Part III, Chapter 9

Finite element generation

In the previous chapter, we have seen how to generate a mesh from a reference cell and a collection of geometric mappings. We now show how to generate a finite element in each mesh cell from a reference finite element. To this purpose, we need one new concept in addition to the geometric mapping: a functional transformation that maps functions defined on the current mesh cell to functions defined on the reference cell. Key examples of such transformations are the Piola transformations. These transformations arise naturally in the chain rule when one investigates how the standard differential operators (gradient, curl, divergence) are transformed by the geometric mapping. The construction presented in this chapter provides the cornerstone for the analysis of the finite element interpolation error to be performed in Chapter 11. Recall that $\|\cdot\|_{\ell^2}$ is the Euclidean norm in $\mathbb{R}^d$ and $\boldsymbol{a}\cdot\boldsymbol{b}$ denotes the corresponding inner product.

9.1 Main ideas

Let $\mathcal{T}_h$ be a mesh generated as described in Chapter 8. This means that we have at hand a reference cell $\widehat{K}$ (recall that $\widehat{K}$ is a polyhedron) and a geometric mapping $\boldsymbol{T}_K : \widehat{K} \to K$ for every mesh cell $K \in \mathcal{T}_h$. Given an integer $q \geq 1$, our goal is now to define a finite element in K composed of $\mathbb{R}^q$-valued functions defined on K. To this purpose, we assume that we have at hand a fixed finite element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$, where $\widehat{P}$ is composed of $\mathbb{R}^q$-valued functions defined on $\widehat{K}$, and $\widehat{\Sigma}$ is the collection of the degrees of freedom (dofs) for these functions.

The triple $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ should not be confused with the geometric finite element $(\widehat{K}, \widehat{P}_{\text{geo}}, \widehat{\Sigma}_{\text{geo}})$ whose only use is to define K, whereas $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ is used to interpolate $\mathbb{R}^q$-valued functions. The interpolation is said to be *isoparametric* whenever $[\widehat{P}_{\text{geo}}]^q = \widehat{P}$ and *subparametric* whenever $[\widehat{P}_{\text{geo}}]^q \subsetneq \widehat{P}$. The most common example of subparametric interpolation consists of using affine geometric mappings together with shape functions that are quadratic or of higher polynomial order.

Definition 9.1 (Reference element). $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ *is called* reference finite element*, and with obvious notation* $\{\widehat{\sigma}_i\}_{i\in\mathcal{N}}$ *and* $\{\widehat{\theta}_i\}_{i\in\mathcal{N}}$ *are called* reference dofs *and* reference shape functions*, respectively.*

Recalling Definition 5.7, we also assume that we have at hand a Banach space $V(\widehat{K}) \subset L^1(\widehat{K};\mathbb{R}^q)$ such that $\widehat{P} \subset V(\widehat{K})$ and such that the linear forms $\{\widehat{\sigma}_i\}_{i\in\mathcal{N}}$ can be extended to $\mathcal{L}(V(\widehat{K});\mathbb{R})$ (we use the same symbol $\widehat{\sigma}_i$ for simplicity). The interpolation operator $\mathcal{I}_{\widehat{K}} : V(\widehat{K}) \to \widehat{P}$ associated with $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ is defined as follows (see (5.7)):

$$\mathcal{I}_{\widehat{K}}(\widehat{v})(\widehat{\boldsymbol{x}}) := \sum_{i\in\mathcal{N}} \widehat{\sigma}_i(\widehat{v})\widehat{\theta}_i(\widehat{\boldsymbol{x}}), \qquad \forall \widehat{\boldsymbol{x}} \in \widehat{K}. \tag{9.1}$$

The operator $\mathcal{I}_{\widehat{K}}$ is called *reference interpolation operator*.

Since our goal is to generate a finite element on K and to build an interpolation operator $\mathcal{I}_K$ acting on functions defined on K, we introduce a counterpart of the space $V(\widehat{K})$ for those functions, say $V(K)$. The new ingredient we need for the construction is a transformation

$$\psi_K : V(K) \to V(\widehat{K}), \tag{9.2}$$

which we assume to be a bounded linear isomorphism. A simple definition of ψ_K is the pullback by the geometric mapping, i.e.,

$$\psi_K(v) := v \circ \boldsymbol{T}_K, \qquad \forall v \in V(K). \tag{9.3}$$

We will see that this definition is well-suited to nodal and modal finite elements. However we will also see that this definition is not adequate when considering vector-valued functions for which the tangential or the normal component at the boundary of K plays specific roles. This is the reason why we use a general notation for the functional transformation ψ_K.

Proposition 9.2 (Finite element generation). *Let* $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ *be the reference element with extended dofs* $\{\widehat{\sigma}_i\}_{i\in\mathcal{N}} \subset \mathcal{L}(V(\widehat{K});\mathbb{R})$*. Let* $K \in \mathcal{T}_h$ *be a mesh cell. Assume that we have at hand a Banach space* $V(K)$ *and a bounded linear isomorphism* $\psi_K \in \mathcal{L}(V(K); V(\widehat{K}))$*. Then the triple* (K, P_K, Σ_K) *s.t.*

$$P_K := \psi_K^{-1}(\widehat{P}) = \{p := \psi_K^{-1}(\widehat{p}) \mid \widehat{p} \in \widehat{P}\}, \tag{9.4a}$$

$$\Sigma_K := \widehat{\Sigma} \circ \psi_K = \{\sigma_{K,i} := \widehat{\sigma}_{i|\widehat{P}} \circ \psi_K\}_{i\in\mathcal{N}} \subset \mathcal{L}(P_K;\mathbb{R}), \tag{9.4b}$$

is a finite element. The dofs in Σ_K *can be extended to* $\mathcal{L}(V(K);\mathbb{R})$ *by setting* $\sigma_{K,i} := \widehat{\sigma}_i \circ \psi_K$ *for all* $i \in \mathcal{N}$*.*

Proof. We apply Remark 5.3 to prove that (K, P_K, Σ_K) is a finite element. Since ψ_K is bijective, we have $\dim(P) = \dim(\widehat{P}) = n_{\rm sh}$. Let $p \in P_K$ be s.t. $\sigma_{K,i}(p) = 0$ for all $i \in \mathcal{N}$. Then $\widehat{\sigma}_i(\psi_K(p)) = 0$ for all $i \in \mathcal{N}$, so that $\psi_K(p) = 0$ by the unisolvence property of $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$. This implies that $p = 0$ since ψ_K is an isomorphism. Finally, since $(\widehat{\sigma}_i \circ \psi_K)_{|P} = \widehat{\sigma}_{i|\widehat{P}} \circ \psi_K$, the linear map $\widehat{\sigma}_i \circ \psi_K : V(K) \to \mathbb{R}$ is an extension of $\sigma_{K,i} : P_K \to \mathbb{R}$ to $V(K)$ (we use the same notation for simplicity), and we have $\sigma_{K,i} \in \mathcal{L}(V(K); \mathbb{R})$ since $|\sigma_{K,i}(v)| \le \|\widehat{\sigma}_i\|_{\mathcal{L}(V(\widehat{K});\mathbb{R})} \|\psi_K\|_{\mathcal{L}(V(K);V(\widehat{K}))} \|v\|_{V(K)}$ for all $v \in V(K)$. □

The linear forms $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$ are called *local dofs*. The following functions, called *local shape functions*:

$$\theta_{K,i} := \psi_K^{-1}(\widehat{\theta}_i), \qquad \forall i \in \mathcal{N}, \tag{9.5}$$

satisfy $\sigma_{K,i}(\theta_{K,j}) = \widehat{\sigma}_i(\psi_K(\theta_{K,j})) = \widehat{\sigma}_i(\widehat{\theta}_j) = \delta_{ij}$ for all $i, j \in \mathcal{N}$. The *local interpolation operator* $\mathcal{I}_K : V(K) \to P_K$ acts as follows:

$$\mathcal{I}_K(v)(\boldsymbol{x}) := \sum_{i\in\mathcal{N}} \sigma_{K,i}(v)\theta_{K,i}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in K. \tag{9.6}$$

The following result plays a key role in the analysis of the interpolation error.

Proposition 9.3 (Commuting diagram). *We have $\mathcal{I}_K = \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} V(K) & \xrightarrow{\ \psi_K\ } & V(\widehat{K}) \\ \downarrow{\scriptstyle \mathcal{I}_K} & & \downarrow{\scriptstyle \mathcal{I}_{\widehat{K}}} \\ P_K & \xrightarrow{\ \psi_K\ } & \widehat{P} \end{array}$$

i.e., P_K is pointwise invariant under $\mathcal{I}_K$, that is, $\mathcal{I}_K(p) = p$ for all $p \in P_K$.

Proof. Let v in $V(K)$. The definition (9.4) of (K, P_K, Σ_K) implies that

$$\mathcal{I}_{\widehat{K}}(\psi_K(v)) = \sum_{i\in\mathcal{N}} \widehat{\sigma}_i(\psi_K(v))\, \widehat{\theta}_i = \sum_{i\in\mathcal{N}} \sigma_{K,i}(v)\, \psi_K(\theta_{K,i}) = \psi_K(\mathcal{I}_K(v)),$$

owing to the linearity of ψ_K. Hence, the above diagram commutes. Let now $p \in P_K$. We have $\mathcal{I}_K(p) = \psi_K^{-1}(\mathcal{I}_{\widehat{K}}(\psi_K(p))) = \psi_K^{-1}(\psi_K(p))$ since $\psi_K(p) \in \widehat{P}$ and $\widehat{P}$ is pointwise invariant under $\mathcal{I}_{\widehat{K}}$. Hence, $\mathcal{I}_K(p) = p$. □

Example 9.4 (Lagrange elements). Let $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ be a Lagrange finite element with nodes $\{\widehat{\boldsymbol{a}}_i\}_{i\in\mathcal{N}}$ and $V(\widehat{K}) := C^0(\widehat{K})$; see §5.4.1. Set $V(K) := C^0(K)$. The map $\psi_K : V(K) \to V(\widehat{K})$ defined in (9.3) is an isomorphism in $\mathcal{L}(V(K); V(\widehat{K}))$. The finite element (K, P_K, Σ_K) constructed in

Proposition 9.2 using ψ_K is also a Lagrange finite element. Indeed, we have $\sigma_{K,i}(p) := \widehat{\sigma}_i(\psi_K(p)) := \psi_K(p)(\widehat{\boldsymbol{a}}_i) = (p \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{a}}_i)$ for all $p \in P_K$. Setting

$$\boldsymbol{a}_{K,i} := \boldsymbol{T}_K(\widehat{\boldsymbol{a}}_i), \qquad \forall i \in \mathcal{N},$$

we infer that $\{\boldsymbol{a}_{K,i}\}_{i\in\mathcal{N}}$ are the Lagrange nodes of (K, P_K, Σ_K). The Lagrange interpolation operator $\mathcal{I}_K^{\mathrm{L}}$ acts as follows:

$$\mathcal{I}_K^{\mathrm{L}}(v)(\boldsymbol{x}) := \sum_{i\in\mathcal{N}} v(\boldsymbol{a}_{K,i})\theta_{K,i}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in K. \tag{9.7}$$

Note that even if $\widehat{P}$ is a polynomial space, $P_K := \{\widehat{p} \circ \boldsymbol{T}_K^{-1},\ \widehat{p} \in \widehat{P}\}$ is not necessarily a polynomial space unless $\boldsymbol{T}_K$ is affine. □

Example 9.5 (Modal elements). Let $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ be a modal finite element with dofs $\widehat{\sigma}_i(\widehat{p}) := \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{\zeta}_i \widehat{p}\,\mathrm{d}\widehat{x}$ for all $\widehat{p} \in \widehat{P}$ and all $i \in \mathcal{N}$, where $\{\widehat{\zeta}_i\}_{i\in\mathcal{N}}$ is a basis of $\widehat{P}$, and let $V(\widehat{K}) := L^1(\widehat{K})$; see §5.4.2. Set $V(K) := L^1(K)$. The map $\psi_K : V(K) \to V(\widehat{K})$ defined in (9.3) is an isomorphism in $\mathcal{L}(V(K); V(\widehat{K}))$. The finite element (K, P_K, Σ_K) constructed in Proposition 9.2 using ψ_K is also a modal finite element. Indeed, we have for all $p \in P_K$,

$$\begin{aligned}
\sigma_{K,i}(p) := \widehat{\sigma}_i(\psi_K(p)) &:= \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{\zeta}_i(\widehat{\boldsymbol{x}})(p\circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{x} \\
&= \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \frac{1}{\alpha_K}(\zeta_{K,i}\circ \boldsymbol{T}_K)(p\circ \boldsymbol{T}_K)\,\mathrm{d}\widehat{x} = \frac{1}{|K|}\int_K \zeta_{K,i}p\,\mathrm{d}x,
\end{aligned}$$

with $\zeta_{K,i} := \alpha_K \widehat{\zeta}_i \circ \boldsymbol{T}_K^{-1}$, $\alpha_K := |\det(\mathbb{J}_K)|^{-1}\frac{|K|}{|\widehat{K}|}$, and $\mathbb{J}_K$ is the Jacobian matrix of $\boldsymbol{T}_K$ defined in (8.3) ($\alpha_K = 1$ if $\boldsymbol{T}_K$ is affine). □

9.2 Differential calculus and geometry

In this section, we present basic identities from differential calculus and geometry showing how the usual differential operators (gradient, curl, and divergence) and normal and tangent vectors are transformed by the geometric mapping. We refer the reader to (4.6) for the definition of the divergence operator and to (4.7) for the definition of the curl operator with $d = 3$ (the material can be adapted to the case $d = 2$ by proceeding as in Remark 4.18).

9.2.1 Transformation of differential operators

Let $\widehat{K}$ be the reference polyhedron in $\mathbb{R}^d$ and let $K \in \mathcal{T}_h$ be a mesh cell. Let $\boldsymbol{T}_K : \widehat{K} \to K$ be the geometric mapping and let $\mathbb{J}_K$ be the Jacobian matrix of $\boldsymbol{T}_K$ (see (8.3)). Recall that we use boldface notation for $\mathbb{R}^d$-valued functions and for functional spaces composed of $\mathbb{R}^d$-valued functions. For instance, we write $\boldsymbol{C}^l(K) := C^l(K;\mathbb{R}^d)$ for all $l \in \mathbb{N}$. The following result is of fundamental importance.

Lemma 9.6 (Differential operators). *Let $v \in C^1(K)$ and $\boldsymbol{v} \in \boldsymbol{C}^1(K)$. The following holds true for all $\widehat{\boldsymbol{x}} \in \widehat{K}$:*

$$\nabla(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}}) = \mathbb{J}_K(\widehat{\boldsymbol{x}})^{\mathsf{T}}(\nabla v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})), \tag{9.8a}$$

$$\nabla{\times}(\mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v} \circ \boldsymbol{T}_K))(\widehat{\boldsymbol{x}}) = \det(\mathbb{J}_K(\widehat{\boldsymbol{x}}))\mathbb{J}_K^{-1}(\widehat{\boldsymbol{x}})(\nabla{\times}\boldsymbol{v})(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})), \tag{9.8b}$$

$$\nabla{\cdot}(\det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\boldsymbol{v} \circ \boldsymbol{T}_K))(\widehat{\boldsymbol{x}}) = \det(\mathbb{J}_K(\widehat{\boldsymbol{x}}))(\nabla{\cdot}\boldsymbol{v})(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})). \tag{9.8c}$$

Proof. (1) Proof of (9.8a). Since the link between the Jacobian matrix of $\boldsymbol{T}_K$ and its Fréchet derivative (see Definition B.1) is that $D\boldsymbol{T}_K(\widehat{\boldsymbol{x}})(\boldsymbol{h}) = \mathbb{J}_K(\widehat{\boldsymbol{x}})\boldsymbol{h}$ for all $\boldsymbol{h} \in \mathbb{R}^d$, we can use Lemma B.4 (chain rule) with $n := 1$ to infer that

$$D(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})(\boldsymbol{h}) = Dv(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))(D\boldsymbol{T}_K(\boldsymbol{x})(\boldsymbol{h})) = Dv(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))(\mathbb{J}_K(\widehat{\boldsymbol{x}})\boldsymbol{h}).$$

Using the gradient to represent the Fréchet derivative yields (9.8a) since

$$\begin{aligned}\nabla(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}}){\cdot}\boldsymbol{h} &= D(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})(\boldsymbol{h}) = Dv(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))(\mathbb{J}_K(\widehat{\boldsymbol{x}})\boldsymbol{h}) \\ &= (\nabla v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}})){\cdot}(\mathbb{J}_K(\widehat{\boldsymbol{x}})\boldsymbol{h}) = (\mathbb{J}_K(\widehat{\boldsymbol{x}})^{\mathsf{T}}(\nabla v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))){\cdot}\boldsymbol{h}.\end{aligned}$$

(2) Proof of (9.8c). This identity is deduced from (9.8a) by integrating by parts. Since $\boldsymbol{T}_K$ is bijective, the ratio $\epsilon_K := \frac{\det(\mathbb{J}_K)}{|\det(\mathbb{J}_K)|}$ is constant over $\widehat{K}$ and is either equal to -1 or 1. Moreover, the volume measure in K at $\boldsymbol{x}$ and in $\widehat{K}$ at $\widehat{\boldsymbol{x}}$ are s.t. $\mathrm{d}x = |\det(\mathbb{J}_K(\widehat{\boldsymbol{x}}))|\,\mathrm{d}\widehat{x}$. Let $q \in C_0^\infty(K)$ be a smooth scalar-valued function compactly supported in K. Integrating by parts and using (9.8a), we infer that

$$\begin{aligned}&\int_{\widehat{K}} (\nabla{\cdot}\boldsymbol{v})(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))q(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\det(\mathbb{J}_K(\widehat{\boldsymbol{x}}))\,\mathrm{d}\widehat{x} = \epsilon_K \int_K (\nabla{\cdot}\boldsymbol{v})(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}x \\ &\quad = -\epsilon_K \int_K (\boldsymbol{v}{\cdot}\nabla q)(\boldsymbol{x})\,\mathrm{d}x = -\epsilon_K \int_{\widehat{K}} (\boldsymbol{v}{\cdot}\nabla q)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\,|\det(\mathbb{J}_K(\widehat{\boldsymbol{x}}))|\,\mathrm{d}\widehat{x} \\ &\quad = -\int_{\widehat{K}} ((\boldsymbol{v} \circ \boldsymbol{T}_K){\cdot}(\mathbb{J}_K^{-\mathsf{T}}\nabla(q \circ \boldsymbol{T}_K)))(\widehat{\boldsymbol{x}})\det(\mathbb{J}_K(\widehat{\boldsymbol{x}}))\,\mathrm{d}\widehat{x} \\ &\quad = -\int_{\widehat{K}} ((\det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\boldsymbol{v} \circ \boldsymbol{T}_K)){\cdot}\nabla(q \circ \boldsymbol{T}_K))(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{x} \\ &\quad = \int_{\widehat{K}} \nabla{\cdot}(\det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\boldsymbol{v} \circ \boldsymbol{T}_K))(\widehat{\boldsymbol{x}})q(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\,\mathrm{d}\widehat{x},\end{aligned}$$

which proves (9.8c) since q is arbitrary.
(3) Proof of (9.8b) in $\mathbb{R}^3$. Let ε be the Levi-Civita symbol ($\varepsilon_{ijk} := 0$ if at least two indices take the same value, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} := 1$, and $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} := -1$). Recall that $\det(\mathbb{J}_K) = \varepsilon_{ijk}(\mathbb{J}_K)_{1i}(\mathbb{J}_K)_{2j}(\mathbb{J}_K)_{3k} = \varepsilon_{ijk}(\mathbb{J}_K)_{i1}(\mathbb{J}_K)_{j2}(\mathbb{J}_K)_{k3}$ and $(\nabla{\times}\boldsymbol{v})_i = \varepsilon_{ijk}\partial_j v_k$, with the Einstein convention

on the summation of repeated indices. For all $i \in \{1{:}d\}$, we have

$$\begin{aligned}(\mathbb{J}_K\nabla\times(\mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v}\circ\boldsymbol{T}_K)))_i &= (\mathbb{J}_K)_{ij}\varepsilon_{jkl}\partial_k(\mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v}\circ\boldsymbol{T}_K))_l \\ &= (\mathbb{J}_K)_{ij}\varepsilon_{jkl}\partial_k((\mathbb{J}_K^{\mathsf{T}})_{lm}(v_m\circ\boldsymbol{T}_K)) \\ &= (\mathbb{J}_K)_{ij}\varepsilon_{jkl}\left(\partial_k(\mathbb{J}_K)_{ml}(v_m\circ\boldsymbol{T}_K)+(\mathbb{J}_K)_{ml}\partial_k(v_m\circ\boldsymbol{T}_K)\right).\end{aligned}$$

Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the two terms on the right-hand side of the above equality. Since $\partial_k(\mathbb{J}_K)_{ml} = \partial_{kl}(\boldsymbol{T}_K)_m = \partial_{lk}(\boldsymbol{T}_K)_m = \partial_l(\mathbb{J}_K)_{mk}$, we infer that $\mathfrak{T}_1 = (\mathbb{J}_K)_{ij}\frac{1}{2}(\varepsilon_{jkl}+\varepsilon_{jlk})\partial_k(\mathbb{J}_K)_{ml}(v_m\circ\boldsymbol{T}_K) = 0$. Moreover, since $\varepsilon_{jkl}(\mathbb{J}_K)_{ij}(\mathbb{J}_K)_{nk}(\mathbb{J}_K)_{ml} = \varepsilon_{inm}\det(\mathbb{J}_K)$, we infer that

$$\begin{aligned}\mathfrak{T}_2 &= (\mathbb{J}_K)_{ij}\varepsilon_{jkl}(\mathbb{J}_K)_{ml}((\partial_n v_m)\circ\boldsymbol{T}_K)(\mathbb{J}_K)_{nk} \\ &= \varepsilon_{jkl}(\mathbb{J}_K)_{ij}(\mathbb{J}_K)_{nk}(\mathbb{J}_K)_{ml}((\partial_n v_m)\circ\boldsymbol{T}_K) \\ &= \varepsilon_{inm}\det(\mathbb{J}_K)((\partial_n v_m)\circ\boldsymbol{T}_K) = \det(\mathbb{J}_K)((\nabla\times\boldsymbol{v})\circ\boldsymbol{T}_K)_i.\end{aligned}$$ □

Remark 9.7 (Literature). See Marsden and Hughes [139, pp. 116-119], Ciarlet [75, p. 39], Monk [145, §3.9], Rognes et al. [168, p. 4134]. □

Definition 9.8 (Piola transformations). *Let $v \in C^0(K)$ and $\boldsymbol{v} \in \boldsymbol{C}^0(K)$. The* Piola transformations *are defined as follows:*

$$\psi_K^{\mathrm{g}}(v) := v\circ\boldsymbol{T}_K, \tag{9.9a}$$

$$\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) := \mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v}\circ\boldsymbol{T}_K), \tag{9.9b}$$

$$\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) := \det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\boldsymbol{v}\circ\boldsymbol{T}_K), \tag{9.9c}$$

$$\psi_K^{\mathrm{b}}(v) := \det(\mathbb{J}_K)(v\circ\boldsymbol{T}_K). \tag{9.9d}$$

ψ_K^{g} *is called* pullback by the geometric mapping, $\boldsymbol{\psi}_K^{\mathrm{c}}$ *is called* covariant Piola transformation*, and* $\boldsymbol{\psi}_K^{\mathrm{d}}$ *is called* contravariant Piola transformation.

Corollary 9.9 (Commuting properties). *The Piola transformations are such that for all $v \in C^1(K)$ and all $\boldsymbol{v} \in \boldsymbol{C}^1(K)$,*

$$\nabla(\psi_K^{\mathrm{g}}(v)) = \boldsymbol{\psi}_K^{\mathrm{c}}(\nabla v), \quad \nabla\times(\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})) = \boldsymbol{\psi}_K^{\mathrm{d}}(\nabla\times\boldsymbol{v}), \quad \nabla\cdot(\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})) = \psi_K^{\mathrm{b}}(\nabla\cdot\boldsymbol{v}).$$

Proof. Apply Lemma 9.6. □

The superscript g (resp., c, d) refers to the fact that the map ψ_K^{g} (resp., $\boldsymbol{\psi}_K^{\mathrm{c}}$, $\boldsymbol{\psi}_K^{\mathrm{d}}$) is used when integrability properties on the gradient (resp., curl, divergence) are required. The superscript b for "broken" means that no integrability with respect to any differential operator is invoked.

9.2.2 Normal and tangent vectors

Another important property of the Piola transformations is that $\boldsymbol{\psi}_K^{\mathrm{c}}$ (resp., $\boldsymbol{\psi}_K^{\mathrm{d}}$) preserves the moments of the tangential (resp., normal) components of fields at the edges (resp., the faces) of the mesh cell K. Let us first motivate this claim by a simple example.

Example 9.10 (Piola transformation vs. pullback). Referring to Figure 9.1, let $\widehat{K}$ be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Let K be the image of $\widehat{K}$ by the geometric mapping $\boldsymbol{T}_K$ defined as the rotation of center $(0,0)$ and of angle $\frac{\pi}{2}$. Let $\widehat{F}_1$ (resp., $\widehat{F}_2$) be the edge of $\widehat{K}$ corresponding to $x_2 = 0$ (resp., $x_1 = 0$), and let F_1 and F_2 be the images of $\widehat{F}_1$ and $\widehat{F}_2$ by $\boldsymbol{T}_K$, respectively. Consider the constant field $\boldsymbol{v}(\boldsymbol{x}) := (1,0)^{\mathsf{T}}$. Note that $\psi_K^{\mathrm{g}}(\boldsymbol{v}) = \boldsymbol{v}$ since $\boldsymbol{v}$ is invariant under the pullback by $\boldsymbol{T}_K$ (applied componentwise). Hence, $\boldsymbol{v}$ is tangent to F_2, whereas $\psi_K^{\mathrm{g}}(\boldsymbol{v})$ is normal to $\widehat{F}_2$. Moreover, $\boldsymbol{v}$ is normal to F_1, whereas $\psi_K^{\mathrm{g}}(\boldsymbol{v})$ is tangent to $\widehat{F}_1$. But $\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) = (0,-1)^{\mathsf{T}}$ is tangent to $\widehat{F}_2$, and $\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) = (0,-1)^{\mathsf{T}}$ is normal to $\widehat{F}_1$. □

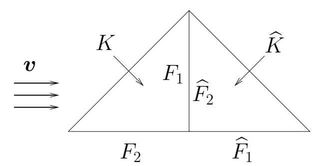

Fig. 9.1 Illustration of Example 9.10.

Our first result identifies how the geometric mapping $\boldsymbol{T}_K : \widehat{K} \to K$ transforms normal and tangent vectors on $\partial\widehat{K}$.

Lemma 9.11 (Normal and tangent). (i) *Let $\widehat{\boldsymbol{n}}_{\widehat{K}}$ be the outward unit normal to $\partial\widehat{K}$ and let $\boldsymbol{n}_K$ be the outward unit normal to ∂K. Let $\widehat{F}$ be a face of $\widehat{K}$ and let $F := \boldsymbol{T}_K(\widehat{F})$ be the corresponding face of K. Let $\widehat{\boldsymbol{x}} \in \mathrm{int}(\widehat{F})$ so that $\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}}(\widehat{\boldsymbol{x}})$ is well defined, and let $\boldsymbol{x} := \boldsymbol{T}_K(\widehat{\boldsymbol{x}}) \in \mathrm{int}(F)$. Then we have*

$$\boldsymbol{n}_{K|F}(\boldsymbol{x}) = \frac{1}{\|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}})(\widehat{\boldsymbol{x}})\|_{\ell^2}}(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}})(\widehat{\boldsymbol{x}}). \tag{9.10}$$

(ii) *Let $\widehat{E}$ be an edge of $\widehat{K}$ and let $E := \boldsymbol{T}_K(\widehat{E})$ be the corresponding edge of K. Let $\widehat{\boldsymbol{x}} \in \mathrm{int}(\widehat{E})$, let $\widehat{\boldsymbol{\tau}}_{\widehat{E}}$ be a unit tangent vector to $\widehat{E}$ at $\widehat{\boldsymbol{x}}$, and let $\boldsymbol{x} := \boldsymbol{T}_K(\widehat{\boldsymbol{x}}) \in E$. Then the vector*

$$\boldsymbol{\tau}_E(\boldsymbol{x}) := \frac{1}{\|(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})\|_{\ell^2}}(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}}) \tag{9.11}$$

is a unit tangent vector to E at $\boldsymbol{x}$.

Proof. (1) Let $\widehat{\psi}$ be the signed distance function to $\widehat{F}$, assumed to be negative inside $\widehat{K}$. Then $\nabla\widehat{\psi}(\widehat{\boldsymbol{x}}) = \widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}}(\widehat{\boldsymbol{x}})$. Defining $\psi := \widehat{\psi} \circ \boldsymbol{T}_K^{-1}$ and using (9.8a), we have $\nabla\psi(\boldsymbol{x}) = \mathbb{J}_K^{-\mathsf{T}}(\widehat{\boldsymbol{x}})\nabla\widehat{\psi}(\widehat{\boldsymbol{x}}) = \mathbb{J}_K^{-\mathsf{T}}(\widehat{\boldsymbol{x}})\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}}(\widehat{\boldsymbol{x}})$. Since ψ is constant

(equal to zero) over $\mathrm{int}(F)$ and takes negative values inside K, the vector $\nabla\psi(\boldsymbol{x})$ is normal to F and points toward the inside of K. This proves (9.10).
(2) Consider an edge $\widehat{E} := \widehat{F}_1 \cap \widehat{F}_2$ of $\widehat{K}$ and let $\widehat{\boldsymbol{x}} \in \widehat{E}$. Since

$$(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}_i})(\widehat{\boldsymbol{x}})\cdot(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}}(\widehat{\boldsymbol{x}})) = \widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}_i}(\widehat{\boldsymbol{x}})\cdot\widehat{\boldsymbol{\tau}}_{\widehat{E}}(\widehat{\boldsymbol{x}}) = 0,$$

we infer from Step (1) that $\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}}(\widehat{\boldsymbol{x}})$ is tangent to $F_i := \boldsymbol{T}_K(\widehat{F}_i)$ for all $i \in \{1,2\}$. Hence, $(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})$ is tangent to $F_i \cap F_j = E = T_K(\widehat{E})$. □

Our next step is to identify how surface and line measures are transformed by the geometric mapping $\boldsymbol{T}_K$. Observe that the unit of $\mathbb{J}_K$ is a length scale and that the unit of $\det(\mathbb{J}_K)$ is a volume. The identity (9.12a) is sometimes called *Nanson's formula* in the continuum mechanics literature; see [149, p. 184] and Truesdell and Toupin [192, p. 249, Eq. (20.8)].

Lemma 9.12 (Surface and line measures). *The surface measures on $\widehat{F}$ at $\widehat{\boldsymbol{x}}$ and on $F := \boldsymbol{T}_K(\widehat{F})$ at $\boldsymbol{x} := \boldsymbol{T}_K(\widehat{\boldsymbol{x}})$ are such that*

$$\mathrm{d}s = |\det(\mathbb{J}_K)(\widehat{\boldsymbol{x}})|\,\|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}})(\widehat{\boldsymbol{x}})\|_{\ell^2}\,\mathrm{d}\widehat{s}, \tag{9.12a}$$

$$\mathrm{d}\widehat{s} = |\det(\mathbb{J}_K^{-1})(\boldsymbol{x})|\,\|(\mathbb{J}_K^{\mathsf{T}}\boldsymbol{n}_{K|F})(\boldsymbol{x})\|_{\ell^2}\,\mathrm{d}s. \tag{9.12b}$$

The line measures on $\widehat{E}$ at $\widehat{\boldsymbol{x}}$ and on $E := \boldsymbol{T}_K(\widehat{E})$ at $\boldsymbol{x} := \boldsymbol{T}_K(\widehat{\boldsymbol{x}})$ are such that

$$\mathrm{d}l = \|(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\boldsymbol{E}})(\widehat{\boldsymbol{x}})\|_{\ell^2}\,\mathrm{d}\widehat{l}, \qquad \mathrm{d}\widehat{l} = \|(\mathbb{J}_K^{-1}\boldsymbol{\tau}_E)(\boldsymbol{x})\|_{\ell^2}\,\mathrm{d}l. \tag{9.13}$$

Proof. Let $q \in C_0^\infty(F)$ and let $\boldsymbol{v} \in \boldsymbol{C}^\infty(K)$ be s.t. $\boldsymbol{v}\cdot\boldsymbol{n}_{K|F} = q$ and $\boldsymbol{v}\cdot\boldsymbol{n}_{K|\partial K\setminus F} = 0$ (this construction is possible since q is compactly supported in F and so vanishes near ∂F where $\boldsymbol{n}_K$ is multivalued). Recall that $\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) := \det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\boldsymbol{v}\circ\boldsymbol{T}_K)$ and that $\epsilon_K := \frac{\det(\mathbb{J}_K)}{|\det(\mathbb{J}_K)|} = \pm 1$. Using (9.8c) and (9.10), we infer that

$$\begin{aligned}
\int_F q(\boldsymbol{x})\,\mathrm{d}s &= \int_{\partial K}(\boldsymbol{v}\cdot\boldsymbol{n}_K)(\boldsymbol{x})\,\mathrm{d}s = \int_K (\nabla\cdot\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}x \\
&= \epsilon_K \int_{\widehat{K}} \nabla\cdot\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{x} = \epsilon_K \int_{\partial\widehat{K}} (\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{n}}_{\widehat{K}})(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{s} \\
&= \int_{\partial\widehat{K}} (\mathbb{J}_K^{-1}\boldsymbol{v})\cdot(\mathbb{J}_K^{\mathsf{T}}\boldsymbol{n}_K)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}})(\widehat{\boldsymbol{x}})\|_{\ell^2}|\det(\mathbb{J}_K)(\widehat{\boldsymbol{x}})|\,\mathrm{d}\widehat{s} \\
&= \int_{\partial\widehat{K}} (\boldsymbol{v}\cdot\boldsymbol{n}_K)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}})(\widehat{\boldsymbol{x}})\|_{\ell^2}|\det(\mathbb{J}_K)(\widehat{\boldsymbol{x}})|\,\mathrm{d}\widehat{s} \\
&= \int_{\widehat{F}} q(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}})(\widehat{\boldsymbol{x}})\|_{\ell^2}|\det(\mathbb{J}_K)(\widehat{\boldsymbol{x}})|\,\mathrm{d}\widehat{s}.
\end{aligned}$$

This yields (9.12a). To prove (9.12b), we use the following identity:

$$\|(\mathbb{J}_K^{\mathsf{T}}\boldsymbol{n}_{K|F})(\boldsymbol{x})\|_{\ell^2} = \|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}})(\widehat{\boldsymbol{x}})\|_{\ell^2}^{-1},$$

which follows from (9.10) and the fact that $\boldsymbol{n}_K$ and $\widehat{\boldsymbol{n}}_{\widehat{K}}$ are unit vectors. We refer the reader to Exercise 9.2 for the transformation of line measures. □

We can now state the main result of this section showing that the Piola transformations $\boldsymbol{\psi}_K^{\mathrm{c}}$ and $\boldsymbol{\psi}_K^{\mathrm{d}}$ are tailored to preserve the moments of the tangential components of fields over edges and the moments of the normal components of fields over faces, respectively. Let $\widehat{F}$ be a face of $\widehat{K}$ and let $\widehat{E}$ be an edge of $\widehat{K}$. Let $F := \boldsymbol{T}_K(\widehat{F})$ and $E := \boldsymbol{T}_K(\widehat{E})$ be the corresponding face and edge of K. Let $\widehat{\boldsymbol{n}}_{\widehat{F}}$ be a unit vector normal to $\widehat{F}$ and let $\widehat{\boldsymbol{\tau}}_{\widehat{E}}$ be a unit vector tangent to $\widehat{E}$. Note that $\widehat{\boldsymbol{n}}_{\widehat{F}}$ can point either toward the inside of $\widehat{K}$ or the outside of $\widehat{K}$, i.e., we only have $\widehat{\boldsymbol{n}}_{\widehat{F}} = \pm\widehat{\boldsymbol{n}}_{\widehat{K}|\widehat{F}}$. Recall that $\epsilon_K := \frac{\det(\mathbb{J}_K)}{|\det(\mathbb{J}_K)|} = \pm 1$. Lemma 9.11 shows that the following unit vectors:

$$\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}})(\boldsymbol{x}) := \epsilon_K \frac{1}{\|(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}})\|_{\ell^2}}(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}}), \tag{9.14a}$$

$$\boldsymbol{\Phi}_K^{\mathrm{c}}(\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\boldsymbol{x}) := \frac{1}{\|(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})\|_{\ell^2}}(\mathbb{J}_K\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}}), \tag{9.14b}$$

are, respectively, normal to F and tangent to E at $\boldsymbol{x} := \boldsymbol{T}_K(\widehat{\boldsymbol{x}})$. The definitions in (9.14) are motivated by the following result.

Lemma 9.13 (Preservation of moments of normal and tangential components). *Let $\boldsymbol{v} \in \boldsymbol{C}^0(K)$ and $q \in C^0(K)$. The following holds true:*

$$\int_F (\boldsymbol{v}\cdot\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}}))(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s = \int_{\widehat{F}} (\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}})\psi_K^{\mathrm{g}}(q)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{s}, \tag{9.15a}$$

$$\int_E (\boldsymbol{v}\cdot\boldsymbol{\Phi}_K^{\mathrm{c}}(\widehat{\boldsymbol{\tau}}_{\widehat{E}}))(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}l = \int_{\widehat{E}} (\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{\tau}}_{\widehat{E}})(\widehat{\boldsymbol{x}})\psi_K^{\mathrm{g}}(q)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{l}. \tag{9.15b}$$

Proof. To prove (9.15a), we use the transformation of surface measures from Lemma 9.12 followed by the definition (9.14a) of $\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}})$ and the definition of the maps $\boldsymbol{\psi}_K^{\mathrm{d}}$ and ψ_K^{g} (see (9.9)) to obtain

$$\begin{aligned}
&\int_F (\boldsymbol{v}\cdot\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}}))(\boldsymbol{x})q(\boldsymbol{x})\,\mathrm{d}s \\
&\quad= \int_{\widehat{F}} (\boldsymbol{v}\cdot\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}}))(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\psi_K^{\mathrm{g}}(q)(\widehat{\boldsymbol{x}})|\det(\mathbb{J}_K)(\widehat{\boldsymbol{x}})|\,\|\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}}(\widehat{\boldsymbol{x}})\|_{\ell^2}\,\mathrm{d}\widehat{s} \\
&\quad= \int_{\widehat{F}} ((\boldsymbol{v}\circ\boldsymbol{T}_K)\cdot(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}_{\widehat{F}}))(\widehat{\boldsymbol{x}})\psi_K^{\mathrm{g}}(q)(\widehat{\boldsymbol{x}})\det(\mathbb{J}_K)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{s} \\
&\quad= \int_{\widehat{F}} (\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{n}}_{\widehat{F}})(\widehat{\boldsymbol{x}})\psi_K^{\mathrm{g}}(q)(\widehat{\boldsymbol{x}})\,\mathrm{d}\widehat{s}.
\end{aligned}$$

The proof of (9.15b) uses similar arguments and is left as an exercise. □

Remark 9.14 (Sign of $\det(\mathbb{J}_K)$). The factor $\epsilon_K = \pm 1$ in the definition (9.14a) is due to the fact that the contravariant Piola transformation

$\boldsymbol{\psi}_K^{\mathrm{d}}$ may transform an outward-pointing field into an inward-pointing field. The definition (9.14a) is such that the sign of $\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{n}_K)(\widehat{\boldsymbol{x}})\cdot\boldsymbol{n}_{\widehat{F}}(\widehat{\boldsymbol{x}})$ and the sign of $\boldsymbol{n}_K(\boldsymbol{x})\cdot\boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}})(\boldsymbol{x})$ are identical. Note that $\epsilon_K = 1$ if $\det(\mathbb{J}_K) > 0$. □

Exercises

Exercise 9.1 (Canonical hybrid element). Consider an affine geometric mapping $\boldsymbol{T}_K$ and the pullback by $\boldsymbol{T}_K$ for ψ_K. Let $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ be the canonical hybrid element of §7.6. Verify that Proposition 9.2 generates the canonical hybrid element in K. Write the dofs.

Exercise 9.2 (Line measure). (i) Prove Lemma 9.12 for line measures. (*Hint*: the change in line measure is $\frac{\mathrm{d}l}{\mathrm{d}\widehat{l}}(\boldsymbol{x}) = \lim_{h\to 0} \frac{\|\boldsymbol{T}_K(\widehat{\boldsymbol{x}}+h\widehat{\boldsymbol{\tau}})-\boldsymbol{T}_K(\widehat{\boldsymbol{x}})\|_{\ell^2}}{\|h\widehat{\boldsymbol{\tau}}\|_{\ell^2}}$.) (ii) Assume that $d = 2$. Show that $|\det(\mathbb{J}_K)|\|\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}\|_{\ell^2(\mathbb{R}^2)} = \|\mathbb{J}_K\widehat{\boldsymbol{\tau}}\|_{\ell^2(\mathbb{R}^2)}$ for any pair of unit vectors $(\widehat{\boldsymbol{n}}, \widehat{\boldsymbol{\tau}})$ that are orthogonal.

Exercise 9.3 (Surface measure). (i) Let $\boldsymbol{T}_F := \boldsymbol{T}_{K|\widehat{F}} : \widehat{F} \to F$ and $\widehat{\boldsymbol{x}} \in \widehat{F}$. Let $\mathbb{J}_F(\widehat{\boldsymbol{x}}) \in \mathbb{R}^{d\times(d-1)}$ be the Jacobian matrix representing the (Fréchet) derivative $D\boldsymbol{T}_F(\widehat{\boldsymbol{x}})$. Let $\mathrm{g}_F(\widehat{\boldsymbol{x}}) = (\mathbb{J}_F(\widehat{\boldsymbol{x}}))^{\mathsf{T}}\mathbb{J}_F(\widehat{\boldsymbol{x}}) \in \mathbb{R}^{(d-1)\times(d-1)}$ be the surface metric tensor at $\widehat{\boldsymbol{x}}$. Prove that $\sqrt{\det(\mathrm{g}_F(\widehat{\boldsymbol{x}}))} = |\det(\mathbb{J}_K)|\,\|\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{n}}\|_{\ell^2}$. (*Hint*: use that $\mathrm{d}s = \sqrt{\det(\mathrm{g}_F(\widehat{\boldsymbol{x}}))}\,\mathrm{d}\widehat{s}$.) (ii) Let $\widehat{K} := \{(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \mathbb{R}^3 \mid 0 \le \widehat{x}_1, \widehat{x}_2, \widehat{x}_3,\ \widehat{x}_1 + \widehat{x}_2 + \widehat{x}_3 \le 1\}$ be the unit simplex in $\mathbb{R}^3$. Let $\boldsymbol{T}_K(\widehat{\boldsymbol{x}}) := (\widehat{x}_1, \widehat{x}_2, \widehat{x}_1^2+\widehat{x}_2^2-\widehat{x}_3)^{\mathsf{T}}$. Let $\widehat{F}$ be the face $\{\widehat{x}_3 = 0\}$ and $F := \boldsymbol{T}_K(\widehat{F})$. Compute $\mathbb{J}_F$, $\mathbb{J}_K$, g_F and verify the identity proved in Step (i).

Exercise 9.4 (Sobolev spaces). Prove that ψ_K^{g} is a bounded isomorphism from $H^1(K)$ to $H^1(\widehat{K})$, that $\boldsymbol{\psi}_K^{\mathrm{c}}$ is a bounded isomorphism from $\boldsymbol{H}(\mathrm{curl}; K)$ to $\boldsymbol{H}(\mathrm{curl}; \widehat{K})$, and that $\boldsymbol{\psi}_K^{\mathrm{d}}$ is a bounded isomorphism from $\boldsymbol{H}(\mathrm{div}; K)$ to $\boldsymbol{H}(\mathrm{div}; \widehat{K})$.

Exercise 9.5 (Transformation of cross products). Let $\mathbb{A}$ be a 3×3 invertible matrix. Prove that $\mathbb{A}^{-\mathsf{T}}(\boldsymbol{x}\times\boldsymbol{y}) = \det(\mathbb{A})^{-1}(\mathbb{A}\boldsymbol{x}\times\mathbb{A}\boldsymbol{y})$ for any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$.

Exercise 9.6 ((9.15b)). Prove (9.15b).