Part III, Chapter 16

Local interpolation in $\boldsymbol{H}(\mathrm{div})$ and $\boldsymbol{H}(\mathrm{curl})$ (I)

In this chapter and the next one, we study the interpolation operators associated with the finite elements introduced in Chapters 14 and 15. We consider a shape-regular sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of affine simplicial meshes with a generation-compatible orientation (this is possible owing to Theorem 10.8). In the present chapter, we show how the degrees of freedom (dofs) attached to the faces and the edges can be extended by using the scale of the Sobolev spaces. On the way, we discover fundamental commuting properties of the interpolation operators embodied in the de Rham complex. In the next chapter, we study a different way of extending the dofs attached to the faces and the edges by requiring some integrability of the divergence or the curl.

16.1 Local interpolation in $\boldsymbol{H}(\mathrm{div})$

The goal of this section is to extend the dofs of the $\mathbb{RT}_{k,d}$ finite element introduced in Chapter 14 and to study the properties of the resulting interpolation operator.

16.1.1 Extending the dofs

Let $K \in \mathcal{T}_h$ be a simplex in $\mathbb{R}^d$ with $d \geq 2$. We generate a $\mathbb{RT}_{k,d}$ finite element in K from the $\mathbb{RT}_{k,d}$ finite element in the reference cell $\widehat{K}$ by using Proposition 14.19. Hence, the dofs in K consist of the following face dofs and cell dofs (if $k \geq 1$): For all $\boldsymbol{v} \in \mathbb{RT}_{k,d}$,

$$\sigma^{\mathrm{f}}_{F,m}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}{\cdot}\boldsymbol{\nu}_F)(\zeta_m \circ \boldsymbol{T}^{-1}_{K,F})\,\mathrm{d}s, \qquad \forall F \in \mathcal{F}_K, \tag{16.1a}$$

$$\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) := \frac{1}{|K|}\int_K (\boldsymbol{v}{\cdot}\boldsymbol{\nu}_{K,j})(\psi_m \circ \boldsymbol{T}^{-1}_{K})\,\mathrm{d}x, \quad \forall j \in \{1{:}d\}, \tag{16.1b}$$

where $\{\zeta_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}$, $\{\psi_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ are bases of $\mathbb{P}_{k,d-1}$, $\mathbb{P}_{k-1,d}$ ($k\ge 1$), respectively, $\boldsymbol{\nu}_F$ is the normal vector orienting F, $\{\boldsymbol{\nu}_{K,j} := |F_j|\boldsymbol{n}_{F_j}\}_{j\in\{1:d\}}$ are the vectors orienting K, and $\boldsymbol{T}_{K,F} : \widehat{S}^{d-1}\to F$, $\boldsymbol{T}_K : \widehat{K}\to K$ are geometric mappings. The local dofs in K are collectively denoted by $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$.

We are going to extend the above dofs to the following functional space:

$$\boldsymbol{V}^{\mathrm{d}}(K) := \boldsymbol{W}^{s,p}(K), \quad sp>1,\ p\in(1,\infty) \text{ or } s=1,\ p=1, \tag{16.2}$$

recalling that $\boldsymbol{W}^{s,p}(K) := W^{s,p}(K;\mathbb{R}^d)$. The idea behind (16.2) is to invoke a trace theorem (Theorem 3.15) to give a meaning to the face dofs. Fixing the real number p in (16.2), one wants to take s as small as possible to make the space $\boldsymbol{V}^{\mathrm{d}}(K)$ as large as possible. Thus, we can assume without loss of generality that $s\le 1$. We can also take $p=\infty$ and $s=1$ in (16.2).

Proposition 16.1 (Extended dofs). *Let $\boldsymbol{V}^{\mathrm{d}}(K)$ be defined in* (16.2)*. Let $\boldsymbol{V}^{\mathrm{d}}(\widehat{K})$ be defined similarly. Then the contravariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{d}}$ is in $\mathcal{L}(\boldsymbol{V}^{\mathrm{d}}(K);\boldsymbol{V}^{\mathrm{d}}(\widehat{K}))$. Moreover, the local dofs are in $\mathcal{L}(\boldsymbol{V}^{\mathrm{d}}(K);\mathbb{R})$ and there is c s.t. for all $\boldsymbol{v}\in\boldsymbol{V}^{\mathrm{d}}(K)$, all $K\in\mathcal{T}_h$, and all $h\in\mathcal{H}$,*

$$\max_{i\in\mathcal{N}} |\sigma_{K,i}(\boldsymbol{v})| \le c\, h_K^{d-1-\frac{d}{p}} \big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K^s|\boldsymbol{v}|_{\boldsymbol{W}^{s,p}(K)}\big). \tag{16.3}$$

Proof. (1) Let $\boldsymbol{v}\in\boldsymbol{V}^{\mathrm{d}}(K)$. Since the mesh is affine and $\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) := \mathbb{A}_K^{\mathrm{d}}(\boldsymbol{v}\circ\boldsymbol{T}_K)$ with $\mathbb{A}_K^{\mathrm{d}} := \det(\mathbb{J}_K)\mathbb{J}_K^{-1}$, we can apply Lemma 11.7 to obtain

$$\|\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(\widehat{K})} \le c\,\|\mathbb{A}_K^{\mathrm{d}}\|_{\ell^2}|\det(\mathbb{J}_K)|^{-\frac{1}{p}}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} \le c'\,h_K^{d-1-\frac{d}{p}}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)},$$

where the second bound follows from the regularity of the mesh sequence. Moreover, letting $\gamma_K := |\det(\mathbb{J}_K)|^{-1}\|\mathbb{J}_K\|_{\ell^2}^d$ if $s<1$ and $\gamma_K := 1$ if $s=1$, as in Lemma 11.7, we obtain

$$\begin{aligned}|\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})|_{\boldsymbol{W}^{s,p}(\widehat{K})} &\le c\,\gamma_K^{\frac{1}{p}}\|\mathbb{A}_K^{\mathrm{d}}\|_{\ell^2}\|\mathbb{J}_K\|_{\ell^2}^s|\det(\mathbb{J}_K)|^{-\frac{1}{p}}|\boldsymbol{v}|_{\boldsymbol{W}^{s,p}(K)}\\ &\le c'\,h_K^{d-1-\frac{d}{p}+s}|\boldsymbol{v}|_{\boldsymbol{W}^{s,p}(K)},\end{aligned}$$

where the second bound follows from the regularity of the mesh sequence. The above bounds show that $\boldsymbol{\psi}_K^{\mathrm{d}}\in\mathcal{L}(\boldsymbol{V}^{\mathrm{d}}(K);\boldsymbol{V}^{\mathrm{d}}(\widehat{K}))$ with

$$\|\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(\widehat{K})} + \ell_{\widehat{K}}^s|\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})|_{\boldsymbol{W}^{s,p}(\widehat{K})} \le c\,h_K^{d-1-\frac{d}{p}}\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K^s|\boldsymbol{v}|_{\boldsymbol{W}^{s,p}(K)}\big),$$

where $\ell_{\widehat{K}} := 1$ is a length scale associated with the reference cell $\widehat{K}$.
(2) Since the local dofs in K are s.t. $\sigma_{K,i} := \widehat{\sigma}_i\circ\boldsymbol{\psi}_K^{\mathrm{d}}$ for all $i\in\mathcal{N}$, we need to bound the reference dofs $\{\widehat{\sigma}_i\}_{i\in\mathcal{N}}$. Let $\widehat{\boldsymbol{v}}\in\boldsymbol{V}^{\mathrm{d}}(\widehat{K})$. If $\widehat{\sigma}_i$ is a cell dof, we have $|\widehat{\sigma}_i(\widehat{\boldsymbol{v}})|\le\widehat{c}\|\widehat{\boldsymbol{v}}\|_{\boldsymbol{L}^p(\widehat{K})}$, whereas if $\widehat{\sigma}_i$ is a face dof, we have $|\widehat{\sigma}_i(\widehat{\boldsymbol{v}})|\le\widehat{c}(\|\widehat{\boldsymbol{v}}\|_{\boldsymbol{L}^p(\widehat{K})}+\ell_{\widehat{K}}^s|\widehat{\boldsymbol{v}}|_{\boldsymbol{W}^{s,p}(\widehat{K})})$ owing to Theorem 3.15 since $sp>1$ if $p\in(1,\infty)$

and $s = 1$ if $p = 1$. The above bound on $\boldsymbol{\psi}_K^{\mathrm{d}}$ shows that the local dofs in K are in $\mathcal{L}(\boldsymbol{V}^{\mathrm{d}}(K); \mathbb{R})$ and that (16.3) holds true. $\square$

16.1.2 Commuting and approximation properties

In this section, we study the properties of the local Raviart–Thomas interpolation operator

$$\mathcal{I}_K^{\mathrm{d}} : \boldsymbol{V}^{\mathrm{d}}(K) \to \mathbb{RT}_{k,d} \tag{16.4}$$

with $\boldsymbol{V}^{\mathrm{d}}(K)$ defined in (16.2). Recall that for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$, $\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})$ is defined as the unique polynomial in $\mathbb{RT}_{k,d}$ s.t. the function $(\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v}) - \boldsymbol{v})$ annihilates all the $\mathbb{RT}_{k,d}$ dofs. Let us start with an important commuting property. Let $\mathcal{I}_K^{\mathrm{b}} : V^{\mathrm{b}}(K) := L^1(K) \to \mathbb{P}_{k,d}$ be the L^2-orthogonal projection onto $\mathbb{P}_{k,d}$, i.e., $\int_K (\mathcal{I}_K^{\mathrm{b}}(\phi) - \phi) q \,\mathrm{d}x = 0$ for all $\phi \in L^1(K)$ and all $q \in \mathbb{P}_{k,d}$; see §11.5.3.

Lemma 16.2 (Commuting with $\nabla\cdot$). *The following diagram commutes:*

$$\begin{array}{ccc}
\check{\boldsymbol{V}}^{\mathrm{d}}(K) & \xrightarrow{\ \nabla\cdot\ } & V^{\mathrm{b}}(K) \\
\downarrow \mathcal{I}_K^{\mathrm{d}} & & \downarrow \mathcal{I}_K^{\mathrm{b}} \\
\mathbb{RT}_{k,d} & \xrightarrow{\ \nabla\cdot\ } & \mathbb{P}_{k,d}
\end{array}$$

where $\check{\boldsymbol{V}}^{\mathrm{d}}(K) := \{\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K) \mid \nabla\cdot\boldsymbol{v} \in V^{\mathrm{b}}(K)\}$. In other words, we have

$$\nabla\cdot(\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})) = \mathcal{I}_K^{\mathrm{b}}(\nabla\cdot\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \check{\boldsymbol{V}}^{\mathrm{d}}(K). \tag{16.5}$$

Proof. Let $\boldsymbol{v} \in \check{\boldsymbol{V}}^{\mathrm{d}}(K)$. Since the divergence operator maps $\mathbb{RT}_{k,d}$ to $\mathbb{P}_{k,d}$ by Lemma 14.9, we have $\nabla\cdot(\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})) \in \mathbb{P}_{k,d}$. Therefore, it suffices to show that $\int_K (\mathcal{I}_K^{\mathrm{b}}(\nabla\cdot\boldsymbol{v}) - \nabla\cdot(\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v}))) q \,\mathrm{d}x = 0$ for all $q \in \mathbb{P}_{k,d}$, and by definition of $\mathcal{I}_K^{\mathrm{b}}$, this amounts to $\int_K (\nabla\cdot\boldsymbol{\zeta}) q \,\mathrm{d}x = 0$ for all $q \in \mathbb{P}_{k,d}$ where $\boldsymbol{\zeta} := \boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})$. Note that by definition $\boldsymbol{\zeta}$ annihilates all the dofs of the $\mathbb{RT}_{k,d}$ element in K. Integrating by parts and decomposing the boundary integral over the faces in $\mathcal{F}_K$, we infer that

$$\int_K (\nabla\cdot\boldsymbol{\zeta}) q \,\mathrm{d}x = -\int_K \boldsymbol{\zeta}\cdot\nabla q \,\mathrm{d}x + \sum_{F \in \mathcal{F}_K} \int_F \boldsymbol{\zeta}\cdot\boldsymbol{n}_{K|F} q_{|F} \,\mathrm{d}s,$$

where $\boldsymbol{n}_K$ is the outward unit normal to K. If $k \geq 1$, we use that $\{\boldsymbol{\nu}_{K,j}\}_{j \in \{1:d\}}$ is a basis of $\mathbb{R}^d$ and $\{\psi_m\}_{m \in \{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ is a basis of $\mathbb{P}_{k-1,d}$ to infer that there are real numbers $\alpha_{j,m}$ s.t. $\nabla q = \sum_{j \in \{1:d\}} \sum_{m \in \{1:n_{\mathrm{sh}}^{\mathrm{c}}\}} \alpha_{j,m} \boldsymbol{\nu}_{K,j} (\psi_m \circ \boldsymbol{T}_K^{-1})$. Recalling that $\boldsymbol{\zeta}$ annihilates all the cell dofs, we obtain

$$\int_K \boldsymbol{\zeta}\cdot\nabla q \,\mathrm{d}x = 0.$$

If $k = 0$, this equality is trivial. Let us now consider the integrals over the faces of K. For all $F \in \mathcal{F}_K$, we use that $\boldsymbol{\nu}_F = |F|\boldsymbol{n}_F$ and $\boldsymbol{n}_F = \pm\boldsymbol{n}_{K|F}$, $q_{|F} \circ \boldsymbol{T}_{K,F}^{-1} \in \mathbb{P}_{k,d-1}$ owing to Lemma 7.10, and that $\boldsymbol{\zeta}$ annihilates all the face dofs attached to F to infer that

$$\int_F \boldsymbol{\zeta}\cdot\boldsymbol{n}_{K|F} q_{|F} \,\mathrm{d}s = 0.$$

This concludes the proof. □

Example 16.3 (Gradient interpolation). Let us set $s = p := 1$ in (16.2). Let $\phi \in W^{2,1}(K)$. Then $\nabla\phi \in \boldsymbol{W}^{1,1}(K) = \boldsymbol{V}^{\mathrm{d}}(K)$, and since $\nabla\cdot(\nabla\phi) \in L^1(K)$, we have $\nabla\phi \in \check{\boldsymbol{V}}^{\mathrm{d}}(K)$. Lemma 16.2 implies that $\nabla\cdot\mathcal{I}_K^{\mathrm{d}}(\nabla\phi) = \mathcal{I}_K^{\mathrm{b}}(\Delta\phi)$. □

Theorem 16.4 (Approximation, $r \geq 1$). *Let $\mathcal{I}_K^{\mathrm{d}}$ be the $\mathbb{RT}_{k,d}$ interpolation operator in K. There is c s.t. for every integers $r \in \{1{:}k+1\}$ and $m \in \{0{:}r\}$, all $p \in [1,\infty]$, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,*

$$|\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,p}(K)} \leq c\, h_K^{r-m} |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}. \tag{16.6}$$

Moreover, for every integers $r \in \{0{:}k+1\}$ and $m \in \{0{:}r\}$, all $p \in [1,\infty]$, all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$ such that $\nabla\cdot\boldsymbol{v} \in W^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, we have

$$|\nabla\cdot(\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v}))|_{W^{m,p}(K)} \leq c\, h_K^{r-m} |\nabla\cdot\boldsymbol{v}|_{W^{r,p}(K)}. \tag{16.7}$$

Proof. Let us start with (16.6). We apply Theorem 11.13. The contravariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{d}}$ is of the form (11.1) with $\mathbb{A}_K^{\mathrm{d}} := \det(\mathbb{J}_K)\mathbb{J}_K^{-1}$, which satisfies the bound (11.12) with $\gamma := 1$. Moreover, we can take $l := 1$ in Theorem 11.13 since $\boldsymbol{W}^{1,p}(\widehat{K}) \hookrightarrow \boldsymbol{V}^{\mathrm{d}}(\widehat{K})$. Since $l \leq k+1$, we can apply the estimate (11.14), which is nothing but (16.6). Finally, to prove (16.7), we use Lemma 16.2 to infer that $\nabla\cdot(\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})) = \nabla\cdot\boldsymbol{v} - \mathcal{I}_K^{\mathrm{b}}(\nabla\cdot\boldsymbol{v})$, and we conclude using Lemma 11.18 ($P_K = \mathbb{P}_{k,d}$ since the mesh is affine). □

Remark 16.5 (Error on the divergence). It is remarkable that the bound on $\nabla\cdot(\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v}))$ only depends on the smoothness of $\nabla\cdot\boldsymbol{v}$. This is a direct consequence of the commuting property stated in Lemma 16.2. □

Theorem 16.6 (Approximation, $r > \frac{1}{p}$). *The estimate* (16.6) *holds true for all $r \in (\frac{1}{p}, 1)$, $m = 0$, all $p \in (1,\infty)$, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, and c can grow unboundedly as $r \downarrow \frac{1}{p}$.*

Proof. We first prove the following stability property:

$$\|\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \leq c\big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K^r |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}\big), \tag{16.8}$$

for all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$ (notice that $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$ since $rp > 1$). The triangle inequality, Proposition 12.5, and the regularity of the mesh sequence imply that

$$\|\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \le \sum_{i\in\mathcal{N}} |\sigma_{K,i}(\boldsymbol{v})|\,\|\boldsymbol{\theta}_{K,i}\|_{\boldsymbol{L}^p(K)} \le c\,h_K^{\frac{d}{p}+1-d} \sum_{i\in\mathcal{N}} |\sigma_{K,i}(\boldsymbol{v})|.$$

Hence, (16.8) follows from the bound (16.3) on the local dofs in K. Since $\mathbb{P}_{0,d} \subset \mathbb{RT}_{k,d}$ is pointwise invariant under $\mathcal{I}_K^{\mathrm{d}}$, we infer that

$$\begin{aligned}\|\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} &\le \inf_{\boldsymbol{q}\in\mathbb{P}_{0,d}} \big(\|\boldsymbol{v}-\boldsymbol{q}\|_{\boldsymbol{L}^p(K)} + \|\mathcal{I}_K^{\mathrm{d}}(\boldsymbol{v}-\boldsymbol{q})\|_{\boldsymbol{L}^p(K)}\big)\\ &\le c \inf_{\boldsymbol{q}\in\mathbb{P}_{0,d}} \big(\|\boldsymbol{v}-\boldsymbol{q}\|_{\boldsymbol{L}^p(K)} + h_K^r|\boldsymbol{v}-\boldsymbol{q}|_{\boldsymbol{W}^{r,p}(K)}\big)\\ &\le c' h_K^r |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)},\end{aligned}$$

where we used (16.8), $|\boldsymbol{v}-\boldsymbol{q}|_{\boldsymbol{W}^{r,p}(K)} = |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}$ since $\boldsymbol{q}$ is constant on K, and the fractional Poincaré–Steklov inequality (12.14) in K. □

16.2 Local interpolation in $\boldsymbol{H}(\mathrm{curl})$

The goal of this section is to extend the dofs of the $\mathbb{N}_{k,d}$ finite element introduced in Chapter 15 for $d = 3$ and to study the properties of the resulting interpolation operator.

16.2.1 Extending the dofs

Let K be a simplex in $\mathbb{R}^d$ with $d = 3$. We generate a $\mathbb{N}_{k,d}$ finite element in K from the $\mathbb{N}_{k,d}$ finite element in the reference cell $\widehat{K}$ by using Proposition 15.20. Hence, the dofs in K consist of the following edge dofs, face dofs (if $k \ge 1$), and cell dofs (if $k \ge 2$): For all $\boldsymbol{v} \in \mathbb{N}_{k,d}$,

$$\sigma_{E,m}^{\mathrm{e}}(\boldsymbol{v}) := \frac{1}{|E|}\int_E (\boldsymbol{v}\cdot\boldsymbol{t}_E)(\mu_m \circ \boldsymbol{T}_{K,E}^{-1})\,\mathrm{d}l, \qquad \forall E \in \mathcal{E}_K, \tag{16.9a}$$

$$\sigma_{F,j,m}^{\mathrm{f}}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})(\zeta_m \circ \boldsymbol{T}_{K,F}^{-1})\,\mathrm{d}s, \quad \forall F \in \mathcal{F}_K,\ \forall j \in \{1,2\}, \tag{16.9b}$$

$$\sigma_{j,m}^{\mathrm{c}}(\boldsymbol{v}) := \frac{1}{|K|}\int_K (\boldsymbol{v}\cdot\boldsymbol{t}_{K,j})(\psi_m \circ \boldsymbol{T}_{K}^{-1})\,\mathrm{d}x, \quad \forall j \in \{1,2,3\}, \tag{16.9c}$$

where $\{\mu_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{e}}\}}$, $\{\zeta_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}$, and $\{\psi_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ are bases of $\mathbb{P}_{k,1}$, $\mathbb{P}_{k-1,2}$ ($k \ge 1$), and $\mathbb{P}_{k-2,3}$ ($k \ge 2$), respectively, $\boldsymbol{t}_E$ is the tangent vector orienting E, $\{\boldsymbol{t}_{F,j}\}_{j\in\{1,2\}}$ the two tangent vectors orienting F, and $\{\boldsymbol{t}_{K,j}\}_{j\in\{1,2,3\}}$ the three vectors orienting K, and $\boldsymbol{T}_{K,E} : \widehat{S}^1 \to E$, $\boldsymbol{T}_{K,F} : \widehat{S}^2 \to F$, and $\boldsymbol{T}_K : \widehat{K} \to K$ are geometric mappings. The local dofs in K are collectively denoted by $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$.

We are going to extend the above dofs to the following functional space:

$$\boldsymbol{V}^{\mathrm{c}}(K) := \boldsymbol{W}^{s,p}(K), \quad sp > 2,\ p \in (1,\infty) \text{ or } s = 2,\ p = 1, \tag{16.10}$$

The idea behind (16.10) is again to use a trace theorem (Theorem 3.15) to give a meaning to the edge (and face) dofs. Fixing the real number p in (16.10), we want to take s as small as possible to make the space $\boldsymbol{V}^{\mathrm{c}}(K)$ as large as possible. Thus, we can assume without loss of generality that $s \le 1$ if $p \in (2,\infty)$ and $s \le 2$ if $p \in [1,2]$. We can also take $p = \infty$ and $s = 1$ in (16.10). We consider the norm $\|\cdot\|_{\boldsymbol{W}^{\widetilde{s},p}(K)}$ defined as follows: If $s \in (0,1]$ (i.e., if $p \in (2,\infty]$), we set

$$\widetilde{s} := 0, \qquad \|\boldsymbol{v}\|_{\boldsymbol{W}^{\widetilde{s},p}(K)} := \|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)}, \tag{16.11a}$$

whereas if $s \in (1,2]$ (i.e., if $p \in [1,2]$), we set

$$\widetilde{s} := 1, \qquad \|\boldsymbol{v}\|_{\boldsymbol{W}^{\widetilde{s},p}(K)} := \|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K |\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(K)}. \tag{16.11b}$$

Proposition 16.7 (Extended dofs). *Let $\boldsymbol{V}^{\mathrm{c}}(K)$ be defined in (16.10). Let $\boldsymbol{V}^{\mathrm{c}}(\widehat{K})$ be defined similarly. Then the covariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{c}}$ is in $\mathcal{L}(\boldsymbol{V}^{\mathrm{c}}(K); \boldsymbol{V}^{\mathrm{c}}(\widehat{K}))$. Moreover, the local dofs are in $\mathcal{L}(\boldsymbol{V}^{\mathrm{c}}(K);\mathbb{R})$, and there is c s.t. for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,*

$$\max_{i\in\mathcal{N}} |\sigma_{K,i}(\boldsymbol{v})| \le c\, h_K^{1-\frac{d}{p}} \big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{\widetilde{s},p}(K)} + h_K^s |\boldsymbol{v}|_{\boldsymbol{W}^{s,p}(K)}\big). \tag{16.12}$$

Proof. (1) Let $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(K)$. Since the mesh is affine and $\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) := \mathbb{A}_K^{\mathrm{c}}(\boldsymbol{v}\circ\boldsymbol{T}_K)$ with $\mathbb{A}_K^{\mathrm{c}} := \mathbb{J}_K^{\mathsf{T}}$, we can proceed as in the proof of Proposition 16.1 and invoke Lemma 11.7 to show that $\boldsymbol{\psi}_K^{\mathrm{c}} \in \mathcal{L}(\boldsymbol{V}^{\mathrm{c}}(K); \boldsymbol{V}^{\mathrm{c}}(\widehat{K}))$ with $\|\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})\|_{\boldsymbol{W}^{\widetilde{s},p}(\widehat{K})} + \ell_{\widehat{K}}^s |\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v})|_{\boldsymbol{W}^{s,p}(\widehat{K})} \le c\, h_K^{1-\frac{d}{p}} \big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{\widetilde{s},p}(K)} + h_K^s |\boldsymbol{v}|_{\boldsymbol{W}^{s,p}(K)}\big)$, where the norm $\|\cdot\|_{\boldsymbol{W}^{\widetilde{s},p}(\widehat{K})}$ is defined similarly to $\|\cdot\|_{\boldsymbol{W}^{\widetilde{s},p}(K)}$ using $\ell_{\widehat{K}} := 1$.
(2) To bound the local dofs, we invoke Theorem 3.15 and proceed again as in the proof of Proposition 16.1. □

16.2.2 Commuting and approximation properties

In this section, we study the properties of the local Nédélec interpolation operator

$$\mathcal{I}_K^{\mathrm{c}} : \boldsymbol{V}^{\mathrm{c}}(K) \to \mathbb{N}_{k,d} \tag{16.13}$$

with $\boldsymbol{V}^{\mathrm{c}}(K)$ defined in (16.10). Recall that for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(K)$, $\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})$ is defined as the unique polynomial in $\mathbb{N}_{k,d}$ such that the function $(\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v}) - \boldsymbol{v})$ annihilates all the $\mathbb{N}_{k,d}$ dofs.

Lemma 16.8 (Commuting with $\nabla\times$). *The following diagram commutes:*

$$\begin{array}{ccc}
\check{\boldsymbol{V}}^{\mathrm{c}}(K) & \xrightarrow{\ \nabla\times\ } & \boldsymbol{V}^{\mathrm{d}}(K) \\
\downarrow{\scriptstyle \mathcal{I}_K^{\mathrm{c}}} & & \downarrow{\scriptstyle \mathcal{I}_K^{\mathrm{d}}} \\
\mathbb{N}_{k,d} & \xrightarrow{\ \nabla\times\ } & \mathbb{RT}_{k,d}
\end{array}$$

where $\check{\boldsymbol{V}}^{\mathrm{c}}(K) := \{\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(K) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)\}$. *In other words, we have*

$$\nabla\times(\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})) = \mathcal{I}_K^{\mathrm{d}}(\nabla\times\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in \check{\boldsymbol{V}}^{\mathrm{c}}(K). \tag{16.14}$$

Proof. Let us first observe that $\nabla\times\mathbb{N}_{k,d} \subset \mathbb{P}_{k,d} \subset \mathbb{RT}_{k,d}$ (see Lemma 15.10), which implies that $\nabla\times$ maps $\mathbb{N}_{k,d}$ to $\mathbb{RT}_{k,d}$. Note also that $\nabla\times$ maps $\check{\boldsymbol{V}}^{\mathrm{c}}(K)$ to $\boldsymbol{V}^{\mathrm{d}}(K)$ by definition of these spaces. Let $\boldsymbol{v} \in \check{\boldsymbol{V}}^{\mathrm{c}}(K)$. The proof of (16.14) consists of showing that $\boldsymbol{\delta} := \nabla\times(\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})) - \mathcal{I}_K^{\mathrm{d}}(\nabla\times\boldsymbol{v}) \in \mathbb{RT}_{k,d}$ annihilates all the dofs of the $\mathbb{RT}_{k,d}$ finite element in K. Let us set $\boldsymbol{\zeta} := \boldsymbol{v} - \mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})$ and $\boldsymbol{\xi} := \nabla\times\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\nabla\times\boldsymbol{v})$, so that we have

$$\boldsymbol{\delta} = \nabla\times(\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})) - \nabla\times\boldsymbol{v} + \nabla\times\boldsymbol{v} - \mathcal{I}_K^{\mathrm{d}}(\nabla\times\boldsymbol{v}) = \boldsymbol{\xi} - \nabla\times\boldsymbol{\zeta}.$$

(1) Let us consider first the dofs attached to K for $k \geq 1$. Let $\boldsymbol{e}$ be a unit vector in $\mathbb{R}^d$ and let $\psi \in \mathbb{P}_{k-1,d}$. We want to show that $\int_K \boldsymbol{\delta}\cdot\boldsymbol{e}\psi \,\mathrm{d}x = 0$. Since $\boldsymbol{\xi}$ annihilates all the cell dofs of the $\mathbb{RT}_{k,d}$ element, we have $\int_K \boldsymbol{\xi}\cdot\boldsymbol{e}\psi \,\mathrm{d}x = 0$, so that $\int_K \boldsymbol{\delta}\cdot\boldsymbol{e}\psi \,\mathrm{d}x = -\int_K (\nabla\times\boldsymbol{\zeta})\cdot\boldsymbol{e}\psi \,\mathrm{d}x$. Using the integration by parts formula (4.8a), we have

$$\int_K (\nabla\times\boldsymbol{\zeta})\cdot\boldsymbol{e}\psi \,\mathrm{d}x = \int_K \boldsymbol{\zeta}\cdot\nabla\times(\boldsymbol{e}\psi) - \sum_{F\in\mathcal{F}_K} \int_F \boldsymbol{\zeta}\cdot(\boldsymbol{n}_{K|F}\times\boldsymbol{e})\psi \,\mathrm{d}s.$$

If $k \geq 2$, we use that $\boldsymbol{\zeta}$ annihilates the cell dofs of the $\mathbb{N}_{k,d}$ element to infer that $\int_K \boldsymbol{\zeta}\cdot\nabla\times(\boldsymbol{e}\psi) = 0$. If $k = 1$, this equality is obvious. Moreover, since $\boldsymbol{\zeta}$ also annihilates the face dofs of the $\mathbb{N}_{k,d}$ element and since the vector $(\boldsymbol{n}_{K|F}\times\boldsymbol{e})$ is tangent to F, we infer that $\int_F \boldsymbol{\zeta}\cdot(\boldsymbol{n}_{K|F}\times\boldsymbol{e})\psi \,\mathrm{d}s = 0$ for all $F \in \mathcal{F}_K$. In conclusion, $\int_K (\nabla\times\boldsymbol{\zeta})\cdot\boldsymbol{e}\psi \,\mathrm{d}x = 0$, so that $\int_K \boldsymbol{\delta}\cdot\boldsymbol{e}\psi \,\mathrm{d}x = 0$.
(2) Let us now consider the dofs attached to a face $F \in \mathcal{F}_K$. We want to show that $\int_F \boldsymbol{\delta}\cdot\boldsymbol{n}_F\psi \,\mathrm{d}s = 0$ for all $\psi \in \mathbb{P}_{k,d}$. This is a sufficient condition to annihilate the $\mathbb{RT}_{k,d}$ dofs attached to F, since for all $q \in \mathbb{P}_{k,d-1}$, there exists $\psi \in \mathbb{P}_{k,d}$ such that $\psi_{|F} = q \circ \boldsymbol{T}_{K,F}^{-1}$ owing to Lemma 7.10. Since $\boldsymbol{\xi}$ annihilates the face dofs of the $\mathbb{RT}_{k,d}$ element, we have $\int_F \boldsymbol{\delta}\cdot\boldsymbol{n}_F\psi \,\mathrm{d}s = -\int_F (\nabla\times\boldsymbol{\zeta})\cdot\boldsymbol{n}_F\psi \,\mathrm{d}s$. Moreover, since $\nabla\times(\psi\boldsymbol{\zeta}) = \nabla\psi\times\boldsymbol{\zeta} + \psi\nabla\times\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}$ annihilates the face dofs of the $\mathbb{N}_{k,d}$ element, we infer that

$$\begin{aligned}
\int_F (\nabla\times\boldsymbol{\zeta})\cdot\boldsymbol{n}_F\psi \,\mathrm{d}s &= \int_F \nabla\times(\psi\boldsymbol{\zeta})\cdot\boldsymbol{n}_F \,\mathrm{d}s - \int_F \boldsymbol{\zeta}\cdot(\boldsymbol{n}_F\times\nabla\psi) \,\mathrm{d}s \\
&= \int_F \nabla\times(\psi\boldsymbol{\zeta})\cdot\boldsymbol{n}_F \,\mathrm{d}s = \int_{\partial F} (\psi\boldsymbol{\zeta})\cdot\boldsymbol{\tau}_F \,\mathrm{d}l = \sum_{E\in\mathcal{E}_F} \int_E \boldsymbol{\zeta}\cdot(\boldsymbol{\tau}_{F|E}\psi) \,\mathrm{d}l,
\end{aligned}$$

where we used the Kelvin–Stokes formula (16.15) with $\boldsymbol{\tau}_F$ being the unit vector tangent to ∂F whose orientation is compatible with that of $\boldsymbol{n}_F$, and where we decomposed the integral over ∂F into the integrals over the edges composing F. Since $\boldsymbol{\tau}_{F|E}$ is tangent to the edge E and $\boldsymbol{\zeta}$ annihilates the edge dofs of the $\mathbb{N}_{k,d}$ element, we obtain $\int_F (\nabla\times\boldsymbol{\zeta})\cdot\boldsymbol{n}_F\psi\,\mathrm{d}s = 0$. Hence, we have $\int_F \boldsymbol{\delta}\cdot\boldsymbol{n}_F\psi\,\mathrm{d}s = 0$, and this concludes the proof. □

Lemma 16.9 (Kelvin–Stokes). *Let K be a simplex in $\mathbb{R}^3$. Let F be a face of K with orientation defined by $\boldsymbol{n}_F$ and with boundary ∂F. Let $\boldsymbol{\tau}_F$ be the unit vector tangent to ∂F whose orientation is compatible with that of $\boldsymbol{n}_F$, i.e., for all $\boldsymbol{x}\in\partial F$, the vector $\boldsymbol{\tau}_F(\boldsymbol{x})\times\boldsymbol{n}_F(\boldsymbol{x})$ points outside of F. The following holds true for all $\boldsymbol{w}\in\check{\boldsymbol{V}}^{\mathrm{c}}(K)$:*

$$\int_F (\nabla\times\boldsymbol{w})\cdot\boldsymbol{n}_F\,\mathrm{d}s = \int_{\partial F} \boldsymbol{w}\cdot\boldsymbol{\tau}_F\,\mathrm{d}l. \tag{16.15}$$

Theorem 16.10 (Approximation, $r\geq 1$ or $r\geq 2$). *Let $\mathcal{I}_K^{\mathrm{c}}$ be the local $\mathbb{N}_{k,d}$ interpolation operator. There is c s.t. the following holds true:*
(i) *If $p\in(2,\infty]$, then we have*

$$|\boldsymbol{v}-\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,p}(K)} \leq c\,h_K^{r-m}|\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}, \tag{16.16}$$

for every integers $r\in\{1{:}k+1\}$ and $m\in\{0{:}r\}$, all $\boldsymbol{v}\in\boldsymbol{W}^{r,p}(K)$, all $K\in\mathcal{T}_h$, and all $h\in\mathcal{H}$.
(ii) *If $p\in[1,2]$, the estimate (16.16) holds true if $k\geq 1$ for every integers $r\in\{2{:}k+1\}$ and $m\in\{0{:}r\}$, all $\boldsymbol{v}\in\boldsymbol{W}^{r,p}(K)$, all $K\in\mathcal{T}_h$, and all $h\in\mathcal{H}$, whereas if $k=0$, we have*

$$|\boldsymbol{v}-\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,p}(K)} \leq c\left(h_K^{1-m}|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(K)} + h_K^{2-m}|\boldsymbol{v}|_{\boldsymbol{W}^{2,p}(K)}\right), \tag{16.17}$$

for all $m\in\{0,1\}$, all $\boldsymbol{v}\in\boldsymbol{W}^{2,p}(K)$, all $K\in\mathcal{T}_h$, and all $h\in\mathcal{H}$.
(iii) *Finally, we have*

$$|\nabla\times(\boldsymbol{v}-\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v}))|_{\boldsymbol{W}^{m,p}(K)} \leq c\,h_K^{r-m}|\nabla\times\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}, \tag{16.18}$$

for every integers $r\in\{1{:}k+1\}$ and $m\in\{0{:}r\}$, all $p\in[1,\infty]$, all $\boldsymbol{v}\in\boldsymbol{V}^{\mathrm{c}}(K)$ such that $\nabla\times\boldsymbol{v}\in\boldsymbol{W}^{r,p}(K)$, all $K\in\mathcal{T}_h$, and all $h\in\mathcal{H}$.

Proof. Let us start with (16.16) and (16.17). We apply Theorem 11.13. The covariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{c}}$ is of the form (11.1) with $\mathbb{A}_K^{\mathrm{c}} := \mathbb{J}_K^{\mathsf{T}}$, which satisfies the bound (11.12) with $\gamma := 1$. Moreover, we can take $l := 2$ if $p\in[1,2]$ and $l := 1$ if $p\in(2,\infty]$ since in both cases we have $\boldsymbol{W}^{l,p}(\widehat{K}) \hookrightarrow \boldsymbol{V}^{\mathrm{c}}(\widehat{K})$. If $p\in(2,\infty]$ or if $p\in[1,2]$ and $k\geq 1$, we have $l\leq k+1$, so that we can apply the estimate (11.14), which is nothing but (16.16). In the case where $p\in[1,2]$ and $k=0$, we apply (11.15), which is nothing but (16.17). Finally, to prove (16.18), we use Lemma 16.8 to infer that $\nabla\times(\boldsymbol{v}-\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})) = \nabla\times\boldsymbol{v}-\mathcal{I}_K^{\mathrm{d}}(\nabla\times\boldsymbol{v})$, and we conclude using Theorem 16.4. □

Remark 16.11 (Error on the curl). It is remarkable that the bound on $\nabla\times(\boldsymbol{v}-\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v}))$ only depends on the smoothness of $\nabla\times\boldsymbol{v}$. This is a direct consequence of the commuting property stated in Lemma 16.8. □

Theorem 16.12 (Approximation, $r > \frac{2}{p}$). *There is c, unbounded as $r \downarrow \frac{2}{p}$, such that:*
(i) *If $p \in (2,\infty)$, the estimate* (16.16) *holds true for all $r \in (\frac{2}{p},1)$, $m=0$, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$.*
(ii) *If $p \in (1,2]$, the estimate* (16.16) *holds true if $k \geq 1$ for all $r \in (\frac{2}{p},2)$, all $m \in \{0,1\}$, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, whereas if $k=0$, we have*

$$|\boldsymbol{v} - \mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,p}(K)} \leq c\left(h_K^{1-m}|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(K)} + h_K^{r-m}|\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}\right), \qquad (16.19)$$

for all $r \in (\frac{2}{p},2)$, all $m \in \{0,1\}$, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$.

Proof. Let us set $l := 2$ if $p \in (1,2]$ and $l := 1$ if $p \in (2,\infty)$. Let $r \in (\frac{2}{p}, l)$, so that $\boldsymbol{W}^{r,p}(K) \hookrightarrow \boldsymbol{V}^{\mathrm{c}}(K)$. Combining the bound from Proposition 12.5, the regularity of the mesh sequence, and the estimate (16.12) on the local dofs, we infer the stability estimate

$$\|\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \leq c\big(\|\boldsymbol{v}\|_{\boldsymbol{W}^{\widetilde{r},p}(K)} + h_K^r|\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}\big),$$

with $\widetilde{r} := 0$ if $r \in (0,1]$ and $\widetilde{r} := 1$ if $r \in (1,2)$.
(i) Assume that $p \in (2,\infty)$. Then $r < 1$ so that $\|\boldsymbol{v}\|_{\boldsymbol{W}^{\widetilde{r},p}(K)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)}$. Since $\mathbb{P}_{0,d} \subset \mathbb{N}_{k,d}$, we infer that

$$\begin{aligned}\|\boldsymbol{v} - \mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} &\leq c \inf_{\boldsymbol{q}\in\mathbb{P}_{0,d}} \big(\|\boldsymbol{v}-\boldsymbol{q}\|_{\boldsymbol{L}^p(K)} + |\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v}-\boldsymbol{q})|_{\boldsymbol{L}^p(K)}\big) \\ &\leq c\Big(\inf_{\boldsymbol{q}\in\mathbb{P}_{0,d}} \|\boldsymbol{v}-\boldsymbol{q}\|_{\boldsymbol{L}^p(K)} + h_K^r|\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}\Big),\end{aligned}$$

where we used that $|\boldsymbol{v}-\boldsymbol{q}|_{\boldsymbol{W}^{r,p}(K)} = |\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}$. The estimate (16.16) with $m=0$ follows from the fractional Poincaré–Steklov inequality (see Lemma 12.12).
(ii) Assume that $p \in (1,2)$. Then $r \in (1,2)$ so that $\|\boldsymbol{v}\|_{\boldsymbol{W}^{\widetilde{r},p}(K)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(K)}$. Let $n := \min(1,k)$. Since $n \leq k$ and $n \leq 1 < r$, proceeding as above, we infer that

$$\|\boldsymbol{v} - \mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \leq c\Big(\inf_{\boldsymbol{q}\in\mathbb{P}_{n,d}} \phi_K(\boldsymbol{v}-\boldsymbol{q}) + h_K^r|\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}\Big),$$

with $\phi_K(\boldsymbol{v}-\boldsymbol{q}) := \|\boldsymbol{v}-\boldsymbol{q}\|_{\boldsymbol{L}^p(K)} + h_K|\boldsymbol{v}-\boldsymbol{q}|_{\boldsymbol{W}^{1,p}(K)}$ Using the inverse inequality $|\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v}-\boldsymbol{q})|_{\boldsymbol{W}^{1,p}(K)} \leq ch_K^{-1}\|\mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v}-\boldsymbol{q})\|_{\boldsymbol{L}^p(K)}$ (see Lemma 12.1) and proceeding again as above, we infer that

$$|\boldsymbol{v} - \mathcal{I}_K^{\mathrm{c}}(\boldsymbol{v})|_{\boldsymbol{W}^{1,p}(K)} \leq c\Big(\inf_{\boldsymbol{q}\in\mathbb{P}_{n,d}} h_K^{-1}\phi_K(\boldsymbol{v}-\boldsymbol{q}) + h_K^{r-1}|\boldsymbol{v}|_{\boldsymbol{W}^{r,p}(K)}\Big).$$

If $k \geq 1$, we have $n=1$, and the estimate (16.16) follows from Corollary 12.13 for all $m \in \{0,1\}$, whereas if $k=0$, we have $n=0$, and the estimate (16.19) for all $m \in \{0,1\}$ follows from the fractional Poincaré–Steklov inequality. □

16.3 The de Rham complex

In this section, we introduce the notion of de Rham complex, and we reinterpret the previous commuting properties from Lemma 16.2 and Lemma 16.8 in this context. We assume that $d = 3$; see Remark 16.17 below to adapt the material when $d = 2$.

Definition 16.13 (Exact cochain complex). *Let $I \geq 2$ be an integer. A* cochain complex *is composed of a sequence of Banach spaces $(V_i)_{i\in\{0:I\}}$ and a sequence of linear operators $(d_i)_{i\in\{1:I\}}$ between these spaces*

$$V_0 \xrightarrow{d_1} V_1 \ldots V_{i-1} \xrightarrow{d_i} V_i \xrightarrow{d_{i+1}} V_{i+1} \ldots V_{I-1} \xrightarrow{d_I} V_I, \tag{16.20}$$

such that for all $i \in \{1:I\}$, $\mathrm{im}(d_i)$ is closed in V_i and if $i < I$, $\mathrm{im}(d_i) \subseteq \ker(d_{i+1})$ (this means that $d_{i+1} \circ d_i = 0$). The cochain complex is said to be exact *if $\mathrm{im}(d_i) = \ker(d_{i+1})$ for all $i \in \{1:I-1\}$.*

The exactness of a cochain complex is useful since it gives a simple way of knowing whether an element $v_i \in V_i$ is in $\mathrm{im}(d_i)$ by checking whether $d_{i+1}(v_i) = 0$. In this book, we focus on one fundamental example of cochain complex, namely the de Rham complex which involves the gradient, curl, and divergence operators.

Proposition 16.14 (de Rham complex). *Let D be a Lipschitz domain in $\mathbb{R}^3$. Assume that D is simply connected and that ∂D is connected. The following cochain complex, called* de Rham complex*, is exact:*

$$\mathbb{R} \xrightarrow{i} H^1(D) \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl}; D) \xrightarrow{\nabla\times} \boldsymbol{H}(\mathrm{div}; D) \xrightarrow{\nabla\cdot} L^2(D) \xrightarrow{o} \{0\}, \tag{16.21}$$

where i maps a real number to a constant function and o is the zero map.

Proof. That $\ker(\nabla) = \mathbb{R}$, $\ker(\nabla\times) = \mathrm{im}(\nabla)$, and $\ker(\nabla\cdot) = \mathrm{im}(\nabla\times)$ are well-known facts from calculus since D is, respectively, connected, simply connected, and has a connected boundary. Finally, that $\mathrm{im}(\nabla\cdot) = L^2(D)$ is proved in Lemma 51.2. □

Proposition 16.15 (Discrete de Rham complex). *Let $\kappa \in \mathbb{N}$. The following cochain complex, called* discrete de Rham complex*, is exact:*

$$\mathbb{R} \xrightarrow{i} \mathbb{P}_{\kappa+1,3} \xrightarrow{\nabla} \mathbf{N}_{\kappa,3} \xrightarrow{\nabla\times} \mathbf{RT}_{\kappa,3} \xrightarrow{\nabla\cdot} \mathbb{P}_{\kappa,3} \xrightarrow{o} \{0\}. \tag{16.22}$$

Proof. $\ker(\nabla) = \mathrm{im}(i)$ is obvious, and $\ker(\nabla\times) = \mathrm{im}(\nabla)$ follows from Lemma 15.10. For $\ker(\nabla\cdot) = \mathrm{im}(\nabla\times)$, $\ker(o) = \mathrm{im}(\nabla\cdot)$; see Exercise 16.6. □

We now connect the above two de Rham complexes by means of interpolation operators. Let K be a simplex in $\mathbb{R}^d$, $d = 3$. Let $p \in [1,\infty)$ and let s be such that $sp > 3$ if $p > 1$ or $s = 3$ if $p = 1$. Recall the following functional spaces where $V^{\mathrm{b}}(K) := L^1(K)$:

$$\check{V}^{\mathrm{g}}(K) := \{f \in W^{s,p}(K) \mid \nabla f \in \boldsymbol{W}^{s-\frac{1}{p},p}(K)\}, \tag{16.23a}$$

$$\check{\boldsymbol{V}}^{\mathrm{c}}(K) := \{\boldsymbol{g} \in \boldsymbol{W}^{s-\frac{1}{p},p}(K) \mid \nabla{\times}\boldsymbol{g} \in \boldsymbol{W}^{s-\frac{2}{p},p}(K)\}, \tag{16.23b}$$

$$\check{\boldsymbol{V}}^{\mathrm{d}}(K) := \{\boldsymbol{g} \in \boldsymbol{W}^{s-\frac{2}{p},p}(K) \mid \nabla{\cdot}\boldsymbol{g} \in V^{\mathrm{b}}(K)\}. \tag{16.23c}$$

Lemma 16.16 (Commuting diagrams). *Let $\kappa \in \mathbb{N}$. Let K be a simplex in $\mathbb{R}^d$, $d = 3$. Let $\mathcal{I}^{\mathrm{g}}_{\kappa+1,K}$ be the interpolation operator associated with the canonical hybrid element of degree $(\kappa+1)$ defined in §7.6. Let $\mathcal{I}^{\mathrm{c}}_{\kappa,K}$ be the $\mathbf{N}_{\kappa,3}$ interpolation operator, let $\mathcal{I}^{\mathrm{d}}_{\kappa,K}$ be the $\mathbf{RT}_{\kappa,d}$ interpolation operator, and let $\mathcal{I}^{\mathrm{b}}_{\kappa,K}$ be the L^2-orthogonal projection onto $\mathbb{P}_{\kappa,d}$. The following diagrams commute:*

$$\begin{array}{ccccccc}
\check{V}^{\mathrm{g}}(K) & \xrightarrow{\nabla} & \check{\boldsymbol{V}}^{\mathrm{c}}(K) & \xrightarrow{\nabla\times} & \check{\boldsymbol{V}}^{\mathrm{d}}(K) & \xrightarrow{\nabla\cdot} & V^{\mathrm{b}}(K) \\
\big\downarrow \mathcal{I}^{\mathrm{g}}_{\kappa+1,K} & & \big\downarrow \mathcal{I}^{\mathrm{c}}_{\kappa,K} & & \big\downarrow \mathcal{I}^{\mathrm{d}}_{\kappa,K} & & \big\downarrow \mathcal{I}^{\mathrm{b}}_{\kappa,K} \\
\mathbb{P}_{\kappa+1,d} & \xrightarrow{\nabla} & \mathbf{N}_{\kappa,d} & \xrightarrow{\nabla\times} & \mathbf{RT}_{\kappa,d} & \xrightarrow{\nabla\cdot} & \mathbb{P}_{\kappa,d}
\end{array}$$

Proof. Recalling Lemma 16.2 and Lemma 16.8, it only remains to prove that the leftmost diagram commutes. This is done in Exercise 16.3. □

Remark 16.17 (2D). There are two possible versions of Lemma 16.16 if $d = 2$, using either the operator $\nabla{\times}\boldsymbol{f} := \partial_1 f_2 - \partial_2 f_1$ or the operator $\nabla^{\perp} f := (-\partial_2 f, \partial_1 f)^{\mathsf{T}}$. One can show that the following two diagrams commute:

$$\begin{array}{ccccc}
\check{V}^{\mathrm{g}}(K) & \xrightarrow{\nabla^{\perp}} & \check{\boldsymbol{V}}^{\mathrm{d}}(K) & \xrightarrow{\nabla\cdot} & V^{\mathrm{b}}(K) \\
\big\downarrow \mathcal{I}^{\mathrm{g}}_{\kappa+1,K} & & \big\downarrow \mathcal{I}^{\mathrm{d}}_{\kappa,K} & & \big\downarrow \mathcal{I}^{\mathrm{b}}_{\kappa,K} \\
\mathbb{P}_{\kappa+1,d} & \xrightarrow{\nabla^{\perp}} & \mathbf{RT}_{\kappa,d} & \xrightarrow{\nabla\cdot} & \mathbb{P}_{\kappa,d}
\end{array}
\qquad
\begin{array}{ccccc}
\check{V}^{\mathrm{g}}(K) & \xrightarrow{\nabla} & \check{\boldsymbol{V}}^{\mathrm{c}}(K) & \xrightarrow{\nabla\times} & V^{\mathrm{b}}(K) \\
\big\downarrow \mathcal{I}^{\mathrm{g}}_{\kappa+1,K} & & \big\downarrow \mathcal{I}^{\mathrm{c}}_{\kappa,K} & & \big\downarrow \mathcal{I}^{\mathrm{b}}_{\kappa,K} \\
\mathbb{P}_{\kappa+1,d} & \xrightarrow{\nabla} & \mathbf{N}_{\kappa,d} & \xrightarrow{\nabla\times} & \mathbb{P}_{\kappa,d}
\end{array}$$

with $\check{V}^{\mathrm{g}}(K)$ defined in (16.23a) with $sp > 2$ if $p \in (1,\infty)$ or $s = 2$ if $p = 1$, $\check{\boldsymbol{V}}^{\mathrm{c}}(K) := \{\boldsymbol{g} \in \boldsymbol{W}^{s-\frac{1}{p},p}(K) \mid \nabla{\times}\boldsymbol{g} \in L^1(K)\}$, and $\check{\boldsymbol{V}}^{\mathrm{d}}(K) := \boldsymbol{R}_{\frac{\pi}{2}}(\check{\boldsymbol{V}}^{\mathrm{c}}(K)) = \{\boldsymbol{g} \in \boldsymbol{W}^{s-\frac{1}{p},p}(K) \mid \nabla{\cdot}\boldsymbol{g} \in V^{\mathrm{b}}(K)\}$, where $\boldsymbol{R}_{\frac{\pi}{2}}$ is the rotation matrix of angle $\frac{\pi}{2}$ in $\mathbb{R}^2$. □

Remark 16.18 (Cuboids). The commuting diagrams from Lemma 16.16 can be adapted when K is a cuboid by using the Cartesian Raviart–Thomas and Nédélec spaces from §14.5.2 and §15.5.2. □

Remark 16.19 (Literature). The construction and analysis of finite elements leading to discrete de Rham complexes has witnessed significant progresses since the early 2000s and has lead to the notion of finite element exterior calculus; see Arnold et al. [11, 12]. Regularity estimates in Sobolev (and other) norms for right inverse operators of the gradient, curl, and divergence can be found in Costabel and McIntosh [83]. □

Exercises

Exercise 16.1 ($\check{\boldsymbol{V}}^{\mathrm{d}}(K)$). Show that $\boldsymbol{V}^{\mathrm{d}}(K)$ defined in (16.2) can be used in the commuting diagram of Lemma 16.2 after replacing $L^1(K)$ by $W^{s-1,p}(K)$. (*Hint*: use Theorem 3.19.)

Exercise 16.2 ($\mathcal{I}_K^{\mathrm{d}}$). Prove that the estimate (16.6) holds true for all $r \in [1, k+1]$, $r \notin \mathbb{N}$, every integer $m \in \{0{:}\lfloor r \rfloor\}$, and all $p \in [1,\infty)$. Prove that (16.7) holds true for all $r \in [0, k+1]$, $r \notin \mathbb{N}$, every integer $m \in \{0{:}\lfloor r \rfloor\}$, and all $p \in [1,\infty)$. (*Hint*: combine $W^{m,p}$-stability with Corollary 12.13.)

Exercise 16.3 (de Rham). Prove that the leftmost diagram in Lemma 16.16 commutes. (*Hint*: verify that $\nabla\mathcal{I}_K^{\mathrm{g}}(v) - \mathcal{I}_K^{\mathrm{c}}(\nabla v)$ annihilates all dofs in $\mathbb{N}_{k,d}$.)

Exercise 16.4 (Poincaré operators). Assume that K is star-shaped with respect to a point $\boldsymbol{a} \in K$. Let f and $\boldsymbol{g}$ be smooth functions on K. Define $P^{\mathrm{g}}(\boldsymbol{g})(\boldsymbol{x}) := (\boldsymbol{x}-\boldsymbol{a}){\cdot}\int_0^1 \boldsymbol{g}(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))\,\mathrm{d}t$, $\boldsymbol{P}^{\mathrm{c}}(\boldsymbol{g})(\boldsymbol{x}) := -(\boldsymbol{x}-\boldsymbol{a}){\times}\int_0^1 \boldsymbol{g}(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))\,\mathrm{d}t$ (if $d=3$), and $\boldsymbol{P}^{\mathrm{d}}(f)(\boldsymbol{x}) := (\boldsymbol{x}-\boldsymbol{a})\int_0^1 f(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))t^{d-1}\,\mathrm{d}t$. Verify that (i) $\nabla P^{\mathrm{g}}(\boldsymbol{g}) = \boldsymbol{g}$ if $\partial_i g_j = \partial_j g_i$ for all $i,j \in \{1{:}d\}$; (ii) $\nabla{\times}\boldsymbol{P}^{\mathrm{c}}(\boldsymbol{g}) = \boldsymbol{g}$ if $\nabla{\cdot}\boldsymbol{g} = 0$; (iii) $\nabla{\cdot}\boldsymbol{P}^{\mathrm{d}}(f) = f$.

Exercise 16.5 (Koszul operator). (i) Let $\boldsymbol{v} \in \mathbb{P}^{\mathrm{H}}_{k,d}$ with $d=3$. Prove that $\nabla(\boldsymbol{x}{\cdot}\boldsymbol{v}) - \boldsymbol{x}{\times}(\nabla{\times}\boldsymbol{v}) = (k+1)\boldsymbol{v}$ and $-\nabla{\times}(\boldsymbol{x}{\times}\boldsymbol{v}) + \boldsymbol{x}(\nabla{\cdot}\boldsymbol{v}) = (k+2)\boldsymbol{v}$. (*Hint*: use Euler's identity from Lemma 14.3.) (ii) Prove that $\mathbb{P}_{k,d} = \nabla\mathbb{P}_{k+1,d} \oplus (\boldsymbol{x}{\times}\mathbb{P}_{k-1,d}) = \nabla{\times}\mathbb{P}_{k+1,d} \oplus (\boldsymbol{x}\mathbb{P}_{k-1,d})$. (*Hint*: establish first these identities for homogeneous polynomials.) *Note*: defining the Koszul operators $\kappa^{\mathrm{g}}(\boldsymbol{v}) := \boldsymbol{x}{\cdot}\boldsymbol{v}$ and $\boldsymbol{\kappa}^{\mathrm{c}}(\boldsymbol{v}) := -\boldsymbol{x}{\times}\boldsymbol{v}$ for vector fields and $\boldsymbol{\kappa}^{\mathrm{d}}(v) := \boldsymbol{x}v$ for scalar fields, one has $\kappa^{\mathrm{g}}(\nabla q) = kq$ (Euler's identity) and $\nabla{\cdot}(\boldsymbol{\kappa}^{\mathrm{d}}(q)) = (k+d)q$ for all $q \in \mathbb{P}^{\mathrm{H}}_{k,d}$, and $\nabla(\kappa^{\mathrm{g}}(\boldsymbol{q})) + \boldsymbol{\kappa}^{\mathrm{c}}(\nabla{\times}\boldsymbol{q}) = (k+1)\boldsymbol{q}$ and $\nabla{\times}(\boldsymbol{\kappa}^{\mathrm{c}}(\boldsymbol{q})) + \boldsymbol{\kappa}^{\mathrm{d}}(\nabla{\cdot}\boldsymbol{q}) = (k+2)\boldsymbol{q}$ for all $\boldsymbol{q} \in \mathbb{P}^{\mathrm{H}}_{k,d}$; see [11, Sec. 3.2].

Exercise 16.6 ($\nabla{\cdot}\mathbb{RT}_{k,d}$ and $\nabla{\times}\mathbb{N}_{k,3}$). (i) Prove that $\nabla{\cdot}\mathbb{RT}_{k,d} = \mathbb{P}_{k,d}$. (*Hint*: prove that $\nabla{\cdot} : \boldsymbol{x}\mathbb{P}_{k,d} \to \mathbb{P}_{k,d}$ is injective using Lemma 14.3.) (ii) Let us set $\mathbb{RT}^{\mathrm{div}=0}_{k,d} := \{\boldsymbol{v} \in \mathbb{RT}_{k,d} \mid \nabla{\cdot}\boldsymbol{v} = 0\}$. Determine $\dim(\mathbb{RT}^{\mathrm{div}=0}_{k,d})$ for $d \in \{2,3\}$. (iii) Show that $\mathbb{RT}^{\mathrm{div}=0}_{k,3} = \nabla{\times}\mathbb{P}_{k+1,3}$. (*Hint*: use Lemma 14.9.) (iv) Prove that $\mathbb{RT}^{\mathrm{div}=0}_{k,3} = \nabla{\times}\mathbb{N}_{k,3}$. (*Hint*: use the rank nullity theorem.)

Exercise 16.7 ($\nabla\mathbb{P}_{k+1,d}$ and $\nabla{\times}\mathbb{P}_{k+1,3}$). Let $k \in \mathbb{N}$. (i) Set $\mathbb{P}^{\mathrm{c}}_{k,d} := \nabla\mathbb{P}_{k+1,d}$. Show that $\dim(\mathbb{P}^{\mathrm{c}}_{k,d}) = \binom{k+d+1}{d} - 1$. (ii) Assume $d=3$. Set $\mathbb{P}^{\mathrm{d}}_{k,3} := \nabla{\times}\mathbb{P}_{k+1,3}$. Show that $\dim(\mathbb{P}^{\mathrm{d}}_{k,3}) = 3\binom{k+4}{3} - \binom{k+5}{3} + 1 = 3\binom{k+3}{3} - \binom{k+2}{3}$ (with the convention that $\binom{2}{3} = 0$). (*Hint*: use the exact cochain complex $\mathbb{P}_{0,d} \xrightarrow{i} \mathbb{P}_{k+2,d} \xrightarrow{\nabla} \mathbb{P}_{k+1,d} \xrightarrow{\nabla\times} \mathbb{P}_{k,d} \xrightarrow{\nabla\cdot} \mathbb{P}_{k-1,d} \xrightarrow{o} \{0\}$.)