Part III, Chapter 15

$\boldsymbol{H}(\mathrm{curl})$ finite elements

The goal of this chapter is to construct $\mathbb{R}^d$-valued finite elements $(K, \boldsymbol{P}, \Sigma)$ with $d \in \{2,3\}$ such that (i) $\mathbb{P}_{k,d} \subset \boldsymbol{P}$ for some $k \geq 0$ and (ii) the degrees of freedom (dofs) in Σ fully determine the tangential components of the polynomials in $\boldsymbol{P}$ on all the faces of K. The first requirement is key for proving convergence rates on the interpolation error. The second one is key for constructing $\boldsymbol{H}(\mathrm{curl})$-conforming finite element spaces (see Chapter 19). The finite elements introduced in this chapter are used, e.g., in Chapter 43 to approximate (simplified forms of) Maxwell's equations which constitute a fundamental model in electromagnetism. The focus here is on defining a reference element and generating finite elements on the mesh cells. The interpolation error analysis is done in Chapters 16 and 17. We detail the construction for the simplicial Nédélec finite elements of the first kind. Some alternative elements are outlined at the end of the chapter.

15.1 The lowest-order case

Let us consider the lowest-order *Nédélec finite element.* Let $d \in \{2,3\}$ be the space dimension, and define the polynomial space

$$\mathbb{N}_{0,d} := \mathbb{P}_{0,d} \oplus \mathbb{S}_{1,d}, \tag{15.1}$$

where $\mathbb{S}_{1,d} := \{\boldsymbol{q} \in \mathbb{P}^{\mathrm{H}}_{1,d} \mid \boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0\}$, i.e.,

$$\mathbb{S}_{1,2} := \mathrm{span}\left\{ \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right\}, \qquad \mathbb{S}_{1,3} := \mathrm{span}\left\{ \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \right\}. \tag{15.2}$$

The sum in (15.1) is indeed direct, so that $\dim(\mathbb{N}_{0,d}) = \frac{d(d+1)}{2} =: d'$ (i.e., $d' = 3$ if $d = 2$ and $d' = 6$ if $d = 3$). Note that d' is the number of edges of a simplex in $\mathbb{R}^d$. The space $\mathbb{N}_{0,d}$ has several interesting properties. (a) One has $\mathbb{P}_{0,d} \subset \mathbb{N}_{0,d}$ in agreement with the first requirement stated above. (b)

The gradient of $\boldsymbol{v} \in \mathbb{N}_{0,d}$ is skew-symmetric. Indeed, only the component $\boldsymbol{q} \in \mathbb{S}_{1,d}$ contributes to the gradient, and the identity $\partial_{x_i x_j}(\boldsymbol{q}(\boldsymbol{x}){\cdot}\boldsymbol{x}) = 0$, $i \neq j$, yields $\partial_i q_j + \partial_j q_i = 0$. (c) If $\boldsymbol{v} \in \mathbb{N}_{0,d}$ is curl-free, then $\boldsymbol{v}$ is constant. Indeed, $\boldsymbol{v}$ being curl-free means that $\nabla\boldsymbol{v}$ is symmetric, which implies $\nabla\boldsymbol{v} = \boldsymbol{0}$ owing to (b). (d) The tangential component of $\boldsymbol{v} \in \mathbb{N}_{0,d}$ along an affine line in $\mathbb{R}^d$ is constant along that line. Let indeed $\boldsymbol{x}, \boldsymbol{y}$ be two distinct points on the line, say L, with tangent vector $\boldsymbol{t}_L$. Then there is $\lambda \in \mathbb{R}$ such that $\boldsymbol{t}_L = \lambda(\boldsymbol{x} - \boldsymbol{y})$ and since $\boldsymbol{v} = \boldsymbol{r} + \boldsymbol{q}$ with $\boldsymbol{r} \in \mathbb{P}_{0,d}$ and $\boldsymbol{q} \in \mathbb{S}_{1,d}$, we infer that $\boldsymbol{v}(\boldsymbol{x}){\cdot}\boldsymbol{t}_L - \boldsymbol{v}(\boldsymbol{y}){\cdot}\boldsymbol{t}_L = (\boldsymbol{q}(\boldsymbol{x}) - \boldsymbol{q}(\boldsymbol{y})){\cdot}\boldsymbol{t}_L = \lambda\boldsymbol{q}(\boldsymbol{x} - \boldsymbol{y}){\cdot}(\boldsymbol{x} - \boldsymbol{y}) = 0$.

Let K be a simplex in $\mathbb{R}^d$ and let $\mathcal{E}_K$ collect the edges of K. Any edge $E \in \mathcal{E}_K$ is oriented by fixing an edge vector $\boldsymbol{t}_E$ s.t. $\|\boldsymbol{t}_E\|_{\ell^2} = |E|$. Conventionally, we set $\boldsymbol{t}_E := \boldsymbol{z}_q - \boldsymbol{z}_p$, where $\boldsymbol{z}_p, \boldsymbol{z}_q$ are the two endpoints of E with $p < q$. We denote by Σ the collection of the following linear forms acting on $\mathbb{N}_{0,d}$:

$$\sigma_E^{\mathrm{e}}(\boldsymbol{v}) := \frac{1}{|E|}\int_E (\boldsymbol{v}{\cdot}\boldsymbol{t}_E)\,\mathrm{d}l, \qquad \forall E \in \mathcal{E}_K. \tag{15.3}$$

Note that the unit of $\sigma_E^{\mathrm{e}}(\boldsymbol{v})$ is a length times the dimension of $\boldsymbol{v}$. A graphic representation of the dofs is shown in Figure 15.1. Each arrow indicates the orientation of the corresponding edge.

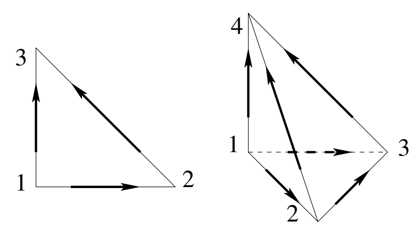

Fig. 15.1 Degrees of freedom of the $\mathbb{N}_{0,d}$ finite element in dimensions two (left) and dimension three (right).

Proposition 15.1 (Face (edge) unisolvence, $d = 2$). *Let $\boldsymbol{v} \in \mathbb{N}_{0,2}$. Let $E \in \mathcal{E}_K$ be an edge of K. Then $\sigma_E^{\mathrm{e}}(\boldsymbol{v}) = 0$ implies that $\boldsymbol{v}_{|E}{\cdot}\boldsymbol{t}_E = 0$.*

Proof. Since we have established above that $\boldsymbol{v}_{|E}{\cdot}\boldsymbol{t}_E$ is constant, the assertion follows readily. $\square$

Proposition 15.2 (Finite element, 2D). $(K, \mathbb{N}_{0,2}, \Sigma)$ *is a finite element.*

Proof. Since $\dim(\mathbb{N}_{0,2}) = \mathrm{card}(\Sigma) = 3$, we just need to verify that the only function $\boldsymbol{v} \in \mathbb{N}_{0,2}$ that annihilates the three dofs in Σ is zero. This follows from Proposition 15.1 since $\mathrm{span}\{\boldsymbol{t}_E\}_{E\in\mathcal{E}_K} = \mathbb{R}^2$. $\square$

The above results hold also true if $d = 3$, but the proofs are more intricate since the tangential component on an affine hyperplane of a function in $\mathbb{N}_{0,3}$ is not necessarily constant. Let $F \in \mathcal{F}_K$ be a face of K and let us fix a unit vector $\boldsymbol{n}_F$ normal to F. There are two ways to define the tangential component of a function $\boldsymbol{v}$ on F: one can define it either as $\boldsymbol{v}\times\boldsymbol{n}_F$ or as $\Pi_F(\boldsymbol{v}) := \boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{n}_F)\boldsymbol{n}_F$. We will use both definitions. The first one is convenient when working with the $\nabla\times$ operator. The second one is more geometric. The two definitions produce ℓ^2-orthogonal vectors since $(\boldsymbol{v}\times\boldsymbol{n}_F)\cdot\Pi_F(\boldsymbol{v}) = 0$ as shown in Figure 15.2.

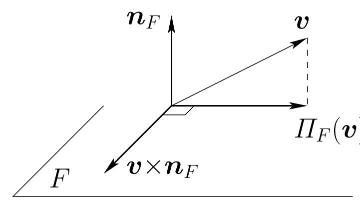

Fig. 15.2 Two possible definitions of the tangential component of a vector.

Proposition 15.3 (Face unisolvence, 3D). *Let $\boldsymbol{v}\in\mathbb{N}_{0,3}$. Let $F\in\mathcal{F}_K$ be a face of K and let $\mathcal{E}_F$ be the collection of the three edges of K forming the boundary of F. Then $\sigma_E^{\mathrm{e}}(\boldsymbol{v})=0$ for all $E\in\mathcal{E}_F$ implies that $\boldsymbol{v}_{|F}\times\boldsymbol{n}_F=\boldsymbol{0}$.*

Proof. Let $\widehat{S}^2$ be the unit simplex in $\mathbb{R}^2$. Let $\boldsymbol{T}_F : \widehat{S}^2 \to F$ be defined by $\boldsymbol{T}_F(0,0) := \boldsymbol{z}_p$, $\boldsymbol{T}_F(1,0) := \boldsymbol{z}_q$, $\boldsymbol{T}_F(0,1) := \boldsymbol{z}_r$, where $\boldsymbol{z}_p, \boldsymbol{z}_q, \boldsymbol{z}_r$ are the three vertices of F enumerated by increasing vertex-index. Let $\mathbb{J}_F$ be the $3{\times}2$ Jacobian matrix of $\boldsymbol{T}_F$. Note that for all $\widehat{\boldsymbol{y}}\in\mathbb{R}^2$ the vector $\mathbb{J}_F\widehat{\boldsymbol{y}}$ is parallel to F and $\boldsymbol{T}_F(\widehat{\boldsymbol{y}})-\boldsymbol{z}_p=\mathbb{J}_F\widehat{\boldsymbol{y}}$. Let $\boldsymbol{v}=\boldsymbol{r}+\boldsymbol{q}$ with $\boldsymbol{r}\in\mathbb{P}_{0,3}$ and $\boldsymbol{q}\in\mathbb{S}_{1,3}$. Let us set $\widehat{\boldsymbol{v}} := \mathbb{J}_F^{\mathsf{T}}\Pi_F(\boldsymbol{v}\circ\boldsymbol{T}_F)$ and let us show that $\widehat{\boldsymbol{v}}\in\mathbb{N}_{0,2}$. For all $\widehat{\boldsymbol{y}}\in\mathbb{R}^2$, we have

$$
\begin{aligned}
\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{y}}) &= \widehat{\boldsymbol{y}}\cdot\left(\mathbb{J}_F^{\mathsf{T}}\Pi_F(\boldsymbol{v}(\boldsymbol{T}_F(\widehat{\boldsymbol{y}})))\right) = \widehat{\boldsymbol{y}}\cdot\left(\mathbb{J}_F^{\mathsf{T}}\Pi_F(\boldsymbol{r}+\boldsymbol{q}(\boldsymbol{T}_F(\widehat{\boldsymbol{y}})))\right)\\
&= \widehat{\boldsymbol{y}}\cdot\left(\mathbb{J}_F^{\mathsf{T}}\Pi_F(\boldsymbol{r}+\boldsymbol{q}(\boldsymbol{z}_p)+\boldsymbol{q}(\mathbb{J}_F\widehat{\boldsymbol{y}}))\right)\\
&= \widehat{\boldsymbol{y}}\cdot(\mathbb{J}_F^{\mathsf{T}}\Pi_F(\boldsymbol{r}+\boldsymbol{q}(\boldsymbol{z}_p))) + (\mathbb{J}_F\widehat{\boldsymbol{y}})\cdot(\boldsymbol{q}(\mathbb{J}_F\widehat{\boldsymbol{y}})).
\end{aligned}
$$

Setting $\widehat{\boldsymbol{c}} := \mathbb{J}_F^{\mathsf{T}}\Pi_F(\boldsymbol{r}+\boldsymbol{q}(\boldsymbol{z}_p))\in\mathbb{R}^2$ and using that $\boldsymbol{q}\in\mathbb{S}_{1,3}$, we infer that $\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{y}})=\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{c}}$. Since $\widehat{\boldsymbol{v}}\in\mathbb{P}_{1,2}$, we have $\widehat{\boldsymbol{v}}=\widehat{\boldsymbol{r}}+\widehat{\boldsymbol{q}}$ where $\widehat{\boldsymbol{r}}\in\mathbb{P}_{0,2}$ and $\widehat{\boldsymbol{q}}\in\mathbb{P}_{1,2}^{\mathrm{H}}$. Then $\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{r}}+\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{q}}(\widehat{\boldsymbol{y}})=\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{c}}$ for all $\widehat{\boldsymbol{y}}\in\mathbb{R}^2$. This implies that the quadratic form $\widehat{\boldsymbol{y}}\cdot\widehat{\boldsymbol{q}}(\widehat{\boldsymbol{y}})$ is zero. Hence, $\widehat{\boldsymbol{v}}\in\mathbb{N}_{0,2}$. Let now $\widehat{E}$ be any of the three edges of $\widehat{S}^2$. Then $E := \boldsymbol{T}_F(\widehat{E})$ is one of the three edges of F. We obtain that

$$
\begin{aligned}
\int_{\widehat{E}}(\widehat{\boldsymbol{v}}\cdot\boldsymbol{t}_{\widehat{E}})\,\mathrm{d}\widehat{l} &= \int_{\widehat{E}}(\mathbb{J}_F^{\mathsf{T}}\Pi_F(\boldsymbol{v}\circ\boldsymbol{T}_F))\cdot\boldsymbol{t}_{\widehat{E}}\,\mathrm{d}\widehat{l}\\
&= \int_{\widehat{E}}(\boldsymbol{v}\circ\boldsymbol{T}_F)\cdot\boldsymbol{t}_E\,\mathrm{d}\widehat{l} = \frac{|\widehat{E}|}{|E|}\int_E \boldsymbol{v}\cdot\boldsymbol{t}_E\,\mathrm{d}l = |\widehat{E}|\sigma_E^{\mathrm{e}}(\boldsymbol{v}) = 0.
\end{aligned}
$$

Since $\widehat{\boldsymbol{v}}\in\mathbb{N}_{0,2}$ annihilates the three edge dofs in $\widehat{S}^2$, Proposition 15.2 implies that $\widehat{\boldsymbol{v}}=\boldsymbol{0}$. After observing that $\mathrm{im}(\Pi_F)$ is orthogonal to $\ker(\mathbb{J}_F^{\mathsf{T}})$, we conclude that the tangential component of $\boldsymbol{v}$ on F is zero. $\square$

Proposition 15.4 (Finite element, 3D). *$(K, \mathbb{N}_{0,3}, \Sigma)$ is a finite element.*

Proof. Since $\dim(\mathbb{N}_{0,3}) = \mathrm{card}(\Sigma) = 6$, we just need to verify that the only function $\boldsymbol{v} \in \mathbb{N}_{0,3}$ that annihilates the six dofs in Σ is zero. Face unisolvence implies that $\boldsymbol{v}_{|F} \times \boldsymbol{n}_F = \boldsymbol{0}$ for all $F \in \mathcal{F}_K$. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be the canonical basis of $\mathbb{R}^3$. Using (4.11), we infer that $\int_K (\nabla \times \boldsymbol{v}) \cdot \boldsymbol{e}_i \, \mathrm{d}x = -\int_{\partial K} (\boldsymbol{v} \times \boldsymbol{n}_K) \cdot \boldsymbol{e}_i \, \mathrm{d}s = 0$, where $\boldsymbol{n}_K$ is the outward unit normal to K. Since $\nabla \times \boldsymbol{v}$ is actually constant on K, we have $\nabla \times \boldsymbol{v} = \boldsymbol{0}$, and we have seen that this implies that $\boldsymbol{v} \in \mathbb{P}_{0,3}$, i.e., $\boldsymbol{v} = \nabla p$ for some $p \in \mathbb{P}_{1,3}$. Integrating ∇p along the edges of K, we infer that p takes the same value at all the vertices of K. Hence, p is constant, which in turn implies that $\boldsymbol{v}$ is zero. $\square$

One can verify that the shape functions are such that

$$\boldsymbol{\theta}_E^{\mathrm{e}}(\boldsymbol{x}) = \lambda_p(\boldsymbol{x}) \nabla \lambda_q - \lambda_q(\boldsymbol{x}) \nabla \lambda_p, \qquad \forall E \in \mathcal{E}_K, \tag{15.4}$$

for all $\boldsymbol{x} \in K$, with $\boldsymbol{t}_E := \boldsymbol{z}_q - \boldsymbol{z}_p$. For every $E' \in \mathcal{E}_K$, we have $\boldsymbol{\theta}_E^{\mathrm{e}} \cdot \boldsymbol{t}_{E'} = \delta_{EE'}$. We refer the reader to Exercise 15.3 for additional properties of the $\mathbb{N}_{0,3}$ shape functions.

15.2 The polynomial space $\mathbb{N}_{k,d}$

Let $k \in \mathbb{N}$ and let $d \in \{2, 3\}$ (the material of this section extends to any dimension $d \geq 2$). Let $\mathbb{P}_{k,d}^{\mathrm{H}}$ be the space of the homogeneous polynomials of degree k (see Definition 14.2). Set $\boldsymbol{\mathbb{P}}_{k,d}^{\mathrm{H}} := [\mathbb{P}_{k,d}^{\mathrm{H}}]^d$ and $\boldsymbol{\mathbb{P}}_{k,d} := [\mathbb{P}_{k,d}]^d$.

Definition 15.5 ($\mathbb{N}_{k,d}$). *We define the following real vector space of $\mathbb{R}^d$-valued polynomials:*

$$\mathbb{N}_{k,d} := \boldsymbol{\mathbb{P}}_{k,d} \oplus \mathbb{S}_{k+1,d}, \quad \textit{with} \quad \mathbb{S}_{k+1,d} := \{\boldsymbol{q} \in \boldsymbol{\mathbb{P}}_{k+1,d}^{\mathrm{H}} \mid \boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0\}. \tag{15.5}$$

Note that the above sum is direct since $\boldsymbol{\mathbb{P}}_{k,d} \cap \mathbb{S}_{k+1,d} \subset \boldsymbol{\mathbb{P}}_{k,d} \cap \boldsymbol{\mathbb{P}}_{k+1,d}^{\mathrm{H}} = \{\boldsymbol{0}\}$.

Example 15.6 (Space $\mathbb{S}_{2,d}$). The set $\{(-x_2^2, x_1x_2)^\mathsf{T}, (x_1x_2, -x_1^2)^\mathsf{T}\}$ is a basis of $\mathbb{S}_{2,2}$, and the set $\{(-x_2^2, x_1x_2, 0)^\mathsf{T}, (-x_3^2, 0, x_1x_3)^\mathsf{T}, (x_1x_2, -x_1^2, 0)^\mathsf{T}, (0, -x_3^2, x_2x_3)^\mathsf{T}, (x_1x_3, 0, -x_1^2)^\mathsf{T}, (0, x_2x_3, -x_2^2)^\mathsf{T}, (x_2x_3, -x_1x_3, 0)^\mathsf{T}, (0, x_1x_3, -x_1x_2)^\mathsf{T}\}$ is a basis of $\mathbb{S}_{2,3}$. Note that $\dim(\mathbb{S}_{2,2}) = 2$ and $\dim(\mathbb{S}_{2,3}) = 8$. $\square$

Lemma 15.7 (Dimension of $\mathbb{N}_{k,d}$). *Let $k \in \mathbb{N}$ and $d \geq 2$. We have*

$$\dim(\mathbb{N}_{k,d}) = \frac{(k+d+1)!}{k!(d-1)!(k+2)}. \tag{15.6}$$

Hence, $\dim(\mathbb{N}_{k,2}) = (k+1)(k+3)$ *and* $\dim(\mathbb{N}_{k,3}) = \frac{1}{2}(k+1)(k+3)(k+4)$.

Proof. (1) Let us first prove that the map $\phi : \mathbb{P}^{\mathrm{H}}_{k,d} \ni \boldsymbol{p} \mapsto \boldsymbol{x}{\cdot}\boldsymbol{p} \in \mathbb{P}^{\mathrm{H}}_{k+1,d}$ is surjective. By linearity, it suffices to prove that for each monomial $q \in \mathbb{P}^{\mathrm{H}}_{k+1,d}$ s.t. $q(\boldsymbol{x}) := \boldsymbol{x}^{\alpha}$ with $|\alpha| := k+1$, there is $\boldsymbol{r} \in \mathbb{P}^{\mathrm{H}}_{k,d}$ such that $q(\boldsymbol{x}) = \boldsymbol{x}{\cdot}\boldsymbol{r}(\boldsymbol{x})$. Let $\{\boldsymbol{e}_i\}_{i\in\{1:d\}}$ be the canonical Cartesian basis of $\mathbb{R}^d$. Since $|\alpha| = k+1 \geq 1$, there exists $i \in \{1:d\}$ s.t. $\alpha_i \geq 1$. Setting $\boldsymbol{r}(\boldsymbol{x}) := x_1^{\alpha_1} \ldots x_i^{\alpha_i-1} \ldots x_d^{\alpha_d}\boldsymbol{e}_i$, we have $\boldsymbol{r} \in \mathbb{P}^{\mathrm{H}}_{k,d}$ and $q(\boldsymbol{x}) = \boldsymbol{x}{\cdot}\boldsymbol{r}(\boldsymbol{x})$.
(2) Observing that $\ker(\phi) = \mathbb{S}_{k,d}$, the rank nullity theorem implies that $\dim(\mathbb{S}_{k,d}) + \dim(\mathrm{im}(\phi)) = \dim \mathbb{P}^{\mathrm{H}}_{k,d}$, i.e., $\dim(\mathbb{S}_{k,d}) = d \dim \mathbb{P}^{\mathrm{H}}_{k,d} - \dim \mathbb{P}^{\mathrm{H}}_{k+1,d} = d\binom{k+d-1}{k} - \binom{k+d}{k+1} = \binom{k+d-1}{k}(d - \frac{k+d}{k+1}) = k\frac{(k+d-1)!}{(k+1)!(d-2)!}$. The sum in (15.5) being direct, we conclude that

$$
\begin{aligned}
\dim(\mathbb{N}_{k,d}) &= d\dim(\mathbb{P}_{k,d}) + \dim(\mathbb{S}_{k+1,d}) \\
&= \frac{(k+d)!}{k!(d-1)!} + (k+1)\frac{(k+d)!}{(k+2)!(d-2)!} = \frac{(k+d+1)!}{k!(d-1)!(k+2)}. \qquad \square
\end{aligned}
$$

Lemma 15.8 (Trace space). *Let H be an affine hyperplane in $\mathbb{R}^d$, let $\boldsymbol{n}_H$ be a unit normal vector to H, and let $\boldsymbol{T}_H : \mathbb{R}^{d-1} \to H$ be an affine bijective mapping with Jacobian matrix $\mathbb{J}_H$. Let $\Pi_H(\boldsymbol{v}) := \boldsymbol{v} - (\boldsymbol{v}{\cdot}\boldsymbol{n}_H)\boldsymbol{n}_H$ be the ℓ^2-orthogonal projection of $\boldsymbol{v}$ onto the tangent space to H (i.e., the linear hyperplane in $\mathbb{R}^d$ parallel to H). Then $\mathbb{J}_H^{\mathsf{T}}\Pi_H(\boldsymbol{v}_{|H}) \in \mathbb{N}_{k,d-1} \circ \boldsymbol{T}_H^{-1}$ for all $\boldsymbol{v} \in \mathbb{N}_{k,d}$.*

Proof. Identical to the proof of Proposition 15.3. $\square$

Lemma 15.9 ($d = 2$). $\mathbb{N}_{k,2} = \boldsymbol{R}_{\frac{\pi}{2}}(\mathbb{RT}_{k,2})$, *where $\boldsymbol{R}_{\frac{\pi}{2}}$ is the rotation of angle $\frac{\pi}{2}$ in $\mathbb{R}^2$, i.e., $\boldsymbol{R}_{\frac{\pi}{2}}\boldsymbol{x} = (-x_2, x_1)^{\mathsf{T}}$ for all $\boldsymbol{x} = (x_1, x_2)^{\mathsf{T}} \in \mathbb{R}^2$.*

Proof. See Exercise 15.4. $\square$

Lemma 15.10 (Curl). *Assume $d \in \{2, 3\}$. Then $\nabla{\times}\boldsymbol{v} \in \mathbb{P}_{k,d}$ for all $\boldsymbol{v} \in \mathbb{N}_{k,d}$, and if $\nabla{\times}\boldsymbol{v} = \boldsymbol{0}$, there is $p \in \mathbb{P}_{k+1,d}$ such that $\boldsymbol{v} = \nabla p$ (that is, $\boldsymbol{v} \in \mathbb{P}_{k,d}$).*

Proof. That $\nabla{\times}\boldsymbol{v} \in \mathbb{P}_{k,d}$ results from $\mathbb{N}_{k,d} \subset \mathbb{P}_{k+1,d}$. The condition $\nabla{\times}\boldsymbol{v} = \boldsymbol{0}$ together with $\boldsymbol{v} \in \mathbb{N}_{k,d} \subset \mathbb{P}_{k+1,d}$ implies that there is $p \in \mathbb{P}_{k+2,d}$ such that $\boldsymbol{v} = \nabla p$. The definition of $\mathbb{N}_{k,d}$ implies that $\boldsymbol{v} = \nabla p_1 + \nabla p_2$ with $p_1 \in \mathbb{P}_{k+1,d}$ and $\nabla p_2 \in \mathbb{S}_{k+1,d}$. We infer that $p_2(\boldsymbol{x}) - p_2(\boldsymbol{0}) = \int_0^1 \nabla p_2(t\boldsymbol{x}){\cdot}(t\boldsymbol{x})t^{-1}\,\mathrm{d}t = 0$, which means that p_2 is constant. Hence, $\boldsymbol{v} = \nabla p_1$ with $p_1 \in \mathbb{P}_{k+1,d}$. $\square$

15.3 Simplicial Nédélec elements

Let $k \in \mathbb{N}$ and let $d \in \{2, 3\}$. Let K be a simplex in $\mathbb{R}^d$. In this section, we define the dofs in order to make the triple $(K, \mathbb{N}_{k,d}, \Sigma)$ a finite element. The construction can be generalized to any dimension.

15.3.1 Two-dimensional case

Let us orient the three edges $E \in \mathcal{E}_K$ of K with the edge vectors $\boldsymbol{t}_E$. Let us orient K with the two vectors $\{\boldsymbol{t}_{K,j}\}_{j\in\{1,2\}}$ which are the edge vectors for the two edges of K sharing the vertex with the lowest enumeration index. Note that $\{\boldsymbol{t}_{K,j}\}_{j\in\{1,2\}}$ is a basis of $\mathbb{R}^2$. Let $\boldsymbol{T}_E$ be an affine bijective mapping from the unit simplex $\widehat{S}^1 := [0,1]$ in $\mathbb{R}$ onto E. We define the dofs of the two-dimensional Nédélec element $(K, \mathbb{N}_{k,2}, \Sigma)$ as follows:

$$\sigma^{\mathrm{e}}_{E,m}(\boldsymbol{v}) := \frac{1}{|E|}\int_E (\boldsymbol{v}{\cdot}\boldsymbol{t}_E)(\mu_m \circ \boldsymbol{T}_E^{-1})\,\mathrm{d}l, \quad \forall E \in \mathcal{E}_K, \tag{15.7a}$$

$$\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) := \frac{1}{|K|}\int_K (\boldsymbol{v}{\cdot}\boldsymbol{t}_{K,j})\psi_m\,\mathrm{d}x, \quad \forall j \in \{1{:}2\}, \tag{15.7b}$$

where $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k,1}$ with $n^{\mathrm{e}}_{\mathrm{sh}} := \dim(\mathbb{P}_{k,1}) = k+1$, and $\{\psi_m\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k-1,2}$ with $n^{\mathrm{c}}_{\mathrm{sh}} := \dim(\mathbb{P}_{k-1,2}) = \frac{1}{2}k(k+1)$ if $k \ge 1$. Since $\mathbb{N}_{k,2} = \boldsymbol{R}_{\frac{\pi}{2}}(\mathbb{RT}_{k,2})$ owing to Lemma 15.9 and since the above dofs are those of the $\mathbb{RT}_{k,2}$ finite element once the edges (faces) are oriented by the vectors $\boldsymbol{\nu}_E := \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{t}_E)$ and K is oriented by the vectors $\boldsymbol{\nu}_{K,j} := \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{t}_{K,j})$, it follows from Proposition 14.15 that the triple $(K, \mathbb{N}_{k,2}, \Sigma)$ is a finite element for all $k \ge 0$. The unit of all the above dofs is a length times the dimension of $\boldsymbol{v}$.

Remark 15.11 (2D Piola transformations). Owing to the identity $\mathbb{A}^{\mathsf{T}} = \det(\mathbb{A})\boldsymbol{R}_{\frac{\pi}{2}}^{-1}\mathbb{A}^{-1}\boldsymbol{R}_{\frac{\pi}{2}}$ for all $\mathbb{A} \in \mathbb{R}^{2\times 2}$, the two-dimensional contravariant and covariant Piola transformations satisfy $\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{\psi}^{\mathrm{c}}_K(\boldsymbol{v})) = \boldsymbol{\psi}^{\mathrm{d}}_K(\boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{v}))$. □

15.3.2 Three-dimensional case

Let K be a simplex (tetrahedron) in $\mathbb{R}^3$. Let $\mathcal{E}_K$ be the collection of the six edges of K and let $\mathcal{F}_K$ be the collection of the four faces of K. Each edge $E \in \mathcal{E}_K$ is oriented by the edge vector $\boldsymbol{t}_E := \boldsymbol{z}_q - \boldsymbol{z}_p$, where $\boldsymbol{z}_p, \boldsymbol{z}_q$ are the two vertices of E with $p < q$ (note that $\|\boldsymbol{t}_E\|_{\ell^2} = |E|$). Each face $F \in \mathcal{F}_K$ is oriented by the two edge vectors $\{\boldsymbol{t}_{F,j}\}_{j\in\{1,2\}}$ with $\boldsymbol{t}_{F,1} := \boldsymbol{z}_q - \boldsymbol{z}_p$, $\boldsymbol{t}_{F,2} := \boldsymbol{z}_r - \boldsymbol{z}_p$, where $\boldsymbol{z}_p, \boldsymbol{z}_q, \boldsymbol{z}_r$ are the three vertices of F with $p < q < r$. Note that the unit normal vector $\boldsymbol{n}_F$ is then defined as $\boldsymbol{t}_{F,1}{\times}\boldsymbol{t}_{F,2}/\|\boldsymbol{t}_{F,1}{\times}\boldsymbol{t}_{F,2}\|_{\ell^2}$; see for instance (10.9). Note also that $\{\boldsymbol{t}_{F,j}\}_{j\in\{1,2\}}$ is a basis of the tangent space of the affine hyperplane supporting F. Finally, the cell K is oriented by the three edge vectors $\{\boldsymbol{t}_{K,j}\}_{j\in\{1:3\}}$ with $\boldsymbol{t}_{K,1} := \boldsymbol{z}_q - \boldsymbol{z}_p$, $\boldsymbol{t}_{K,2} := \boldsymbol{z}_r - \boldsymbol{z}_p$, $\boldsymbol{t}_{K,3} := \boldsymbol{z}_s - \boldsymbol{z}_p$, where $\boldsymbol{z}_p, \boldsymbol{z}_q, \boldsymbol{z}_r, \boldsymbol{z}_s$ are the four vertices of K with $p < q < r < s$. Note that $\{\boldsymbol{t}_{K,j}\}_{j\in\{1:3\}}$ is a basis of $\mathbb{R}^3$. In order to define dofs using moments on the edges and moments on the faces of K, we introduce affine bijective mappings $\boldsymbol{T}_F : \widehat{S}^2 \to F$ and $\boldsymbol{T}_E : \widehat{S}^1 \to E$, where $\widehat{S}^2$ is the unit simplex in $\mathbb{R}^2$ and $\widehat{S}^1$ is the unit simplex in $\mathbb{R}$; see Figure 15.3. For instance, after enumerating the vertices of $\widehat{S}^1$, $\widehat{S}^2$, these mappings can be constructed by using the increasing vertex-index enumeration technique of §10.2.

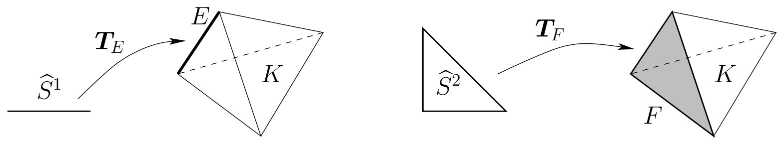

Fig. 15.3 Reference edge $\widehat{S}^1$ and reference face $\widehat{S}^2$ with the corresponding mappings.

Definition 15.12 (dofs). *The set Σ is defined to be the collection of the following linear forms acting on $\mathbb{N}_{k,3}$:*

$$\sigma^{\mathrm{e}}_{E,m}(\boldsymbol{v}) := \frac{1}{|E|}\int_E (\boldsymbol{v}{\cdot}\boldsymbol{t}_E)(\mu_m \circ \boldsymbol{T}_E^{-1})\,\mathrm{d}l, \qquad \forall E \in \mathcal{E}_K \tag{15.8a}$$

$$\sigma^{\mathrm{f}}_{F,j,m}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}{\cdot}\boldsymbol{t}_{F,j})(\zeta_m \circ \boldsymbol{T}_F^{-1})\,\mathrm{d}s, \quad \forall F \in \mathcal{F}_K, \forall j \in \{1,2\}, \tag{15.8b}$$

$$\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) := \frac{1}{|K|}\int_K (\boldsymbol{v}{\cdot}\boldsymbol{t}_{K,j})\psi_m\,\mathrm{d}x, \qquad \forall j \in \{1,2,3\}, \tag{15.8c}$$

where $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k,1}$ with $n^{\mathrm{e}}_{\mathrm{sh}} := k+1$, $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k-1,2}$ with $n^{\mathrm{f}}_{\mathrm{sh}} := \frac{1}{2}(k+1)k$ if $k \geq 1$, and $\{\psi_m\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k-2,3}$ with $n^{\mathrm{c}}_{\mathrm{sh}} := \frac{1}{6}(k+1)k(k-1)$ if $k \geq 2$. We regroup the dofs as follows:

$$\Sigma^{\mathrm{e}}_E := \{\sigma^{\mathrm{e}}_{E,m}\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}, \qquad \forall E \in \mathcal{E}_K, \tag{15.9a}$$

$$\Sigma^{\mathrm{f}}_F := \{\sigma^{\mathrm{f}}_{F,j,m}\}_{(j,m)\in\{1,2\}\times\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}, \quad \forall F \in \mathcal{F}_K, \tag{15.9b}$$

$$\Sigma^{\mathrm{c}} := \{\sigma^{\mathrm{c}}_{j,m}\}_{(j,m)\in\{1:3\}\times\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}. \tag{15.9c}$$

Remark 15.13 (dofs). The unit of all the dofs is a length times the dimension of $\boldsymbol{v}$. For the cell dofs, we could also have written $\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) := \ell_K^{-2}\int_K(\boldsymbol{v}{\cdot}\boldsymbol{e}_j)\psi_m\,\mathrm{d}x$, where ℓ_K is a length scale associated with K and $\{\boldsymbol{e}_j\}_{j\in\{1:d\}}$ is the canonical Cartesian basis of $\mathbb{R}^d$. We will see that the definition (15.8c) is more natural when using the covariant Piola transformation to generate Nédélec finite elements. The dofs are defined here on $\mathbb{N}_{k,d}$. Their extension to some larger space $\boldsymbol{V}(K)$ is addressed in Chapters 16 and 17. □

Lemma 15.14 (Invariance). *Assume that every affine bijective mapping $\boldsymbol{S} : \widehat{S}^1 \to \widehat{S}^1$ (resp., $\boldsymbol{S} : \widehat{S}^2 \to \widehat{S}^2$) leaves the basis $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ (resp., $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$) globally invariant. Then for all $E \in \mathcal{E}_K$ and all $F \in \mathcal{F}_K$, the set Σ^{e}_E and Σ^{f}_F are independent of the affine bijective mapping $\boldsymbol{T}_E$ and $\boldsymbol{T}_F$, respectively.*

Proof. Similar to that of Lemma 14.12; see also Example 14.13 for the invariance w.r.t. vertex permutation. □

The following result is important in view of $\boldsymbol{H}(\mathrm{curl})$-conformity.

Lemma 15.15 (Face unisolvence). *Let $\boldsymbol{v} \in \mathbb{N}_{k,3}$ and let $F \in \mathcal{F}_K$ be a face of K. Let $\mathcal{E}_F$ be the collection of the (three) edges forming the boundary of F, let $\Sigma_F^{\mathrm{e}} := \bigcup_{E\in\mathcal{E}_F} \Sigma_E^{\mathrm{e}}$, and let $\boldsymbol{n}_F$ be a unit normal to F. We have the following equivalence:*

$$[\,\sigma(\boldsymbol{v}) = 0, \quad \forall \sigma \in \Sigma_F^{\mathrm{f}} \cup \Sigma_F^{\mathrm{e}}\,] \iff [\,\boldsymbol{v}_{|F}\times\boldsymbol{n}_F = \boldsymbol{0}\,]. \tag{15.10}$$

Moreover, both assertions in (15.10) *imply that* $(\nabla\times\boldsymbol{v})_{|F}\cdot\boldsymbol{n}_F = 0$.

Proof. We only need to prove the implication in (15.10) since the converse is evident. The proof is an extension of that of Proposition 15.3 accounting for the richer structure of the dofs. We introduce $\widehat{\boldsymbol{v}} := \mathbb{J}_F^{\mathsf{T}} \Pi_F(\boldsymbol{v}\circ\boldsymbol{T}_F)$. It can be shown that $\widehat{\boldsymbol{v}} \in \mathbb{N}_{k,2}$; see Exercise 15.6. The unit simplex $\widehat{S}^2$ is oriented by the two edge vectors $\{\widehat{\boldsymbol{t}}_j\}_{j\in\{1,2\}}$ s.t. $\mathbb{J}_F\widehat{\boldsymbol{t}}_j = \boldsymbol{t}_{F,j}\circ\boldsymbol{T}_F$ for all $j \in \{1,2\}$. For the face dofs, we have

$$\begin{aligned}
\frac{1}{|\widehat{S}^2|}\int_{\widehat{S}^2} (\widehat{\boldsymbol{v}}\cdot\widehat{\boldsymbol{t}}_j)\zeta_m \,\mathrm{d}\widehat{s} &= \frac{1}{|\widehat{S}^2|}\int_{\widehat{S}^2} ((\mathbb{J}_F^{\mathsf{T}}(\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n}_F)\boldsymbol{n}_F)\circ\boldsymbol{T}_F)\cdot\widehat{\boldsymbol{t}}_j)\zeta_m \,\mathrm{d}\widehat{s} \\
&= \frac{1}{|\widehat{S}^2|}\int_{\widehat{S}^2} (((\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n}_F)\boldsymbol{n}_F)\cdot\boldsymbol{t}_{F,j})\circ\boldsymbol{T}_F)\zeta_m \,\mathrm{d}\widehat{s} \\
&= \frac{1}{|\widehat{S}^2|}\int_{\widehat{S}^2} ((\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})\circ\boldsymbol{T}_F)\zeta_m \,\mathrm{d}\widehat{s} \\
&= \frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})(\zeta_m\circ\boldsymbol{T}_F^{-1}) \,\mathrm{d}s = \sigma_{F,j,m}^{\mathrm{f}}(\boldsymbol{v}) = 0.
\end{aligned}$$

One proves similarly that the edge dofs vanish. This proves that $\widehat{\boldsymbol{v}} = \boldsymbol{0}$ because $\widehat{\boldsymbol{v}} \in \mathbb{N}_{k,2}$. Since $\mathbb{J}_F^{\mathsf{T}}$ has full rank, we infer that $\Pi_F(\boldsymbol{v}_{|F}) = \boldsymbol{0}$, which implies that $\boldsymbol{v}_{|F}\times\boldsymbol{n}_F = \Pi_F(\boldsymbol{v}_{|F})\times\boldsymbol{n}_F = \boldsymbol{0}$. Finally, $(\nabla\times\boldsymbol{v})_{|F}\cdot\boldsymbol{n}_F = 0$ immediately follows from $\boldsymbol{v}_{|F}\times\boldsymbol{n}_F = \boldsymbol{0}$. $\square$

Proposition 15.16 (Finite element). $(K, \mathbb{N}_{k,3}, \Sigma)$ *is a finite element.*

Proof. Observe first that the cardinality of Σ can be evaluated as follows:

$$\begin{aligned}
\mathrm{card}(\Sigma) &= 3n_{\mathrm{sh}}^{\mathrm{c}} + 2\times 4n_{\mathrm{sh}}^{\mathrm{f}} + 6n_{\mathrm{sh}}^{\mathrm{e}} = 3\binom{k+1}{3} + 8\binom{k+1}{2} + 6(k+1) \\
&= \frac{1}{2}(k+1)(k+3)(k+4) = \dim(\mathbb{N}_{k,3}).
\end{aligned}$$

Hence, the assertion will be proved once it is established that zero is the only function in $\mathbb{N}_{k,3}$ that annihilates all the dofs in Σ. Let $\boldsymbol{v} \in \mathbb{N}_{k,3}$ be such that $\sigma(\boldsymbol{v}) = 0$ for all σ in Σ. We are going to show that $\boldsymbol{v} = \boldsymbol{0}$. Owing to Lemma 15.15, $\boldsymbol{v}_{|F}\times\boldsymbol{n}_F = \boldsymbol{0}$ and $(\nabla\times\boldsymbol{v})_{|F}\cdot\boldsymbol{n}_F = 0$ for every face $F \in \mathcal{F}_K$.
(1) Let us prove that $\boldsymbol{w} := \nabla\times\boldsymbol{v} = \boldsymbol{0}$. Since $\boldsymbol{w} \in \mathbb{P}_{k,3} \subset \mathbb{RT}_{k,3}$, it suffices to prove that $\boldsymbol{w}$ annihilates all the dofs of the $\mathbb{RT}_{k,3}$ element. Since $\boldsymbol{w}_{|F}\cdot\boldsymbol{n}_F = 0$, $\boldsymbol{w}$ annihilates all the dofs associated with the faces of K. In addition, if $k \geq 1$,

we observe that $\int_K \boldsymbol{w}\cdot\boldsymbol{q}\,\mathrm{d}x = \int_K \nabla\times\boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x = \int_K \boldsymbol{v}\cdot\nabla\times\boldsymbol{q}\,\mathrm{d}x$ for all $\boldsymbol{q} \in \mathbb{P}_{k-1,3}$, since $\boldsymbol{v}\times\boldsymbol{n}_K = \mathbf{0}$ on ∂K, where $\boldsymbol{n}_K$ is the outward unit normal to K. This in turn implies that $\int_K \boldsymbol{w}\cdot\boldsymbol{q}\,\mathrm{d}x = 0$ since $\nabla\times\boldsymbol{q} \in \mathbb{P}_{k-2,3}$ and $\sigma(\boldsymbol{v}) = 0$ for all $\sigma \in \Sigma^{\mathrm{c}}$ if $k \geq 2$. The statement is obvious if $k = 1$. In conclusion, $\nabla\times\boldsymbol{v} = \boldsymbol{w} = \mathbf{0}$.
(2) Using Lemma 15.10, we infer that there is $p \in \mathbb{P}_{k+1,3}$ such that $\boldsymbol{v} = \nabla p$. The condition $\boldsymbol{v}\times\boldsymbol{n}_K = \mathbf{0}$ on ∂K implies that p is constant on ∂K. Without loss of generality, we take this constant equal to zero. This in turn implies that $p = 0$ if $k \leq 2$ (see Exercise 7.5(iii)), so that it remains to consider the case $k \geq 3$. In this case, we infer that $p = \lambda_0 \ldots \lambda_3 r$ where λ_i, $i \in \{0{:}3\}$, are the barycentric coordinates in K and $r \in \mathbb{P}_{k-3,3}$. Writing this polynomial in the form $r(\boldsymbol{x}) = \sum_{|\alpha|\leq k-3} a_\alpha \boldsymbol{x}^\alpha$, we consider the field $\boldsymbol{q}(\boldsymbol{x}) := \sum_{|\alpha|\leq k-3} \frac{1}{\alpha_1+1} a_\alpha x_1 \boldsymbol{x}^\alpha \boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is the first vector of the canonical Cartesian basis of $\mathbb{R}^3$. Since $\boldsymbol{q} \in \mathbb{P}_{k-2,3}$, the fact that $\sigma(\boldsymbol{v}) = 0$ for all $\sigma \in \Sigma^{\mathrm{c}}$ implies that $\int_K \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x = 0$. Integration by parts and the fact that $p_{|\partial K} = 0$ yield $0 = \int_K \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x = -\int_K p\nabla\cdot\boldsymbol{q}\,\mathrm{d}x = -\int_K \lambda_0 \ldots \lambda_3 r^2\,\mathrm{d}x$. In conclusion, $r = 0$, so that $\boldsymbol{v} = \nabla p = \mathbf{0}$. □

The shape functions $\{\theta_i\}_{i\in\mathcal{N}}$ associated with the dofs $\{\sigma_i\}_{i\in\mathcal{N}}$ defined in (15.8) can be constructed by choosing a basis $\{\phi_i\}_{i\in\mathcal{N}}$ of the polynomial space $\mathbb{N}_{k,3}$ and by inverting the corresponding generalized Vandermonde matrix as explained in Proposition 5.5. Recall that this matrix has entries $\mathcal{V}_{ij} = \sigma_j(\phi_i)$ and that the i-th line of $\mathcal{V}^{-1}$ gives the components of the shape function θ_i in the basis $\{\phi_i\}_{i\in\mathcal{N}}$. The basis $\{\phi_i\}_{i\in\mathcal{N}}$ chosen in Bonazzoli and Rapetti [31] (built by dividing the simplex into smaller sub-simplices following the ideas in Rapetti and Bossavit [163], Christiansen and Rapetti [70]) is particularly interesting since the entries of $\mathcal{V}^{-1}$ are integers. One could also choose $\{\phi_i\}_{i\in\mathcal{N}}$ to be the hierarchical basis of $\mathbb{N}_{k,d}$ constructed in Fuentes et al. [103, §7.2]. This basis can be organized into functions attached to the the edges of K, the faces of K, and to K itself, in such a way that the generalized Vandermonde matrix $\mathcal{V}$ is block-triangular (notice though that this matrix is not block-diagonal). For earlier work on shape functions and basis functions for the $\mathbb{N}_{k,3}$ element, see Webb [197], Gopalakrishnan et al. [109].

Remark 15.17 (Dof independence). As in Remark 14.16, the results from Exercise 5.2 imply that the interpolation operator $\mathcal{I}_K^{\mathrm{c}}$ associated with the $\mathbb{N}_{k,3}$ element is independent of the bases $\{\mu_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{e}}\}}$, $\{\zeta_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}$, and $\{\psi_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ that are used to define the dofs in (15.8). The interpolation operator is also independent of the mappings $\boldsymbol{T}_E$, $\boldsymbol{T}_F$ and of the orientation vectors $\{\boldsymbol{t}_E\}_{E\in\mathcal{E}_K}$, $\{\boldsymbol{t}_{F,j}\}_{F\in\mathcal{F}_K, j\in\{1,2\}}$, and $\{\boldsymbol{t}_{K,j}\}_{j\in\{1,2,3\}}$. □

Remark 15.18 (Literature). The $\mathbb{N}_{k,d}$ finite element has been introduced by Nédélec [151]; see also Weil [198], Whitney [199] for $k = 0$. It is an accepted practice in the literature to call this element *edge element* or *Nédélec element*. See also Bossavit [36, Chap. 3], Hiptmair [117], Monk [145, Chap. 5]. □

15.4 Generation of Nédélec elements

Let $\widehat{K}$ be the reference simplex in $\mathbb{R}^3$. Let $\mathcal{T}_h$ be an affine simplicial mesh. Let $K = \boldsymbol{T}_K(\widehat{K})$ be a mesh cell where $\boldsymbol{T}_K : \widehat{K} \to K$ is the geometric mapping, and let $\mathbb{J}_K$ be the Jacobian matrix of $\boldsymbol{T}_K$. Let $F \in \mathcal{F}_K$ be a face of K. We have $F = \boldsymbol{T}_K(\widehat{F})$ where $\widehat{F} \in \mathcal{F}_{\widehat{K}}$ is a face of $\widehat{K}$. Similarly, let $E \in \mathcal{E}_K$ be an edge of K. We have $E = \boldsymbol{T}_K(\widehat{E})$ where $\widehat{E} \in \mathcal{E}_{\widehat{K}}$ is an edge of $\widehat{K}$. Using the increasing vertex-index enumeration, Theorem 10.8 shows that it is possible to orient the edges E and $\widehat{E}$ in a way that is compatible with the geometric mapping $\boldsymbol{T}_K$. This means that the unit tangent vectors $\boldsymbol{\tau}_E$ and $\widehat{\boldsymbol{\tau}}_{\widehat{E}}$ satisfy (10.6a), i.e., $\boldsymbol{\tau}_E = \boldsymbol{\Phi}_K^{\mathrm{c}}(\widehat{\boldsymbol{\tau}}_{\widehat{E}})$ with $\boldsymbol{\Phi}_K^{\mathrm{c}}$ defined in (9.14b). In other words, we have

$$\boldsymbol{\tau}_E \circ \boldsymbol{T}_{K|\widehat{E}} = \frac{1}{\|\mathbb{J}_K \widehat{\boldsymbol{\tau}}_{\widehat{E}}\|_{\ell^2}} \mathbb{J}_K \widehat{\boldsymbol{\tau}}_{\widehat{E}}. \tag{15.11}$$

Since $\boldsymbol{t}_E := |E|\boldsymbol{\tau}_E$, $\widehat{\boldsymbol{\tau}}_{\widehat{E}} := |\widehat{E}|\widehat{\boldsymbol{t}}_{\widehat{E}}$ and since $|E| = \|\mathbb{J}_K \widehat{\boldsymbol{\tau}}_{\widehat{E}}\|_{\ell^2}|\widehat{E}|$ owing to Lemma 9.12, we infer that

$$\boldsymbol{t}_E \circ \boldsymbol{T}_{K|\widehat{E}} = \mathbb{J}_K \widehat{\boldsymbol{t}}_{\widehat{E}}. \tag{15.12}$$

We also orient the faces of K by using the two edge vectors originating from the vertex with the lowest index in each face. We finally orient K by using the three edge vectors originating from the vertex with the lowest index in K. Reasoning as above, we infer that

$$\boldsymbol{t}_{F,j} \circ \boldsymbol{T}_{K|\widehat{F}} = \mathbb{J}_K \widehat{\boldsymbol{t}}_{\widehat{F},j},\ \forall j \in \{1,2\} \quad \boldsymbol{t}_{K,j} \circ \boldsymbol{T}_K = \mathbb{J}_K \widehat{\boldsymbol{t}}_{\widehat{K},j},\ \forall j \in \{1{:}3\}. \tag{15.13}$$

Recall the covariant Piola transformation introduced in (9.9b) such that

$$\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) := \mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v} \circ \boldsymbol{T}_K), \tag{15.14}$$

and the pullback by the geometric mapping such that $\psi_K^{\mathrm{g}}(q) := q \circ \boldsymbol{T}_K$.

Lemma 15.19 (Transformation of dofs). *Let $\boldsymbol{v} \in \boldsymbol{C}^0(K)$ and let $q \in C^0(K)$. The following holds true:*

$$\begin{aligned}
\frac{1}{|E|}\int_E (\boldsymbol{v}\cdot\boldsymbol{t}_E) q \,\mathrm{d}l &= \frac{1}{|\widehat{E}|}\int_{\widehat{E}} (\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{t}}_{\widehat{E}})\psi_K^{\mathrm{g}}(q)\,\mathrm{d}\widehat{l}, && \forall E \in \mathcal{E}_K,\\
\frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{t}_{F,j}) q \,\mathrm{d}s &= \frac{1}{|\widehat{F}|}\int_{\widehat{F}} (\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{t}}_{\widehat{F},j})\psi_K^{\mathrm{g}}(q)\,\mathrm{d}\widehat{s}, && \forall F \in \mathcal{F}_K,\ j \in \{1,2\},\\
\frac{1}{|K|}\int_K (\boldsymbol{v}\cdot\boldsymbol{t}_{K,j}) q \,\mathrm{d}x &= \frac{1}{|\widehat{K}|}\int_{\widehat{K}} (\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{t}}_{\widehat{K},j})\psi_K^{\mathrm{g}}(q)\,\mathrm{d}\widehat{x}, && \forall j \in \{1{:}3\}.
\end{aligned}$$

Proof. The first identity is nothing but (10.7b) from Lemma 10.4, which itself is a reformulation of (9.15b) from Lemma 9.13 (the fact that $\boldsymbol{T}_K$ is affine is not used here). The proof of the other two identities is similar to (9.15b) using (15.13) and the fact that $\mathrm{d}s = \frac{|F|}{|\widehat{F}|}\,\mathrm{d}\widehat{s}$, $\mathrm{d}x = \frac{|K|}{|\widehat{K}|}\,\mathrm{d}\widehat{x}$ since $\boldsymbol{T}_K$ is affine. For instance, we have

$$\begin{aligned}
\frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})q\,\mathrm{d}s &= \frac{1}{|F|}\int_{\widehat{F}} (\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})\circ\boldsymbol{T}_{K|\widehat{F}}(q\circ\boldsymbol{T}_{K|\widehat{F}})\frac{|F|}{|\widehat{F}|}\,\mathrm{d}\widehat{s} \\
&= \frac{1}{|\widehat{F}|}\int_{\widehat{F}} ((\mathbb{J}_K^{\mathsf{T}}\boldsymbol{v})\cdot(\mathbb{J}_K^{-1}\boldsymbol{t}_{F,j}))\circ\boldsymbol{T}_{K|\widehat{F}}\psi_K^{\mathrm{g}}(q)\,\mathrm{d}\widehat{s} \\
&= \frac{1}{|\widehat{F}|}\int_{\widehat{F}} (\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{t}}_{\widehat{F},j})\psi_K^{\mathrm{g}}(q)\,\mathrm{d}\widehat{s}. \qquad \square
\end{aligned}$$

Proposition 15.20 (Generation). *Let $(\widehat{K},\widehat{\boldsymbol{P}},\widehat{\Sigma})$ be a simplicial Nédélec element with edge, face, and cell dofs defined by using the polynomial bases $\{\mu_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{e}}\}}$, $\{\zeta_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}$ (if $k\ge 1$), and $\{\psi_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ (if $k\ge 2$) of $\mathbb{P}_{k,1}$, $\mathbb{P}_{k-1,2}$, and $\mathbb{P}_{k-2,3}$, respectively, as in* (15.8). *Assume that the geometric mapping $\boldsymbol{T}_K$ is affine and that* (15.12)-(15.13) *hold true. Then the finite element $(K,\boldsymbol{P}_K,\Sigma_K)$ generated using Proposition* 9.2 *with the covariant Piola transformation* (15.14) *is a simplicial Nédélec element with dofs*

$$\sigma_{E,m}^{\mathrm{e}}(\boldsymbol{v}) = \frac{1}{|E|}\int_E (\boldsymbol{v}\cdot\boldsymbol{t}_E)(\mu_m\circ\boldsymbol{T}_{K,E}^{-1})\,\mathrm{d}l, \quad \forall E\in\mathcal{E}_K, \tag{15.15a}$$

$$\sigma_{F,j,m}^{\mathrm{f}}(\boldsymbol{v}) = \frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})(\zeta_m\circ\boldsymbol{T}_{K,F}^{-1})\,\mathrm{d}s, \quad \forall F\in\mathcal{F}_K,\ \forall j\in\{1,2\}, \tag{15.15b}$$

$$\sigma_{j,m}^{\mathrm{c}}(\boldsymbol{v}) = \frac{1}{|K|}\int_K (\boldsymbol{v}\cdot\boldsymbol{t}_{K,j})(\psi_m\circ\boldsymbol{T}_K^{-1})\,\mathrm{d}x, \ \forall j\in\{1,2,3\}, \tag{15.15c}$$

where $\boldsymbol{T}_{K,E} := \boldsymbol{T}_{K|\widehat{E}}\circ\boldsymbol{T}_{\widehat{E}} : \widehat{S}^1\to E$ and $\boldsymbol{T}_{K,F} := \boldsymbol{T}_{K|\widehat{F}}\circ\boldsymbol{T}_{\widehat{F}} : \widehat{S}^2\to F$ are the affine bijective mappings that map vertices with increasing indices.

Proof. Let us first prove that $\boldsymbol{P}_K = \mathbb{N}_{k,3}$. We can write $\boldsymbol{T}_K(\widehat{\boldsymbol{x}}) := \mathbb{J}_K\widehat{\boldsymbol{x}} + \boldsymbol{b}_K$ with $\mathbb{J}_K\in\mathbb{R}^{3\times 3}$ and $\boldsymbol{b}_K\in\mathbb{R}^3$. Let $\boldsymbol{v}$ be a member of $\boldsymbol{P}_K$. Then $\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) = \widehat{\boldsymbol{p}}+\widehat{\boldsymbol{q}}$ with $\widehat{\boldsymbol{p}}\in\mathbb{P}_{k,3}$ and $\widehat{\boldsymbol{q}}\in\mathbb{S}_{k+1,3}$, yielding $\boldsymbol{v} = \mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{p}}\circ\boldsymbol{T}_K^{-1} + \mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{q}}\circ\boldsymbol{T}_K^{-1}$. Since each component of $\widehat{\boldsymbol{q}}$ is in $\mathbb{P}_{k+1,3}^{\mathrm{H}}$, we infer that $\widehat{\boldsymbol{q}}\circ\boldsymbol{T}_K^{-1}(\boldsymbol{x}) = \widehat{\boldsymbol{q}}(\mathbb{J}_K^{-1}\boldsymbol{x} - \mathbb{J}_K^{-1}\boldsymbol{b}_K) = \widehat{\boldsymbol{q}}(\mathbb{J}_K^{-1}\boldsymbol{x}) + \boldsymbol{r}(\boldsymbol{x})$, where $\boldsymbol{r}\in\mathbb{P}_{k,3}$; see Exercise 14.4. As a result, $\boldsymbol{v} = (\boldsymbol{p}+\boldsymbol{r})+\boldsymbol{q}$, where $\boldsymbol{p} = \mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{p}}\circ\boldsymbol{T}_K^{-1}\in\mathbb{P}_{k,3}$ and $\boldsymbol{q} = \mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{q}}\circ\mathbb{J}_K^{-1}$. Note that $\boldsymbol{p}+\boldsymbol{r}\in\mathbb{P}_{k,3}$ and $\widehat{\boldsymbol{q}}\circ\mathbb{J}_K^{-1}$ is a member of $\mathbb{P}_{k+1,3}^{\mathrm{H}}$, which implies that $\boldsymbol{q}$ is also in $\mathbb{P}_{k+1,3}^{\mathrm{H}}$. Moreover, $\boldsymbol{x}\cdot(\mathbb{J}_K^{-\mathsf{T}}\widehat{\boldsymbol{q}}(\mathbb{J}_K^{-1}\boldsymbol{x})) = (\mathbb{J}_K^{-1}\boldsymbol{x})\cdot\widehat{\boldsymbol{q}}(\mathbb{J}_K^{-1}\boldsymbol{x}) = 0$ which in turn implies that $\boldsymbol{q}\in\mathbb{S}_{k+1,3}$. In conclusion, $\boldsymbol{v}\in\mathbb{N}_{k,3}$, meaning that $\boldsymbol{P}_K\subset\mathbb{N}_{k,3}$. The converse statement follows from a dimension argument. Finally, the definition of the dofs results from Lemma 15.19, and the properties of the mappings $\boldsymbol{T}_{K,E}$ and $\boldsymbol{T}_{K,F}$ from those of $\boldsymbol{T}_K$, $\boldsymbol{T}_{\widehat{E}}$, and $\boldsymbol{T}_{\widehat{F}}$. $\square$

Remark 15.21 (Unit). The shape functions scale like the reciprocal of a length unit. $\square$

Remark 15.22 (Nonaffine meshes). Proposition 9.2 together with the map (15.14) can still be used to generate a finite element $(K, \boldsymbol{P}_K, \Sigma_K)$ if the geometric mapping $\boldsymbol{T}_K$ is nonaffine. The function space $\boldsymbol{P}_K$ and the dofs in Σ_K then differ from those of the $\mathbb{N}_{k,3}$ element. □

15.5 Other $\boldsymbol{H}(\mathrm{curl})$ finite elements

15.5.1 Nédélec elements of the second kind

Nédélec elements of the second kind [152] offer an interesting alternative to those investigated in §15.3 (and often called Nédélec elements of the first kind) since in this case the polynomial space is $\boldsymbol{P} := \mathbb{P}_{k,d} \subsetneq \mathbb{N}_{k,d}$, $k \geq 1$. This space is optimal from the approximation viewpoint. The price to pay for this simplification is that the curl operator maps onto $\mathbb{P}_{k-1,d}$. This is not a limitation if the functions to be interpolated are curl-free.

Let K be a simplex in $\mathbb{R}^3$. The dofs are attached to the edges of K, its faces (for $k \geq 2$), and to K itself (for $k \geq 3$). The edge dofs are defined in (15.8a) as for the elements of the first kind, whereas the face dofs are moments on each face of K of the tangential component against a set of basis functions of $\mathbb{RT}_{k-2,2}$ up to a contravariant Piola transformation (instead of basis functions of $\mathbb{P}_{k-1,2}$ for the elements of the first kind), and the cell dofs are moments against a set of basis functions of $\mathbb{RT}_{k-3,3}$ (instead of basis functions of $\mathbb{P}_{k-2,3}$ for the elements of the first kind). It is shown in [152] that the triple $(K, \boldsymbol{P}, \Sigma)$ is a finite element. Hierarchical basis functions for the Nédélec element of the second kind are constructed in Ainsworth and Coyle [6], Schöberl and Zaglmayr [176].

15.5.2 Cartesian Nédélec elements

The Cartesian version of Nédélec elements have been introduced in Nédélec [151, pp. 330-333]. Let us briefly review these elements (see Exercise 15.8 for the proofs). We focus on the case $d = 3$, since two-dimensional Cartesian Nédélec elements can be built by a rotation of the two-dimensional Cartesian Raviart–Thomas elements from §14.5.2. Let $k \in \mathbb{N}$ and define

$$\mathbb{N}^{\square}_{k,3} := \mathbb{Q}_{k,k+1,k+1} \times \mathbb{Q}_{k+1,k,k+1} \times \mathbb{Q}_{k+1,k+1,k}, \tag{15.16}$$

where the anisotropic polynomial spaces $\mathbb{Q}_{\alpha_1,\alpha_2,\alpha_3}$ are defined in §14.5.2. Since the three anisotropic spaces in (15.16) have dimension $(k+1)(k+2)^2$, we have $\dim(\mathbb{N}^{\square}_{k,3}) = 3(k+1)(k+2)^2$.

Let $K := (0,1)^3$ be the unit cube in $\mathbb{R}^3$. Let $\mathcal{F}_K$ collect the six faces of K, and let $\mathcal{E}_K$ collect the twelve edges of K. Let $\boldsymbol{T}_F$, $F \in \mathcal{F}_K$ (resp., $\boldsymbol{T}_E$, $E \in \mathcal{E}_K$) be an affine geometric mapping from $[0,1]^2$ onto F (resp., $[0,1]$ onto E). Let $\widehat{t}^{\mathrm{e}} := 1$ be the canonical basis of $\mathbb{R}$. We orient $E \in \mathcal{E}_K$ using $\boldsymbol{t}_E := \mathbb{J}_E \widehat{t}^{\mathrm{e}}$, where $\mathbb{J}_E$ is the Jacobian matrix of $\boldsymbol{T}_E$. Let $\{\boldsymbol{t}_j^{\mathrm{f}}\}_{j\in\{1,2\}}$ be the canonical basis of $\mathbb{R}^2$. We orient $F \in \mathcal{F}_K$ by using $\boldsymbol{t}_{F,j} := \mathbb{J}_F \widehat{\boldsymbol{t}}_j^{\mathrm{f}}$ for all $j \in \{1,2\}$, where $\mathbb{J}_F$ is the Jacobian matrix of $\boldsymbol{T}_F$. We orient K by using the canonical basis $\{\boldsymbol{t}_{K,j} := \boldsymbol{e}_j\}_{j\in\{1:3\}}$ of $\mathbb{R}^3$. Let Σ be the set composed of the following linear forms:

$$\sigma^{\mathrm{e}}_{E,m}(\boldsymbol{v}) := \frac{1}{|E|}\int_E (\boldsymbol{v}\cdot\boldsymbol{t}_E)(\mu_m \circ \boldsymbol{T}_E^{-1})\,\mathrm{d}l, \quad \forall E \in \mathcal{E}_K, \tag{15.17a}$$

$$\sigma^{\mathrm{f}}_{F,j,m}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})(\zeta_{j,m} \circ \boldsymbol{T}_F^{-1})\,\mathrm{d}s, \ \forall F \in \mathcal{F}_K,\ \forall j \in \{1,2\}, \tag{15.17b}$$

$$\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) := \frac{1}{|K|}\int_K (\boldsymbol{v}\cdot\boldsymbol{t}_{K,j})\psi_{j,m}\,\mathrm{d}x, \quad \forall j \in \{1,2,3\}, \tag{15.17c}$$

where $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k,1}$ with $n^{\mathrm{e}}_{\mathrm{sh}} := k+1$, $\{\zeta_{j,m}\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ is a basis of the space $\mathbb{Q}_{k,k-1}$ if $j=1$ and $\mathbb{Q}_{k-1,k}$ if $j=2$, with $n^{\mathrm{f}}_{\mathrm{sh}} := (k+1)k$ (if $k \ge 1$), and $\{\psi_{j,m}\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$ is a basis of the space $\mathbb{Q}_{k,k-1,k-1}$ if $j=1$, $\mathbb{Q}_{k-1,k,k-1}$ if $j=2$, and $\mathbb{Q}_{k-1,k-1,k}$ if $j=3$, with $n^{\mathrm{c}}_{\mathrm{sh}} := (k+1)k^2$ (if $k \ge 1$).

Proposition 15.23 (Finite element). *$(K, \mathbb{N}^{\square}_{k,3}, \Sigma)$ is a finite element.*

Cartesian Nédélec elements can be generated for all the mesh cells of an affine mesh composed of parallelotopes by using affine geometric mappings and the covariant Piola transformation. Recall however that orienting such meshes requires some care; see Theorem 10.10.

Exercises

Exercise 15.1 ($\mathbb{S}_{1,d}$). (i) Prove that for all $\boldsymbol{q} \in \mathbb{S}_{1,d}$, there is a unique skew-symmetric matrix $\mathbb{Q}$ s.t. $\boldsymbol{q}(\boldsymbol{x}) = \mathbb{Q}\boldsymbol{x}$. (ii) Propose a basis of $\mathbb{S}_{1,d}$. (iii) Show that $\boldsymbol{q} \in \mathbb{S}_{1,3}$ if and only if there is $\boldsymbol{b} \in \mathbb{R}^3$ such that $\boldsymbol{q}(\boldsymbol{x}) = \boldsymbol{b}\times\boldsymbol{x}$.

Exercise 15.2 (Cross product). (i) Prove that $(\mathbb{A}\boldsymbol{b})\times(\mathbb{A}\boldsymbol{c}) = \mathbb{A}(\boldsymbol{b}\times\boldsymbol{c})$ for every rotation matrix $\mathbb{A} \in \mathbb{R}^{3\times 3}$ and all $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$. (*Hint*: use Exercise 9.5.) (ii) Show that $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$. (*Hint*: $(\boldsymbol{a}\times\boldsymbol{b})_k = \varepsilon_{ikj}a_ib_j$ with Levi-Civita tensor ε_{ikj}; see also the proof of Lemma 9.6.) (iii) Prove that $-(\boldsymbol{b}\times\boldsymbol{n})\times\boldsymbol{n} + (\boldsymbol{b}\cdot\boldsymbol{n})\boldsymbol{n} = \boldsymbol{b}$ if $\boldsymbol{n}$ is a unit vector.

Exercise 15.3 ($\mathbb{N}_{0,3}$). (i) Prove (15.4). (*Hint*: verify that $\boldsymbol{t}_E\cdot\nabla\lambda_q = 1$ and $\boldsymbol{t}_E\cdot\nabla\lambda_p = -1$.) (ii) Prove that $\boldsymbol{v} = \langle\boldsymbol{v}\rangle_K + \frac{1}{2}(\nabla\times\boldsymbol{v})\times(\boldsymbol{x}-\boldsymbol{c}_K)$ for all $\boldsymbol{v} \in \mathbb{N}_{0,3}$, where $\langle\boldsymbol{v}\rangle_K$ is the mean value of $\boldsymbol{v}$ on K and $\boldsymbol{c}_K$ is the barycenter of K. (*Hint*: $\nabla\times(\boldsymbol{b}\times\boldsymbol{x}) = 2\boldsymbol{b}$ for $\boldsymbol{b} \in \mathbb{R}^3$.) (iii) Let $\boldsymbol{\theta}^{\mathrm{e}}_E$ be the shape function associated with the edge $E \in \mathcal{E}_K$. Let $F \in \mathcal{F}_K$ with unit normal $\boldsymbol{n}_{K|F}$ pointing outward K. Prove that $(\boldsymbol{\theta}^{\mathrm{e}}_E)_{|F}\times\boldsymbol{n}_{K|F} = \boldsymbol{0}$ if E is not an edge of F, and $\int_F \boldsymbol{\theta}^{\mathrm{e}}_E\times\boldsymbol{n}_{K|F}\,\mathrm{d}s =$

$\iota_{E,F}(\boldsymbol{c}_E - \boldsymbol{c}_F)$ otherwise, where $\boldsymbol{c}_E$ is the barycenter of E, $\boldsymbol{c}_F$ that of F, and $\iota_{E,F} = -1$ if $\boldsymbol{n}_{K|F}\times\boldsymbol{t}_E$ points outward F, $\iota_{E,F} = 1$ otherwise. (*Hint*: use Lemma 15.15 and Exercise 14.1(ii).) (iv) Let $\mathcal{F}_E$ collect the two faces sharing $E \in \mathcal{E}_K$. Prove that $\int_K \boldsymbol{\theta}_E^{\mathrm{e}}\,\mathrm{d}x = \frac{1}{2}\sum_{F\in\mathcal{F}_E}\iota_{E,F}(\boldsymbol{c}_F - \boldsymbol{c}_K)\times(\boldsymbol{c}_E - \boldsymbol{c}_F)$. (*Hint*: take the inner product with an arbitrary vector $\boldsymbol{e} \in \mathbb{R}^3$ and introduce the function $\boldsymbol{\psi}(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{e}\times(\boldsymbol{x} - \boldsymbol{c}_K)$.)

Exercise 15.4 (Rotated $\mathbb{RT}_{k,2}$). Prove Lemma 15.9. (*Hint*: observe that $\boldsymbol{R}_{\frac{\pi}{2}}(\mathbb{P}_{k,2}) = \mathbb{P}_{k,2}$ and $\mathbb{S}_{k+1,2} = \boldsymbol{R}_{\frac{\pi}{2}}(\boldsymbol{x})\mathbb{P}^{\mathrm{H}}_{k,2}$.)

Exercise 15.5 (Hodge decomposition). Prove that for all $k \in \mathbb{N}$,

$$\mathbb{P}_{k+1,d} = \mathbb{N}_{k,d} \oplus \nabla\mathbb{P}^{\mathrm{H}}_{k+2,d}.$$

(*Hint*: compute $\mathbb{N}_{k,d} \cap \nabla\mathbb{P}^{\mathrm{H}}_{k+2,d}$, and use a dimension argument.)

Exercise 15.6 (Face element). We use the notation from the proof of Lemma 15.15. Let $F \in \mathcal{F}_K$. Let $\boldsymbol{T}_F : \widehat{S}^2 \to F$ be an affine bijective mapping. Let $\mathbb{J}_F$ be the Jacobian matrix of $\boldsymbol{T}_F$. Let $\boldsymbol{v} \in \mathbb{N}_{k,3}$ and let $\widehat{\boldsymbol{v}} := \mathbb{J}_F^{\mathsf{T}}(\mathbb{I}_3 - \boldsymbol{n}_F\otimes\boldsymbol{n}_F)(\boldsymbol{v}\circ\boldsymbol{T}_F)$. Show that $\widehat{\boldsymbol{v}} \in \mathbb{N}_{k,2}$. (*Hint*: compute $\widehat{\boldsymbol{y}}^{\mathsf{T}}\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{y}})$ and apply the result from Exercise 14.4.)

Exercise 15.7 (Geometric mapping $\boldsymbol{T}_A$). Let A be an affine subspace of $\mathbb{R}^d$ of dimension $l \in \{1{:}d-1\}$, $d \geq 2$. Let $\boldsymbol{a} \in A$ and let $\boldsymbol{P}_A(\boldsymbol{x}) := \boldsymbol{a} + \Pi_A(\boldsymbol{x} - \boldsymbol{a})$ be the orthogonal projection onto A, where $\Pi_A \in \mathbb{R}^{d\times d}$. (i) Let $\boldsymbol{n} \in \mathbb{R}^d$ be such that $\boldsymbol{n}\cdot(\boldsymbol{x} - \boldsymbol{y}) = 0$ for all $\boldsymbol{x}, \boldsymbol{y} \in A$ (we say that $\boldsymbol{n}$ is normal to A). Show that $\Pi_A\boldsymbol{n} = 0$. Let $\boldsymbol{t} \in \mathbb{R}^d$ be such that $\boldsymbol{a} + \boldsymbol{t} \in A$ (we say that $\boldsymbol{t}$ is tangent to A). Show that $\Pi_A(\boldsymbol{t}) = \boldsymbol{t}$. (ii) Let $q \in \mathbb{P}_{k,l}$ and let $\tilde{q}(\boldsymbol{x}) := q(\boldsymbol{T}_A^{-1}\circ\boldsymbol{P}_A(\boldsymbol{x}))$. Compute $\nabla\tilde{q}$. (iii) Show that there are $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_l$ tangent vectors and $q_1, \ldots, q_l$ polynomials in $\mathbb{P}_{k,l}$ such that $\nabla\tilde{q}(\boldsymbol{x}) = \sum_{s\in\{1:l\}} q_s(\boldsymbol{T}_A^{-1}(\boldsymbol{x}))\boldsymbol{t}_s$ for all $\boldsymbol{x} \in A$. (iv) Let $\boldsymbol{t}$ be a tangent vector. Show that there is $\mu \in \mathbb{P}_{k,l}$ such that $\boldsymbol{t}\cdot\nabla\tilde{q}(\boldsymbol{x}) = \mu(\boldsymbol{T}_A^{-1}(\boldsymbol{x}))$.

Exercise 15.8 (Cartesian Nédélec element). (i) Propose a basis for $\mathbb{N}^{\square}_{0,3}$. (ii) Prove Proposition 15.23. (*Hint*: accept as a fact that any field $\boldsymbol{v} \in \mathbb{N}^{\square}_{k,3}$ annihiliating all the edge and faces dofs defined in (15.17) satisfies $\boldsymbol{v}_{|F}\times\boldsymbol{n}_F = \boldsymbol{0}$ for all $F \in \mathcal{F}_K$; then adapt the proof of Lemma 15.16 by using the $\mathbb{RT}^{\square}_{k,3}$ finite element defined in §14.5.2.)