respectively, but not on the whole gradient. The underlying idea is quite general and consists of defining the traces in a Sobolev space of negative order at the boundary by extending a suitable integration by parts formula valid for smooth functions. Recall that for any field $\boldsymbol{v} = (v_i)_{i\in\{1:d\}} \in \boldsymbol{L}^1_{\text{loc}}(D) := L^1_{\text{loc}}(D;\mathbb{R}^d)$, the divergence is defined by

$$\nabla\cdot\boldsymbol{v} := \sum_{i\in\{1:d\}} \partial_i v_i, \tag{4.6}$$

and for $d = 3$, the curl $\nabla\times\boldsymbol{v}$ is the column vector in $\mathbb{R}^3$ with components $(\nabla\times\boldsymbol{v})_i := \sum_{j,k\in\{1:3\}} \varepsilon_{ijk}\partial_j v_k$ for all $i \in \{1:3\}$, where ε_{ijk} denotes the Levi-Civita symbol ($\varepsilon_{ijk} := 0$ if at least two indices take the same value, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} := 1$ (i.e., for even permutations), and $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} := -1$ (i.e., for odd permutations)). In component form, we have

$$\nabla\times\boldsymbol{v} := (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)^{\mathsf{T}}. \tag{4.7}$$

Recall also that the following integration by parts formulas hold true for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{C}^1(\overline{D})$ and all $q \in C^1(\overline{D})$:

$$\int_{\partial D} (\boldsymbol{v}\times\boldsymbol{n})\cdot\boldsymbol{w}\,\mathrm{d}s = \int_D \boldsymbol{v}\cdot\nabla\times\boldsymbol{w}\,\mathrm{d}x - \int_D (\nabla\times\boldsymbol{v})\cdot\boldsymbol{w}\,\mathrm{d}x, \tag{4.8a}$$

$$\int_{\partial D} (\boldsymbol{v}\cdot\boldsymbol{n})q\,\mathrm{d}s = \int_D \boldsymbol{v}\cdot\nabla q\,\mathrm{d}x + \int_D (\nabla\cdot\boldsymbol{v})q\,\mathrm{d}x. \tag{4.8b}$$

Let $p \in (1,\infty)$ and let us consider the following Banach spaces:

$$\boldsymbol{Z}^{\mathrm{c},p}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^p(D)\}, \tag{4.9a}$$

$$\boldsymbol{Z}^{\mathrm{d},p}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \nabla\cdot\boldsymbol{v} \in L^p(D)\}. \tag{4.9b}$$

For $p = 2$, we write

$$\boldsymbol{H}(\mathrm{curl}; D) := \boldsymbol{Z}^{\mathrm{c},2}(D), \qquad \boldsymbol{H}(\mathrm{div}; D) := \boldsymbol{Z}^{\mathrm{d},2}(D). \tag{4.10}$$

Let $\langle\cdot,\cdot\rangle_{\partial D}$ denote the duality pairing between $\boldsymbol{W}^{-\frac{1}{p},p}(\partial D)$ and $\boldsymbol{W}^{\frac{1}{p},p'}(\partial D)$. The trace operator $\gamma^{\mathrm{g}} : W^{1,p'}(D) \longrightarrow W^{\frac{1}{p},p'}(\partial D)$ being surjective (see Theorem 3.10), we infer that there is $c_{\gamma^{\mathrm{c}}}$ such that for all $\boldsymbol{l} \in \boldsymbol{W}^{\frac{1}{p},p'}(\partial D)$, there is $\boldsymbol{w}(\boldsymbol{l}) \in \boldsymbol{W}^{1,p'}(D)$ s.t. $\gamma^{\mathrm{g}}(\boldsymbol{w}(\boldsymbol{l})) = \boldsymbol{l}$ and $\|\boldsymbol{w}(\boldsymbol{l})\|_{\boldsymbol{W}^{1,p'}(D)} \le c_{\gamma^{\mathrm{c}}}\|\boldsymbol{l}\|_{\boldsymbol{W}^{\frac{1}{p},p'}(\partial D)}$. We then define the linear map $\gamma^{\mathrm{c}} : \boldsymbol{Z}^{\mathrm{c},p}(D) \to \boldsymbol{W}^{-\frac{1}{p},p}(\partial D)$ by

$$\langle \gamma^{\mathrm{c}}(\boldsymbol{v}), \boldsymbol{l}\rangle_{\partial D} := \int_D \boldsymbol{v}\cdot\nabla\times\boldsymbol{w}(\boldsymbol{l})\,\mathrm{d}x - \int_D (\nabla\times\boldsymbol{v})\cdot\boldsymbol{w}(\boldsymbol{l})\,\mathrm{d}x, \tag{4.11}$$

for all $\boldsymbol{v} \in \boldsymbol{Z}^{\mathrm{c},p}(D)$ and all $\boldsymbol{l} \in \boldsymbol{W}^{\frac{1}{p},p'}(\partial D)$. Note that (4.8a) shows that $\gamma^{\mathrm{c}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\times\boldsymbol{n}$ when $\boldsymbol{v}$ is smooth. A direct verification invoking Hölder's inequality shows that the map γ^{c} is bounded. Moreover, the definition (4.11) is independent of the choice of $\boldsymbol{w}(\boldsymbol{l})$; see Exercise 4.5.

We also define the linear map $\gamma^{\mathrm{d}} : \boldsymbol{Z}^{\mathrm{d},p}(D) \to W^{-\frac{1}{p},p}(\partial D)$ by

$$\langle \gamma^{\mathrm{d}}(\boldsymbol{v}), l\rangle_{\partial D} := \int_D \boldsymbol{v}\cdot\nabla q(l)\,\mathrm{d}x + \int_D (\nabla\cdot\boldsymbol{v})q(l)\,\mathrm{d}x, \tag{4.12}$$

for all $\boldsymbol{v} \in \boldsymbol{Z}^{\mathrm{d},p}(D)$ and all $l \in W^{\frac{1}{p},p'}(\partial D)$, where $q(l) \in W^{1,p'}(D)$ is such that $\gamma^{\mathrm{g}}(q(l)) = l$, and $\langle\cdot,\cdot\rangle_{\partial D}$ now denotes the duality pairing between $W^{-\frac{1}{p},p}(\partial D)$ and $W^{\frac{1}{p},p'}(\partial D)$. Reasoning as above, one can verify that: $\gamma^{\mathrm{d}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{n}$ when $\boldsymbol{v}$ is smooth; the map γ^{d} is bounded; the definition (4.12) is independent of the choice of $q(l)$.

Theorem 4.15 (Normal/tangential component). *Let $p \in (1,\infty)$. Let D be a Lipschitz domain in $\mathbb{R}^d$. Let $\gamma^{\mathrm{c}} : \boldsymbol{Z}^{\mathrm{c},p}(D) \to \boldsymbol{W}^{-\frac{1}{p},p}(\partial D)$ and $\gamma^{\mathrm{d}} : \boldsymbol{Z}^{\mathrm{d},p}(D) \to W^{-\frac{1}{p},p}(\partial D)$ be defined in* (4.11) *and* (4.12)*, respectively. The following holds true:*

(i) $\gamma^{\mathrm{c}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\times\boldsymbol{n}$ *and* $\gamma^{\mathrm{d}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{n}$ *whenever* $\boldsymbol{v}$ *is smooth.*
(ii) γ^{d} *is surjective.*
(iii) *Density: setting* $\boldsymbol{Z}_0^{\mathrm{c},p}(D) := \overline{\boldsymbol{C}_0^\infty(D)}^{\boldsymbol{Z}^{\mathrm{c},p}(D)}$, $\boldsymbol{Z}_0^{\mathrm{d},p}(D) := \overline{\boldsymbol{C}_0^\infty(D)}^{\boldsymbol{Z}^{\mathrm{d},p}(D)}$, *we have*

$$\boldsymbol{Z}_0^{\mathrm{c},p}(D) = \ker(\gamma^{\mathrm{c}}), \qquad \boldsymbol{Z}_0^{\mathrm{d},p}(D) = \ker(\gamma^{\mathrm{d}}). \tag{4.13}$$

Proof. Item (i) is a simple consequence of the definition of γ^{c} and γ^{d}. See Tartar [189, Lem. 20.2] for item (ii) when $p = 2$. See [96, Thm. 4.7] for item (iii) (see also Exercise 23.9). □

Example 4.16 (Normal derivative). In the context of elliptic PDEs, one often deals with functions $v \in H^1(D)$ such that $\nabla\cdot(\nabla v) \in L^2(D)$. For these functions we have $\nabla v \in \boldsymbol{H}(\mathrm{div}; D)$. Owing to Theorem 4.15 with $p = 2$, one can then give a meaning to the normal derivative of v at the boundary as $\gamma^{\mathrm{d}}(\nabla v) \in H^{-\frac{1}{2}}(\partial D)$. Assuming more smoothness on v, e.g., $v \in H^s(D)$, $s > \frac{3}{2}$, and some smoothness of ∂D, one can instead invoke Theorem 3.16 to infer that $\gamma^{\partial_n}(v) \in H^{s-\frac{3}{2}}(\partial D) \hookrightarrow L^2(\partial D)$, i.e., the normal derivative is integrable. However, this smoothness assumption is often too strong for elliptic PDEs, and one has to use $\gamma^{\mathrm{d}}(\nabla v)$ to define the normal derivative. □

Example 4.17 (Whitney's paradox). Let us show by a counterexample (see [199, p. 100]) that the normal component of a vector field with integrable divergence over D may not be integrable over ∂D. The two-dimensional field

$\boldsymbol{v}(x_1, x_2) := (\frac{-x_2}{x_1^2+x_2^2}, \frac{x_1}{x_1^2+x_2^2})^\mathsf{T}$ in $D := (0,1)^2$ satisfies $\|\boldsymbol{v}(\boldsymbol{x})\|_{\ell^2} = \|\boldsymbol{x}\|_{\ell^2}^{-1}$, $\boldsymbol{v} \in \boldsymbol{L}^p(D)$ for all $p \in [1, 2)$, and $\nabla\cdot\boldsymbol{v} = 0$. However, $\boldsymbol{v}\cdot\boldsymbol{n}$ is not integrable, i.e., $\boldsymbol{v}\cdot\boldsymbol{n} \notin L^1(\partial D)$. □

Remark 4.18 (2D). In dimension two ($d = 2$), the tangential component is defined using the linear map $\gamma^\mathrm{c} : \boldsymbol{Z}^{\mathrm{c},p}(D) \to W^{-\frac{1}{p},p}(\partial D)$ as follows:

$$\langle \gamma^\mathrm{c}(\boldsymbol{v}), l\rangle_{\partial D} := \int_D \boldsymbol{v}\cdot\nabla^\perp w(l)\,\mathrm{d}x + \int_D (\nabla\times\boldsymbol{v}) w(l)\,\mathrm{d}x,$$

for all $\boldsymbol{v} \in \boldsymbol{Z}^{\mathrm{c},p}(D)$ and all $l \in W^{\frac{1}{p},p'}(\partial D)$, where $w(l) \in W^{1,p'}(D)$ is such that $\gamma^\mathrm{g}(w(l)) = l$. Here, $\nabla^\perp v := (-\partial_2 v, \partial_1 v)^\mathsf{T}$ and $\nabla\times\boldsymbol{v} := \partial_1 v_2 - \partial_2 v_1$. Note that $\nabla^\perp v = \boldsymbol{R}_{\frac{\pi}{2}}(\nabla v)$ and $\nabla\times\boldsymbol{v} = -\nabla\cdot(\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{v}))$, where $\boldsymbol{R}_{\frac{\pi}{2}}$ is the rotation of angle $\frac{\pi}{2}$ in $\mathbb{R}^2$ (i.e., the matrix of $\boldsymbol{R}_{\frac{\pi}{2}}$ relative to the canonical basis of $\mathbb{R}^2$ is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$). Whenever $\boldsymbol{v}$ is smooth, we have $\gamma^\mathrm{c}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{t}$ where $\boldsymbol{t} := \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{n})$ is a unit tangent vector to ∂D. □

Exercises

Exercise 4.1 (Distributions). Let D be an open set in $\mathbb{R}^d$. Let v be a distribution in D. (i) Let $\psi \in C^\infty(D)$. Show that the map $C_0^\infty(D) \ni \varphi \mapsto \langle v, \psi\varphi\rangle$ defines a distribution in D (this distribution is usually denoted by ψv). (ii) Let $\alpha, \beta \in \mathbb{N}^d$. Prove that $\partial^\alpha(\partial^\beta v) = \partial^\beta(\partial^\alpha v)$ in the distribution sense.

Exercise 4.2 (Dirac measure on a manifold). Let D be a smooth bounded and open set in $\mathbb{R}^d$. Let $u \in C^2(D;\mathbb{R})$ and assume that $u_{|\partial D} = 0$. Let $\tilde{u}$ be the extension by zero of u over $\mathbb{R}^d$. Compute $\nabla\cdot(\nabla\tilde{u}) = \partial_{11}u + \ldots + \partial_{dd}u$ in the distribution sense.

Exercise 4.3 (P.V. $\frac{1}{x}$). Let $D := (-1, 1)$. Prove that the linear map $T : C_0^\infty(D) \to \mathbb{R}$ defined by $\langle T, \varphi\rangle := \lim_{\epsilon\to 0} \int_{|x|>|\epsilon|} \frac{1}{x}\varphi(x)\,\mathrm{d}x$ is a distribution.

Exercise 4.4 (Integration by parts). Prove the two identities in (4.8) by using the divergence formula $\int_D \nabla\cdot\boldsymbol{\phi}\,\mathrm{d}x = \int_{\partial D}(\boldsymbol{\phi}\cdot\boldsymbol{n})\,\mathrm{d}s$ for all $\boldsymbol{\phi} \in \boldsymbol{C}^1(\overline{D})$.

Exercise 4.5 (Definition (4.11)). Verify that the right-hand side of (4.11) is independent of the choice of $\boldsymbol{w}(\boldsymbol{l})$. (*Hint*: consider two functions $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{W}^{1,p'}(D)$ s.t. $\gamma^\mathrm{g}(\boldsymbol{w}_1) = \gamma^\mathrm{g}(\boldsymbol{w}_2) = \boldsymbol{l}$ and use the density of $\boldsymbol{C}_0^\infty(D)$ in $\boldsymbol{W}_0^{1,p'}(D)$.)