Part IV, Chapter 19

Main properties of the conforming subspaces

In this chapter, we continue the study of the interpolation properties of the conforming finite element subspaces introduced in the previous chapter. Recall that

$$P_k^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q) := \{v_h \in \boldsymbol{P}_k^{\mathrm{x,b}}(\mathcal{T}_h;\mathbb{R}^q) \mid [\![v_h]\!]_F^{\mathrm{x}} = 0,\ \forall F \in \mathcal{F}_h^{\circ}\},$$

where $\boldsymbol{P}_k^{\mathrm{x,b}}(\mathcal{T}_h;\mathbb{R}^q)$ is a broken finite element space, with $q \in \{1, d\}$ depending on the superscript $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}, \mathrm{d}\}$, and the jump operator $[\![\cdot]\!]_F^{\mathrm{x}}$ is defined in (18.7). Recall that the H^1-conforming subspace $P_k^{\mathrm{g}}(\mathcal{T}_h)$ $(q = 1)$ is built using a Lagrange element or a canonical hybrid element of degree $k \geq 1$, the $\boldsymbol{H}(\mathrm{curl})$-conforming subspace $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ $(q = d = 3)$ is built using a Nédélec element of degree $k \geq 0$, and the $\boldsymbol{H}(\mathrm{div})$-conforming subspace $\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$ $(q = d)$ is built using a Raviart–Thomas element of degree $k \geq 0$. The cornerstone of the construction, which is presented in a unified way for $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}, \mathrm{d}\}$, is a connectivity array with ad hoc clustering properties of the local degrees of freedom (dofs). In the present chapter, we postulate the existence of the connectivity array and show how it allows us to build global shape functions and a global interpolation operator in $P_k^{\mathrm{g}}(\mathcal{T}_h)$. The actual construction of this mapping is undertaken in Chapters 20 and 21. In this book, we shall implicitly assume that the mesh $\mathcal{T}_h$ is matching (see Definition 8.11) when the conforming space $P_k^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ is invoked.

19.1 Global shape functions and dofs

For all $K \in \mathcal{T}_h$, the local dofs are $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$, and the local shape functions are $\{\theta_{K,i}\}_{i\in\mathcal{N}}$. Recall that $\{\theta_{K,i}\}_{i\in\mathcal{N}}$ is a basis of P_K and that $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$ is a basis of $\mathcal{L}(P_K;\mathbb{R})$. We start by organizing all the dofs and shape functions

$$\{\sigma_{K,i}\}_{(K,i)\in\mathcal{T}_h\times\mathcal{N}}, \qquad \{\theta_{K,i}\}_{(K,i)\in\mathcal{T}_h\times\mathcal{N}},$$

by grouping them into clusters, which we are going to call *connectivity classes*. We assume that we have at hand a nonzero natural number I and a *connectivity array*

$$\texttt{j\_dof} : \mathcal{T}_h \times \mathcal{N} \to \mathcal{A}_h := \{1{:}I\}. \tag{19.1}$$

Without loss of generality we assume that the mapping $\texttt{j\_dof}$ is surjective, i.e., for every connectivity class $a \in \mathcal{A}_h$, there exists $(K, i) \in \mathcal{T}_h \times \mathcal{N}$ s.t. $\texttt{j\_dof}(K, i) = a$. This hypothesis is nonessential and can always be satisfied by rearranging the codomain of $\texttt{j\_dof}$.

Definition 19.1 (Connectivity class). *Two pairs* $(K, i), (K', i') \in \mathcal{T}_h \times \mathcal{N}$ *are said to be in the same connectivity class if* $\texttt{j\_dof}(K, i) = \texttt{j\_dof}(K', i')$.

We require that the mapping $\texttt{j\_dof}$ satisfies two key properties.
(1) The first one is that for all $v_h \in P_k^{\mathrm{x,b}}(\mathcal{T}_h)$,

$$[\, v_h \in P_k^{\mathrm{x}}(\mathcal{T}_h) \,] \iff \begin{bmatrix} \text{For all } (K, i), (K', i') \text{ in the same} \\ \text{connectivity class, we have} \\ \sigma_{K,i}(v_{h|K}) = \sigma_{K',i'}(v_{h|K'}) \end{bmatrix}. \tag{19.2}$$

Thus, (19.2) means that for every function v_h in the broken finite element space $P_k^{\mathrm{x,b}}(\mathcal{T}_h)$, a *necessary and sufficient* condition for v_h to be a member of the conforming subspace $P_k^{\mathrm{x}}(\mathcal{T}_h)$ is that for all $a \in \mathcal{A}_h$, the quantity $\sigma_{K,i}(v_{h|K})$ is independent of the choice of the pair (K, i) in the preimage $\texttt{j\_dof}^{-1}(a) := \{(K', i') \in \mathcal{T}_h \times \mathcal{N} \mid \texttt{j\_dof}(K', i') = a\}$.
(2) The second key property is that

$$\forall K \in \mathcal{T}_h, \qquad \texttt{j\_dof}(K, \cdot) : \mathcal{N} \to \mathcal{A}_h \text{ is injective}, \tag{19.3}$$

i.e., if (K, i) and (K, i') are in the same connectivity class, then $i = i'$.

We now construct global dofs and shape functions in $P_k^{\mathrm{x}}(\mathcal{T}_h)$. Since for all $a \in \mathcal{A}_h$ and all $v_h \in P_k^{\mathrm{x}}(\mathcal{T}_h)$, (19.2) implies that the value of $\sigma_{K,i}(v_{h|K})$ is independent of the choice of the pair (K, i) in the connectivity class a, it is legitimate to introduce the following definition: For all $a \in \mathcal{A}_h$, we define the linear form $\sigma_a : P_k^{\mathrm{x}}(\mathcal{T}_h) \to \mathbb{R}$ s.t. for all $v_h \in P_h^{\mathrm{x}}(\mathcal{T}_h)$,

$$\sigma_a(v_h) := \sigma_{K,i}(v_{h|K}), \qquad \forall (K, i) \in \texttt{j\_dof}^{-1}(a), \tag{19.4}$$

i.e., $\sigma_a(v_h) := \sigma_{K,i}(v_{h|K})$ for every pair (K, i) in the connectivity class a. Observe that $\sigma_a \in \mathcal{L}(P_k^{\mathrm{x}}(\mathcal{T}_h); \mathbb{R})$. We now define the function $\varphi_a : \overline{D} \to \mathbb{R}^q$ for all $a \in \mathcal{A}_h$ by

$$\varphi_{a|K} := \begin{cases} \theta_{K,i} & \text{if there exists } i \in \mathcal{N} \text{ s.t. } (K, i) \in \texttt{j\_dof}^{-1}(a), \\ 0 & \text{otherwise.} \end{cases} \tag{19.5}$$

This definition makes sense since if $(K, i) \in \texttt{j\_dof}^{-1}(a)$ and $(K, i') \in \texttt{j\_dof}^{-1}(a)$, then $i = i'$ owing to (19.3).

Definition 19.2 (Global shape functions and dofs). *The functions* φ_a *are called* global shape functions*, and the linear forms* σ_a *are called* global degrees of freedom (dofs).

For all $a \in \mathcal{A}_h$, let us introduce the following collection of cells:

$$\mathcal{T}_a := \{K \in \mathcal{T}_h \mid \exists i \in \mathcal{N},\ (K,i) \in \mathtt{j\_dof}^{-1}(a)\}, \tag{19.6}$$

i.e., $\mathcal{T}_a = \{K \in \mathcal{T}_h \mid a \in \mathtt{j\_dof}(K,\mathcal{N})\}$. A direct consequence of the definition (19.5) is that

$$\mathrm{supp}(\varphi_a) = \bigcup_{K\in\mathcal{T}_a} K. \tag{19.7}$$

Lemma 19.3 (Conformity). *For all* a *in* $\mathcal{A}_h$*,* $\varphi_a \in P_k^{\mathrm{x}}(\mathcal{T}_h)$ *and*

$$\sigma_a(\varphi_{a'}) = \delta_{aa'}, \qquad \forall a' \in \mathcal{A}_h. \tag{19.8}$$

Proof. Let $a \in \mathcal{A}_h$ and let us prove that $\varphi_a \in P_k^{\mathrm{x}}(\mathcal{T}_h)$. Since $\varphi_a \in P_k^{\mathrm{x,b}}(\mathcal{T}_h)$, we prove the assertion by checking that the property on the right-hand side of (19.2) holds true. Let a' be arbitrary in $\mathcal{A}_h$. We need to show that the quantity $\sigma_{K,i}(\varphi_{a|K})$ is independent of the pair $(K,i) \in \mathtt{j\_dof}^{-1}(a')$.
(1) Assume first that $a' = a$. Let (K,i) be an arbitrary pair in $\mathtt{j\_dof}^{-1}(a')$. Then $\mathtt{j\_dof}(K,i) = a' = a$, and the definition of φ_a implies that $\varphi_{a|K} = \theta_{K,i}$. Hence, $\sigma_{K,i}(\varphi_{a|K}) = \sigma_{K,i}(\theta_{K,i}) = 1$ for all $(K,i) \in \mathtt{j\_dof}^{-1}(a')$.
(2) Assume now that $a' \neq a$. Let (K,i) be an arbitrary pair in $\mathtt{j\_dof}^{-1}(a')$. If there exists $j \in \mathcal{N}$ s.t. $\mathtt{j\_dof}(K,j) = a$, then $\varphi_{a|K} = \theta_{K,j}$. Notice that $j \neq i$ owing to (19.3), since $\mathtt{j\_dof}(K,j) = a \neq a' = \mathtt{j\_dof}(K,i)$. We infer in this case that $\sigma_{K,i}(\varphi_{a|K}) = \sigma_{K,i}(\theta_{K,j}) = 0$ since $j \neq i$. If there is no $j \in \mathcal{N}$ s.t. $\mathtt{j\_dof}(K,j) = a$, then $\varphi_{a|K} = 0$ and again $\sigma_{K,i}(\varphi_{a|K}) = 0$. To sum up, $\sigma_{K,i}(\varphi_{a|K}) = 0$ for all $(K,i) \in \mathtt{j\_dof}^{-1}(a')$.
(3) In conclusion, the above argument shows that $\sigma_a(\varphi_a) = 1$ and $\sigma_{a'}(\varphi_a) = 0$ if $a' \neq a$, i.e., i $\sigma_{K,i}(\varphi_{a|K})$ is independent of the pair $(K,i) \in \mathtt{j\_dof}^{-1}(a')$ for all $a' \in \mathcal{A}_h$, and (19.8) holds true. □

Proposition 19.4 (Basis). $\{\varphi_a\}_{a\in\mathcal{A}_h}$ *is a basis of* $P_k^{\mathrm{x}}(\mathcal{T}_h)$*, and* $\{\sigma_a\}_{a\in\mathcal{A}_h}$ *is a basis of* $\mathcal{L}(P_k^{\mathrm{x}}(\mathcal{T}_h);\mathbb{R})$.

Proof. Assume that $\sum_{a\in\mathcal{A}_h} \lambda_a\varphi_a$ vanishes identically on $\overline{D}$ for some real numbers $\{\lambda_a\}_{a\in\mathcal{A}_h}$. Using the linearity of σ_a and (19.8) yields

$$0 = \sigma_{a'}(0) = \sigma_{a'}\left(\sum_{a\in\mathcal{A}_h} \lambda_a\varphi_a\right) = \sum_{a\in\mathcal{A}_h} \lambda_a\sigma_{a'}(\varphi_a) = \lambda_{a'}.$$

Hence, $\lambda_{a'} = 0$ for all $a' \in \mathcal{A}_h$, i.e., $\{\varphi_a\}_{a\in\mathcal{A}_h}$ is linearly independent. To show that $\{\varphi_a\}_{a\in\mathcal{A}_h}$ is a spanning set of $P_k^{\mathrm{x}}(\mathcal{T}_h)$, let $v_h \in P_k^{\mathrm{x}}(\mathcal{T}_h)$ and let us set $\delta_h := v_h - \sum_{a'\in\mathcal{A}_h} \sigma_{a'}(v_h)\varphi_{a'}$. We are going to prove that $\delta_{h|K} = 0$ for all $K \in \mathcal{T}_h$, and since $\delta_{h|K} \in P_K$, we do so by showing that $\delta_{h|K}$ annihilates all the local dofs in K, i.e., $\sigma_{K,i}(\delta_{h|K}) = 0$ for all $i \in \mathcal{N}$. Let K be an arbitrary cell in $\mathcal{T}_h$, let i be an arbitrary index in $\mathcal{N}$, and let $a := \mathtt{j\_dof}(K,i)$. Then

$$\sigma_{K,i}(\delta_{h|K}) = \sigma_a(\delta_h) = \sigma_a(v_h) - \sigma_a(v_h) = 0,$$

where the first equality follows from the fact that $\delta_h \in P_k^{\mathrm{x}}(\mathcal{T}_h)$ and the second one from (19.8). We have thus proved that $\delta_{h|K} = 0$ for all $K \in \mathcal{T}_h$, and hence that δ_h vanishes identically because K is arbitrary. In conclusion, $\{\varphi_a\}_{a\in\mathcal{A}_h}$ is a basis of $P_k^{\mathrm{x}}(\mathcal{T}_h)$. Since $\{\varphi_a\}_{a\in\mathcal{A}_h}$ is a basis of $P_k^{\mathrm{x}}(\mathcal{T}_h)$, the identity (19.8) implies that $\{\sigma_a\}_{a\in\mathcal{A}_h}$ is a basis of $\mathcal{L}(P_k^{\mathrm{x}}(\mathcal{T}_h);\mathbb{R})$. □

To sum up, we have shown that provided we have at hand a connectivity array $\mathtt{j\_dof} : \mathcal{T}_h \times \mathcal{N} \to \mathcal{A}_h$ satisfying the properties (19.2) and (19.3), we can build in a simple manner the global basis functions and the global dofs in the conforming finite element subspace $P_k^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$. The actual construction of the mapping `j_dof` will be undertaken in the following two chapters.

Remark 19.5 (Connectivity class). Another way to formalize the grouping of the dofs consists of introducing the equivalence relation $\mathcal{R}$ in $\mathcal{T}_h\times\mathcal{N}$ defined by $(K,i)\,\mathcal{R}\,(K',i')$ iff $\mathtt{j\_dof}(K,i) = \mathtt{j\_dof}(K',i')$. One can then redefine $\mathcal{A}_h$ to be the set of the equivalence classes for $\mathcal{R}$. The elements of $\mathcal{A}_h$ are then sets and are called connectivity classes. In this case, we write $(K,i) \in a$ instead of $\mathtt{j\_dof}(K,i) = a$. We are going to adopt this equivalent viewpoint from Chapter 20 onward. □

19.2 Examples

In this section, we illustrate the concepts developed in §19.1 for the spaces $P_k^{\mathrm{g}}(\mathcal{T}_h)$, $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$, and $\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$.

19.2.1 H^1-conforming subspace $P_k^{\mathrm{g}}(\mathcal{T}_h)$

Let $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$ be one of the scalar-valued Lagrange elements of degree $k \geq 1$ introduced in §6.4 or §7.4, or one of the canonical hybrid finite elements of degree $k \geq 1$ introduced in §7.6. The broken finite element space is

$$P_k^{\mathrm{g,b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in L^\infty(D) \mid \psi_K^{\mathrm{g}}(v_h) \in \widehat{P}^{\mathrm{g}},\ \forall K \in \mathcal{T}_h\}, \tag{19.9}$$

where $\psi_K^{\mathrm{g}}(v) := v \circ \boldsymbol{T}_K$ is the pullback by the geometric mapping, and the corresponding H^1-conforming subspace is

$$P_k^{\mathrm{g}}(\mathcal{T}_h) := \{v_h \in P_k^{\mathrm{g,b}}(\mathcal{T}_h) \mid [\![v_h]\!]_F = 0, \forall F \in \mathcal{F}_h^\circ\}. \tag{19.10}$$

We have $P_k^{\mathrm{g}}(\mathcal{T}_h) \subset Z^{\mathrm{g},p}(D) := W^{1,p}(D) = \{v \in L^p(D) \mid \nabla v \in \boldsymbol{L}^p(D)\}$ for all $p \in [1,\infty]$ (note that $Z^{\mathrm{g},2}(D) := H^1(D)$). We show in Figure 19.1 the connectivity classes generated by j_dof on a mesh composed of four triangles with $\mathbb{P}_{2,2}$ Lagrange elements.

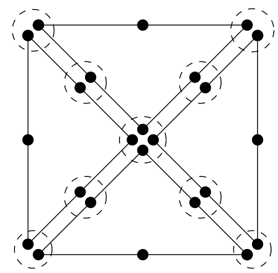

Fig. 19.1 $\mathbb{P}_{2,2}$ Lagrange nodes in the same connectivity class for a mesh composed of four triangles (drawn slightly apart).

The Lagrange and the canonical hybrid finite elements of the same degree generate the same space $P_k^{\mathrm{g}}(\mathcal{T}_h)$, but the shape functions and dofs differ for $k \geq 2$. Some global shape functions in $\mathbb{P}_1^{\mathrm{g}}(\mathcal{T}_h)$ and $\mathbb{P}_2^{\mathrm{g}}(\mathcal{T}_h)$ in dimension 2 are shown in Figure 19.2 for Lagrange elements. The function shown in the left panel is continuous and piecewise affine, and it takes the value 1 at one mesh vertex and the value 0 at all the other mesh vertices. Because its graph is reminiscent of a hat, this function is often called *hat basis function* (and sometimes also *Courant basis functions* [84]). The functions shown in the central and right panels are continuous and piecewise quadratic. The function on the central panel takes the value 1 at one mesh vertex and the value 0 at all the other mesh vertices, and it takes the value 0 at all the edge midpoints. The function in the right panel takes the value 0 at all the mesh vertices, and it takes the value 1 at one edge midpoint and the value 0 at the midpoint of all the other edges.

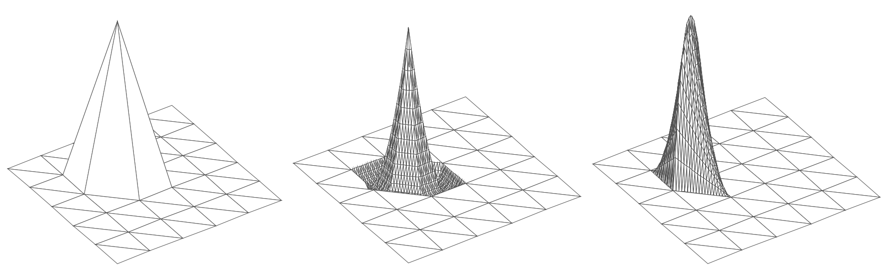

Fig. 19.2 Global shape functions in dimension 2: $\mathbb{P}_{1,2}$ (left) and $\mathbb{P}_{2,2}$ (center and right) Lagrange finite elements.

Let N_{v}, N_{e}, N_{f}, N_{c} be the number of vertices, edges, faces, and cells in the mesh $\mathcal{T}_h$ (recall that $\mathcal{T}_h$ is assumed to be a matching mesh). For a simplicial Lagrange element, the number of Lagrange nodes per edge that are not located at the extremities of the edge is $\binom{k-1}{1}$ (if $k \geq 2$), the number of Lagrange nodes per face that are not located at the boundary of the face is $\binom{k-1}{2}$ (if $k \geq 3$), and the number of Lagrange nodes per cell that not located at the boundary of the cell is $\binom{k-1}{3}$ (if $k \geq 4$). These numbers are the same for the canonical hybrid finite element. We will establish in Chapter 21 that

$$\dim(P_k^{\mathrm{g}}(\mathcal{T}_h)) = N_{\mathrm{v}} + \binom{k-1}{1}N_{\mathrm{e}} + \binom{k-1}{2}N_{\mathrm{f}} + \binom{k-1}{3}N_{\mathrm{c}} \quad \text{if } d = 3, \tag{19.11a}$$

$$\dim(P_k^{\mathrm{g}}(\mathcal{T}_h)) = N_{\mathrm{v}} + \binom{k-1}{1}N_{\mathrm{e}} + \binom{k-1}{2}N_{\mathrm{c}} \quad \text{if } d = 2, \tag{19.11b}$$

with the convention that for natural numbers n, m, $\binom{n}{m} := 0$ if $n < m$. In the lowest-order case ($k = 1$), we have $\dim(P_1^{\mathrm{g}}(\mathcal{T}_h)) = N_{\mathrm{v}}$, and the connectivity array `j_dof` coincides with the double-entry array `j_cv` defined in §8.3.

19.2.2 $\boldsymbol{H}(\mathrm{curl})$-conforming subspace $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$

Let $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{c}}, \widehat{\Sigma}^{\mathrm{c}})$ be one of the Nédélec finite elements of degree $k \geq 0$ described in Chapter 15. The broken finite element space is

$$\boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^{\infty}(D) \mid \boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{c}},\ \forall K \in \mathcal{T}_h\}, \tag{19.12}$$

with the covariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) := \mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v} \circ \boldsymbol{T}_K)$, and the corresponding $\boldsymbol{H}(\mathrm{curl})$-conforming subspace is

$$\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F \times \boldsymbol{n}_F = \boldsymbol{0},\ \forall F \in \mathcal{F}_h^{\circ}\}. \tag{19.13}$$

We have $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) \subset \boldsymbol{Z}^{\mathrm{c},p}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^p(D)\}$ for all $p \in [1, \infty]$ (note that $\boldsymbol{Z}^{\mathrm{c},2}(D) := \boldsymbol{H}(\mathrm{curl}; D)$). A global shape function attached to an edge is shown in the left panel of Figure 19.3 for the $\mathbb{N}_{0,2}$ element. Notice that the tangential component is continuous across the interface, but the normal component is not.

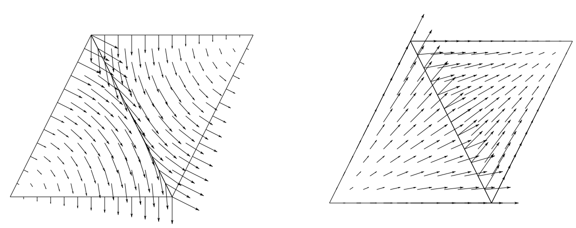

Fig. 19.3 Global shape functions for the lowest-order Nédélec (left) and Raviart–Thomas (right) elements in dimension 2.

Let N_{e}, N_{f}, N_{c} be the number of edges, faces, and cells in $\mathcal{T}_h$. We will show in Chapter 21 that

$$\dim(\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)) = \binom{k+1}{1}N_{\mathrm{e}} + 2\binom{k+1}{2}N_{\mathrm{f}} + 3\binom{k+1}{3}N_{\mathrm{c}}, \qquad \text{if } d=3,$$
$$\dim(\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)) = \binom{k+1}{1}N_{\mathrm{e}} + 2\binom{k+1}{2}N_{\mathrm{c}}, \qquad \text{if } d=2,$$

with the convention that $\binom{n}{m} := 0$ if $n < m$. In the lowest-order case ($k = 0$), we have $\dim(\boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h)) = N_{\mathrm{e}}$, and the connectivity array `j_dof` coincides with the double-entry array `j_ce` defined in §8.3.

19.2.3 $\boldsymbol{H}(\mathrm{div})$-conforming subspace $\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$

Let $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{d}}, \widehat{\Sigma}^{\mathrm{d}})$ be one of the Raviart–Thomas finite elements of degree $k \geq 0$ introduced in Chapter 14. The broken finite element space is

$$\boldsymbol{P}_k^{\mathrm{d,b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^1(D) \mid \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{d}},\ \forall K \in \mathcal{T}_h\}, \tag{19.15}$$

with the contravariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) := \det(\mathbb{J}_K)\,\mathbb{J}_K^{-1}(\boldsymbol{v}\circ\boldsymbol{T}_K)$. The corresponding $\boldsymbol{H}(\mathrm{div})$-conforming subspace is

$$\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{d,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F\cdot\boldsymbol{n}_F = 0,\ \forall F \in \mathcal{F}_h^{\circ}\}. \tag{19.16}$$

We have $\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h) \subset \boldsymbol{Z}^{\mathrm{d},p}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \nabla\cdot\boldsymbol{v} \in L^p(D)\}$ for all $p \in [1,\infty]$ (note that $\boldsymbol{Z}^{\mathrm{d},2}(D) := \boldsymbol{H}(\mathrm{div}; D)$). A global shape function attached to a face is shown in the right panel of Figure 19.3 for the $\mathbb{RT}_{0,2}$ element (the normal component is continuous across the interface, but the tangential component is not). We will establish in Chapter 21 that

$$\dim(\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)) = \binom{k+2}{2}N_{\mathrm{f}} + 3\binom{k+2}{3}N_{\mathrm{c}}, \qquad \text{if } d=3, \tag{19.17a}$$
$$\dim(\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)) = \binom{k+2}{1}N_{\mathrm{f}} + 2\binom{k+2}{2}N_{\mathrm{c}}, \qquad \text{if } d=2, \tag{19.17b}$$

with the convention that $\binom{n}{m} := 0$ if $n < m$. Notice that the spaces $\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h)$ and $\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$ have the same dimension when $d = 2$. In the lowest-order case ($k = 0$), we have $\dim(\boldsymbol{P}_0^{\mathrm{d}}(\mathcal{T}_h)) = N_{\mathrm{f}}$, and the connectivity array `j_dof` coincides with the double-entry array `j_cf` defined in §8.3.

19.3 Global interpolation operators

The goal of this section is to study the commuting and approximation properties of the global interpolation operators in the conforming finite element subspaces $P_k^{\mathrm{x}}(\mathcal{T}_h; \mathbb{R}^q)$ with $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}, \mathrm{d}\}$. Recall that $q = 1$ if $\mathrm{x} = \mathrm{g}$ and $q = d$ if $\mathrm{x} \in \{\mathrm{c}, \mathrm{d}\}$ (and $d = 3$ if $\mathrm{x} = \mathrm{c}$). We start by introducing the global spaces

$$V^{\mathrm{x,b}}(D) := \{v \in L^1(D; \mathbb{R}^q) \mid v_{|K} \in V^{\mathrm{x}}(K),\ \forall K \in \mathcal{T}_h\}, \tag{19.18a}$$
$$V^{\mathrm{x}}(D) := \{v \in V^{\mathrm{x,b}}(D) \mid [\![v]\!]_F^{\mathrm{x}} = 0,\ \forall F \in \mathcal{F}_h^{\circ}\}, \tag{19.18b}$$

where $V^{\mathrm{x}}(K)$ is the domain of the local interpolation operator $\mathcal{I}_K^{\mathrm{x}}$ (see Definition 5.7). For instance, owing to Theorem 18.8 and Theorem 18.10 and letting $p \in [1,\infty)$, admissible choices for these spaces are as follows:

$$V^{\mathrm{g}}(D) := W^{s,p}(D), \quad \text{with } s > \tfrac{d}{p} \text{ if } p > 1 \text{ or } s = d \text{ if } p = 1, \tag{19.19a}$$
$$\boldsymbol{V}^{\mathrm{c}}(D) := \boldsymbol{W}^{s,p}(D), \quad \text{with } s > \tfrac{2}{p} \text{ if } p > 1 \text{ or } s = 2 \text{ if } p = 1, \tag{19.19b}$$
$$\boldsymbol{V}^{\mathrm{d}}(D) := \boldsymbol{W}^{s,p}(D), \quad \text{with } s > \tfrac{1}{p} \text{ if } p > 1 \text{ or } s = 1 \text{ if } p = 1. \tag{19.19c}$$

Recall that since Chapter 5 we have abused the notation regarding the definition of the dofs. In particular, we have used the same symbols to denote the dofs in $\mathcal{L}(P_K;\mathbb{R})$ and the extended dofs in $\mathcal{L}(V(K);\mathbb{R})$. We are going to be a little bit more careful in this chapter and in Chapters 20 and 21. More precisely, we are going to use the symbol $\sigma_{K,i}$ to denote dofs acting on functions in P_K and the symbol $\widetilde{\sigma}_{K,i}$ to denote the extension of $\sigma_{K,i}$ acting on functions in $V^{\mathrm{x}}(K)$. This means that the local interpolation operator $\mathcal{I}_K : V^{\mathrm{x}}(K) \to P_K$ is s.t.

$$\mathcal{I}_K(v)(\boldsymbol{x}) := \sum_{i\in\mathcal{N}} \widetilde{\sigma}_{K,i}(v)\theta_{K,i}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in K. \tag{19.20}$$

We assume that the extension of the dofs is done in such a way that the following property holds true (compare with (19.2)): For all $v \in V^{\mathrm{x,b}}(D)$,

$$[\, v \in V^{\mathrm{x}}(D)\,] \implies \begin{bmatrix} \text{For all } (K,i),(K',i') \text{ in the same} \\ \text{connectivity class, we have} \\ \widetilde{\sigma}_{K,i}(v_{|K}) = \widetilde{\sigma}_{K',i'}(v_{|K'}) \end{bmatrix}. \tag{19.21}$$

In other words, for every function v in $V^{\mathrm{x,b}}(D)$, a *necessary* condition for v to be a member of the subspace $V^{\mathrm{x}}(D)$ is that, for all $a \in \mathcal{A}_h$, the quantity $\widetilde{\sigma}_{K,i}(v_{|K})$ is independent of the choice of the pair (K,i) in $\mathtt{j\_dof}^{-1}(a)$. (This condition is not sufficient since the knowledge of the values of $\{\widetilde{\sigma}_{K,i}(v_{|K})\}_{i\in\mathcal{N}}$ does not uniquely determine the function $v_{|K}$.) We then define the global interpolation operator $\mathcal{I}_h^{\mathrm{x}} : V^{\mathrm{x}}(D) \to P_k^{\mathrm{x}}(\mathcal{T}_h)$ s.t.

$$\mathcal{I}_h^{\mathrm{x}}(v)(\boldsymbol{x}) := \sum_{a\in\mathcal{A}_h} \widetilde{\sigma}_a(v)\varphi_a(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \overline{D}, \tag{19.22}$$

where $\widetilde{\sigma}_a(v)$ is defined by setting $\widetilde{\sigma}_a(v) := \widetilde{\sigma}_{K,i}(v_{|K})$ for all (K,i) in the connectivity class a, i.e., $\mathtt{j\_dof}(K,i) = a$, which makes sense owing to (19.21). The definitions of $\widetilde{\sigma}_a$ and φ_a imply that

$$\mathcal{I}_h^{\mathrm{x}}(v)_{|K} = \sum_{i\in\mathcal{N}} \widetilde{\sigma}_{K,i}(v_{|K})\theta_{K,i} = \mathcal{I}_K^{\mathrm{x}}(v_{|K}), \qquad \forall K \in \mathcal{T}_h. \tag{19.23}$$

The above construction leads to the global interpolation operators:

$$\mathcal{I}^{\mathrm{L}}_{k,h} : V^{\mathrm{g}}(D) \to P^{\mathrm{g}}_k(\mathcal{T}_h), \qquad \mathcal{I}^{\mathrm{g}}_{k,h} : V^{\mathrm{g}}(D) \to P^{\mathrm{g}}_k(\mathcal{T}_h), \tag{19.24a}$$

$$\mathcal{I}^{\mathrm{c}}_{k,h} : \boldsymbol{V}^{\mathrm{c}}(D) \to \boldsymbol{P}^{\mathrm{c}}_k(\mathcal{T}_h), \qquad \mathcal{I}^{\mathrm{d}}_{k,h} : \boldsymbol{V}^{\mathrm{d}}(D) \to \boldsymbol{P}^{\mathrm{d}}_k(\mathcal{T}_h), \tag{19.24b}$$

for Lagrange, canonical hybrid, Nédélec, and Raviart–Thomas elements, respectively. We indicate explicitly the degree of the underlying finite element in the notation to avoid ambiguities. (Recall that $k \ge 1$ in (19.24a) and $k \ge 0$ in (19.24b).) Let us consider for $k \ge 0$ the L^2-orthogonal projection

$$\mathcal{I}^{\mathrm{b}}_{k,h} : V^{\mathrm{b}}(D) \to P^{\mathrm{b}}_k(\mathcal{T}_h) := \{v_h \in L^\infty(D) \mid \psi^{\mathrm{b}}_K(v_{h|K}) \in \widehat{P}^{\mathrm{b}}, \forall K \in \mathcal{T}_h\}, \tag{19.25}$$

where $V^{\mathrm{b}}(D) := L^1(D)$, $\psi^{\mathrm{b}}_K(v) := \det(\mathbb{J}_K)(v \circ \boldsymbol{T}_K)$, and $\widehat{P}^{\mathrm{b}} := \mathbb{P}_{k,d}$ if $\widehat{K}$ is a simplex and $\widehat{P}^{\mathrm{b}} := \mathbb{Q}_{k,d}$ is $\widehat{K}$ is a cuboid. Note that since the mesh is affine, the factor $\det(\mathbb{J}_K)$ is irrelevant in the definition of $P^{\mathrm{b}}_k(\mathcal{T}_h)$.

Lemma 19.6 (de Rham complex). *Let us set*

$$\check{V}^{\mathrm{g}}(D) := \{f \in V^{\mathrm{g}}(D) \mid \nabla f \in \boldsymbol{V}^{\mathrm{c}}(D)\}, \tag{19.26a}$$

$$\check{\boldsymbol{V}}^{\mathrm{c}}(D) := \{\boldsymbol{g} \in \boldsymbol{V}^{\mathrm{c}}(D) \mid \nabla{\times}\boldsymbol{g} \in \boldsymbol{V}^{\mathrm{d}}(D)\}, \tag{19.26b}$$

$$\check{\boldsymbol{V}}^{\mathrm{d}}(D) := \{\boldsymbol{g} \in \boldsymbol{V}^{\mathrm{d}}(D) \mid \nabla{\cdot}\boldsymbol{g} \in V^{\mathrm{b}}(D)\}. \tag{19.26c}$$

Let $\kappa \in \mathbb{N}$. The following diagrams commute:

$$\begin{array}{ccccccc}
\check{V}^{\mathrm{g}}(D) & \xrightarrow{\nabla} & \check{\boldsymbol{V}}^{\mathrm{c}}(D) & \xrightarrow{\nabla\times} & \check{\boldsymbol{V}}^{\mathrm{d}}(D) & \xrightarrow{\nabla\cdot} & V^{\mathrm{b}}(D) \\
\downarrow \mathcal{I}^{\mathrm{g}}_{\kappa+1,h} & & \downarrow \mathcal{I}^{\mathrm{c}}_{\kappa,h} & & \downarrow \mathcal{I}^{\mathrm{d}}_{\kappa,h} & & \downarrow \mathcal{I}^{\mathrm{b}}_{\kappa,h} \\
P^{\mathrm{g}}_{\kappa+1}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{P}^{\mathrm{c}}_{\kappa}(\mathcal{T}_h) & \xrightarrow{\nabla\times} & \boldsymbol{P}^{\mathrm{d}}_{\kappa}(\mathcal{T}_h) & \xrightarrow{\nabla\cdot} & P^{\mathrm{b}}_{\kappa}(\mathcal{T}_h)
\end{array} \tag{19.27}$$

Proof. Combine Lemma 16.16 (and Remark 16.18) with (19.23). □

Remark 19.7 (Interpolation with extended domain). The commuting diagram (19.27) shows that we can extend the domain of $\mathcal{I}^{\mathrm{c}}_{\kappa,h}$ to $\widetilde{\boldsymbol{V}}^{\mathrm{c}}(D) := \boldsymbol{V}^{\mathrm{c}}(D) + \nabla V^{\mathrm{g}}(D)$, that of $\mathcal{I}^{\mathrm{d}}_{\kappa,h}$ to $\widetilde{\boldsymbol{V}}^{\mathrm{d}}(D) := \boldsymbol{V}^{\mathrm{d}}(D) + \nabla{\times}\boldsymbol{V}^{\mathrm{c}}(D)$, and that of $\mathcal{I}^{\mathrm{b}}_{\kappa,h}$ to $\widetilde{V}^{\mathrm{b}}(D) := \boldsymbol{V}^{\mathrm{d}}(D) + \nabla{\cdot}V^{\mathrm{b}}(D)$. Keeping the same notation for the differential operators, this leads to the following commuting diagrams:

$$\begin{array}{ccccccc}
V^{\mathrm{g}}(D) & \xrightarrow{\nabla} & \widetilde{\boldsymbol{V}}^{\mathrm{c}}(D) & \xrightarrow{\nabla\times} & \widetilde{\boldsymbol{V}}^{\mathrm{d}}(D) & \xrightarrow{\nabla\cdot} & \widetilde{V}^{\mathrm{b}}(D) \\
\downarrow \mathcal{I}^{\mathrm{g}}_{\kappa+1,h} & & \downarrow \mathcal{I}^{\mathrm{c}}_{\kappa,h} & & \downarrow \mathcal{I}^{\mathrm{d}}_{\kappa,h} & & \downarrow \mathcal{I}^{\mathrm{b}}_{\kappa,h} \\
P^{\mathrm{g}}_{\kappa+1}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{P}^{\mathrm{c}}_{\kappa}(\mathcal{T}_h) & \xrightarrow{\nabla\times} & \boldsymbol{P}^{\mathrm{d}}_{\kappa}(\mathcal{T}_h) & \xrightarrow{\nabla\cdot} & P^{\mathrm{b}}_{\kappa}(\mathcal{T}_h)
\end{array} \tag{19.28}$$

For instance, for all $\boldsymbol{v} = \boldsymbol{w} + \nabla\psi \in \widetilde{\boldsymbol{V}}^{\mathrm{c}}(D)$ with $\boldsymbol{w} \in \boldsymbol{V}^{\mathrm{c}}(D)$ and $\psi \in V^{\mathrm{g}}(D)$, we set $\mathcal{I}^{\mathrm{c}}_{\kappa,h}(\boldsymbol{v}) := \mathcal{I}^{\mathrm{c}}_{\kappa,h}(\boldsymbol{w}) + \nabla\mathcal{I}^{\mathrm{g}}_{\kappa+1,h}(\psi)$. To verify that $\mathcal{I}^{\mathrm{c}}_{\kappa,h}(\boldsymbol{v})$ is well defined, we observe that if $\boldsymbol{v} = \boldsymbol{w}_1 + \nabla\psi_1 = \boldsymbol{w}_2 + \nabla\psi_2$, then $\psi_1 - \psi_2 \in \check{V}^{\mathrm{g}}(D)$ so that $\nabla(\mathcal{I}^{\mathrm{g}}_{\kappa+1,h}(\psi_1-\psi_2)) = \mathcal{I}^{\mathrm{c}}_{\kappa,h}(\nabla(\psi_1-\psi_2)) = \mathcal{I}^{\mathrm{c}}_{\kappa,h}(\boldsymbol{w}_2-\boldsymbol{w}_1)$. Thus, we have $\mathcal{I}^{\mathrm{c}}_{\kappa,h}(\boldsymbol{w}_1) + \nabla\mathcal{I}^{\mathrm{g}}_{\kappa+1,h}(\psi_1) = \mathcal{I}^{\mathrm{c}}_{\kappa,h}(\boldsymbol{w}_2) + \nabla\mathcal{I}^{\mathrm{g}}_{\kappa+1,h}(\psi_2)$. □

Let us now turn to the approximation properties of the global interpolation operators defined in (19.24). Henceforth, the subscript k is omitted when the context is unambiguous. The following results follow from the localization property (19.23) combined with the corresponding local interpolation results, and from Lemma 19.6 for the approximation properties on the divergence and the curl.

Corollary 19.8 (H^1-conforming interpolation). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine matching meshes. Let $p \in [1,\infty]$. Let l be the smallest integer such that $l > \frac{d}{p}$ if $p > 1$ or $l := d$ if $p = 1$. The following estimates hold true, uniformly w.r.t. p, with either $\mathcal{I}_h = \mathcal{I}^{\mathrm{g}}_h : V^{\mathrm{g}}(D) \to P^{\mathrm{g}}_k(\mathcal{T}_h)$ or $\mathcal{I}_h = \mathcal{I}^{\mathrm{L}}_h : V^{\mathrm{g}}(D) \to P^{\mathrm{g}}_k(\mathcal{T}_h)$, $k \geq 1$:*
(i) *If $l \leq k+1$, then for every integers $r \in \{l{:}k+1\}$ and $m \in \{0{:}r\}$, all $v \in W^{r,p}(D)$, and all $h \in \mathcal{H}$,*

$$|v - \mathcal{I}_h(v)|_{W^{m,p}(\mathcal{T}_h)} \leq c \left(\sum_{K\in\mathcal{T}_h} h_K^{p(r-m)} |v|^p_{W^{r,p}(K)} \right)^{\frac{1}{p}}, \tag{19.29}$$

for $p < \infty$, and $|v - \mathcal{I}_h(v)|_{W^{m,\infty}(\mathcal{T}_h)} \leq c \max_{K\in\mathcal{T}_h} h_K^{r-m} |v|_{W^{r,\infty}(K)}$.
(ii) *If $l > k+1$, then for every integer $m \in \{0{:}k+1\}$, all $v \in W^{l,p}(D)$, and all $h \in \mathcal{H}$,*

$$|v - \mathcal{I}_h(v)|_{W^{m,p}(\mathcal{T}_h)} \leq c \left(\sum_{K\in\mathcal{T}_h} \sum_{n\in\{k+1:l\}} h_K^{p(n-m)} |v|^p_{W^{n,p}(K)} \right)^{\frac{1}{p}}, \tag{19.30}$$

for $p < \infty$, and $|v - \mathcal{I}_h(v)|_{W^{m,\infty}(\mathcal{T}_h)} \leq c \max_{K\in\mathcal{T}_h, n\in\{k+1:l\}} h_K^{n-m} |v|_{W^{n,\infty}(K)}$.

Corollary 19.9 ($\boldsymbol{H}$(curl)-conforming interpolation). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine matching meshes. Let $p \in [1,\infty]$ and let $l := 1$ if $p > 2$ and $l := 2$ if $p \in [1,2]$. The following holds true, uniformly w.r.t. p, with $\mathcal{I}^{\mathrm{c}}_h : \boldsymbol{V}^{\mathrm{c}}(D) \to \boldsymbol{P}^{\mathrm{c}}_k(\mathcal{T}_h)$, $k \geq 0$:*
(i) *If $p > 2$ or if $p \in [1,2]$ and $k \geq 1$, then for every integers $r \in \{l{:}k+1\}$ and $m \in \{0{:}r\}$, all $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, and all $h \in \mathcal{H}$,*

$$|\boldsymbol{v} - \mathcal{I}^{\mathrm{c}}_h(\boldsymbol{v})|_{\boldsymbol{W}^{m,p}(\mathcal{T}_h)} \leq c \left(\sum_{K\in\mathcal{T}_h} h_K^{p(r-m)} |\boldsymbol{v}|^p_{\boldsymbol{W}^{r,p}(K)} \right)^{\frac{1}{p}}, \tag{19.31}$$

for $p < \infty$, and $|\boldsymbol{v} - \mathcal{I}^{\mathrm{c}}_h(\boldsymbol{v})|_{\boldsymbol{W}^{m,\infty}(\mathcal{T}_h)} \leq c \max_{K\in\mathcal{T}_h} h_K^{r-m} |\boldsymbol{v}|_{\boldsymbol{W}^{r,\infty}(K)}$.
(ii) *If $p \in [1,2]$ and $k = 0$, then for every integer $m \in \{0{:}1\}$, all $\boldsymbol{v} \in \boldsymbol{W}^{2,p}(D)$, and all $h \in \mathcal{H}$,*

$$|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,p}(\mathcal{T}_h)} \le c \left(\sum_{K\in\mathcal{T}_h} \sum_{n\in\{1,2\}} h_K^{p(n-m)} |\boldsymbol{v}|^p_{\boldsymbol{W}^{n,p}(K)} \right)^{\frac{1}{p}}, \tag{19.32}$$

for $p < \infty$, *and* $|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,\infty}(\mathcal{T}_h)} \le c \max_{K\in\mathcal{T}_h, n\in\{1,2\}} h_K^{n-m} |\boldsymbol{v}|_{\boldsymbol{W}^{n,\infty}(K)}$.
(iii) *For every integers* $r \in \{1{:}k+1\}$ *and* $m \in \{0{:}r\}$, *all* $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(D)$ *with* $\nabla{\times}\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, *and all* $h \in \mathcal{H}$,

$$|\nabla{\times}(\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c}}(\boldsymbol{v}))|_{\boldsymbol{W}^{m,p}(\mathcal{T}_h)} \le c \left(\sum_{K\in\mathcal{T}_h} h_K^{p(r-m)} |\nabla{\times}\boldsymbol{v}|^p_{\boldsymbol{W}^{r,p}(K)} \right)^{\frac{1}{p}}, \tag{19.33}$$

for $p < \infty$, *and* $|\nabla{\times}(\boldsymbol{v} - \mathcal{I}_h^{\mathrm{c}}(\boldsymbol{v}))|_{\boldsymbol{W}^{m,\infty}(\mathcal{T}_h)} \le c \max_{K\in\mathcal{T}_h} h_K^{r-m} |\nabla{\times}\boldsymbol{v}|_{\boldsymbol{W}^{r,\infty}(K)}$.

Corollary 19.10 ($\boldsymbol{H}$(div)-conforming interpolation). *Let* $(\mathcal{T}_h)_{h\in\mathcal{H}}$ *be a shape-regular sequence of affine matching meshes. Let* $p \in [1,\infty]$. *The following holds true, uniformly w.r.t.* p, *with* $\mathcal{I}_h^{\mathrm{d}} : \boldsymbol{V}^{\mathrm{d}}(D) \to \boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h)$, $k \ge 0$:
(i) *For every integers* $r \in \{1{:}k+1\}$ *and* $m \in \{0{:}r\}$, *all* $\boldsymbol{v} \in \boldsymbol{W}^{r,p}(D)$, *and all* $h \in \mathcal{H}$,

$$|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{d}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,p}(\mathcal{T}_h)} \le c \left(\sum_{K\in\mathcal{T}_h} h_K^{p(r-m)} |\boldsymbol{v}|^p_{\boldsymbol{W}^{r,p}(K)} \right)^{\frac{1}{p}}, \tag{19.34}$$

for $p < \infty$, *and* $|\boldsymbol{v} - \mathcal{I}_h^{\mathrm{d}}(\boldsymbol{v})|_{\boldsymbol{W}^{m,\infty}(\mathcal{T}_h)} \le c \max_{K\in\mathcal{T}_h} h_K^{r-m} |\boldsymbol{v}|_{\boldsymbol{W}^{r,\infty}(K)}$.
(ii) *For every integers* $r \in \{0{:}k+1\}$ *and* $m \in \{0{:}r\}$, *all* $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(D)$ *with* $\nabla{\cdot}\boldsymbol{v} \in W^{r,p}(D)$, *and all* $h \in \mathcal{H}$,

$$|\nabla{\cdot}(\boldsymbol{v} - \mathcal{I}_h^{\mathrm{d}}(\boldsymbol{v}))|_{W^{m,p}(\mathcal{T}_h)} \le c \left(\sum_{K\in\mathcal{T}_h} h_K^{p(r-m)} |\nabla{\cdot}\boldsymbol{v}|^p_{W^{r,p}(K)} \right)^{\frac{1}{p}}, \tag{19.35}$$

for $p < \infty$, *and* $|\nabla{\cdot}(\boldsymbol{v} - \mathcal{I}_h^{\mathrm{d}}(\boldsymbol{v}))|_{W^{m,\infty}(\mathcal{T}_h)} \le c \max_{K\in\mathcal{T}_h} h_K^{r-m} |\nabla{\cdot}\boldsymbol{v}|_{W^{\infty,p}(K)}$.

19.4 Subspaces with zero boundary trace

In this section, we briefly review the main changes to be applied when one wishes to enforce homogeneous boundary conditions to the functions in $P_k^{\mathrm{x}}(\mathcal{T}_h)$. Let $p \in [1,\infty)$ and let $s > \frac{d}{p}$ if $p > 1$ and $s = d$ if $p = 1$. We consider the trace operator $\gamma^{\mathrm{x}} : W^{s,p}(\mathcal{T}_h;\mathbb{R}^q) \longrightarrow L^1(\partial D;\mathbb{R}^t)$ defined by

$$\gamma^{\mathrm{g}}(v) := v_{|\partial D} \qquad (q = t = 1), \tag{19.36a}$$
$$\gamma^{\mathrm{c}}(\boldsymbol{v}) := \boldsymbol{v}_{|\partial D}{\times}\boldsymbol{n} \qquad (q = t = d = 3), \tag{19.36b}$$
$$\gamma^{\mathrm{d}}(\boldsymbol{v}) := \boldsymbol{v}_{|\partial D}{\cdot}\boldsymbol{n} \qquad (q = d, t = 1), \tag{19.36c}$$

where $\boldsymbol{n}$ is the outward unit normal to D. Notice that $\gamma^{\mathrm{x}}(v)_{|F} = \gamma^{\mathrm{x}}_{K_l,F}(v_{|K_l})$ for all $F \in \mathcal{F}_h^\partial$ with $F := \partial K_l \cap \partial D$ and $\gamma^{\mathrm{x}}_{K_l,F}$ is the operator defined in (18.7) for the mesh cell K_l. We are interested in the following subspace of $P_k^{\mathrm{x}}(\mathcal{T}_h)$:

$$P_{k,0}^{\mathrm{x}}(\mathcal{T}_h) := \{v_h \in P_k^{\mathrm{x}}(\mathcal{T}_h) \mid \gamma^{\mathrm{x}}(v) = 0\}. \tag{19.37}$$

Definition 19.11 (Boundary & internal classes). *We say that a connectivity class $a \in \mathcal{A}_h$ is a* boundary connectivity class *if and only if $\sigma_a(v) = 0$ for all $v \in P_{k,0}^{\mathrm{x}}(\mathcal{T}_h)$. The collection of boundary connectivity classes is denoted by $\mathcal{A}_h^\partial$. The classes in $\mathcal{A}_h^\circ := \mathcal{A}_h \setminus \mathcal{A}_h^\partial$ are called* internal connectivity classes*.*

We assume that the following properties hold true:

$$\forall v_h \in P_k^{\mathrm{x}}(\mathcal{T}_h), \qquad [\,\gamma^{\mathrm{x}}(v_h) = 0\,] \iff [\,\sigma_a(v_h) = 0,\ \forall a \in \mathcal{A}_h^\partial\,], \tag{19.38a}$$

$$\forall v \in V^{\mathrm{x}}(D), \qquad [\,\gamma^{\mathrm{x}}(v) = 0\,] \implies [\,\widetilde{\sigma}_a(v) = 0,\ \forall a \in \mathcal{A}_h^\partial\,]. \tag{19.38b}$$

We are going to show in Chapters 20 and 21 that these properties are indeed satisfied by most of the finite elements considered in this book.

Example 19.12 ($\mathcal{A}_h^\partial$). For Lagrange elements, $a \in \mathcal{A}_h^\partial$ iff σ_a is an evaluation at a node located on ∂D. For canonical hybrid elements, $a \in \mathcal{A}_h^\partial$ iff σ_a is an evaluation at a vertex located on ∂D, or σ_a is an integral over an edge or a face located on ∂D. For Nédélec elements, $a \in \mathcal{A}_h^\partial$ iff σ_a is an integral over an edge or a face located on ∂D, and for Raviart–Thomas elements, $a \in \mathcal{A}_h^\partial$ iff σ_a is an integral over a face located on ∂D. □

Proposition 19.13 (Basis). *$\{\varphi_a\}_{a\in\mathcal{A}_h^\circ}$ is a basis of $P_{k,0}^{\mathrm{x}}(\mathcal{T}_h)$, and $\{\sigma_a\}_{a\in\mathcal{A}_h^\circ}$ is a basis of $\mathcal{L}(P_{k,0}^{\mathrm{x}}(\mathcal{T}_h);\mathbb{R})$.*

Proof. See Exercise 19.3. □

Let $V^{\mathrm{x}}(D)$ be defined in (19.19). Since functions in $V^{\mathrm{x}}(D)$ have a γ^{x}-trace on ∂D, it is legitimate to set

$$V_0^{\mathrm{x}}(D) := \{v \in V^{\mathrm{x}}(D) \mid \gamma^{\mathrm{x}}(v) = 0\}. \tag{19.39}$$

The interpolation operator with prescribed boundary conditions $\mathcal{I}_{h0}^{\mathrm{x}} : V_0^{\mathrm{x}}(D) \to P_{k,0}^{\mathrm{x}}(\mathcal{T}_h)$ acts as follows:

$$\mathcal{I}_{h0}^{\mathrm{x}}(v)(\boldsymbol{x}) := \sum_{a\in\mathcal{A}_h^\circ} \widetilde{\sigma}_a(v)\varphi_a(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \overline{D}, \tag{19.40}$$

and (19.38b) implies that

$$\mathcal{I}_{h0}^{\mathrm{x}}(v) = \mathcal{I}_h^{\mathrm{x}}(v), \qquad \forall v \in V_0^{\mathrm{x}}(D). \tag{19.41}$$

Hence, the approximation properties of $\mathcal{I}_{h0}^{\mathrm{x}}$ are identical to those of the restriction of $\mathcal{I}_h^{\mathrm{x}}$ to $V_0^{\mathrm{x}}(D)$. Moreover, we have the following commuting properties.

Lemma 19.14 (de Rham complex with boundary prescription). *Let* $\check{V}_0^{\mathrm{x}}(D) := \{v \in \check{V}^{\mathrm{x}}(D) \mid \gamma^{\mathrm{x}}(v) = 0\}$ *with* $\check{V}^{\mathrm{x}}(D)$ *defined in* (19.26)*, and*

$$V_0^{\mathrm{b}}(D) := \{v \in V^{\mathrm{b}}(D) := L^1(D) \mid (v,1)_{L^2(D)} = 0\}, \tag{19.42a}$$

$$P_{\kappa,0}^{\mathrm{b}}(\mathcal{T}_h) := \{v_h \in P_\kappa^{\mathrm{b}}(\mathcal{T}_h) \mid (v_h,1)_{L^2(D)} = 0\}. \tag{19.42b}$$

Let $\kappa \in \mathbb{N}$*. The following diagrams commute:*

$$\begin{array}{ccccccc} \check{V}_0^{\mathrm{g}}(D) & \xrightarrow{\nabla} & \check{\boldsymbol{V}}_0^{\mathrm{c}}(D) & \xrightarrow{\nabla\times} & \check{\boldsymbol{V}}_0^{\mathrm{d}}(D) & \xrightarrow{\nabla\cdot} & V_0^{\mathrm{b}}(D) \\ \downarrow \mathcal{I}_{\kappa+1,h0}^{\mathrm{g}} & & \downarrow \mathcal{I}_{\kappa,h0}^{\mathrm{c}} & & \downarrow \mathcal{I}_{\kappa,h0}^{\mathrm{d}} & & \downarrow \mathcal{I}_h^{\mathrm{b}} \\ P_{\kappa+1,0}^{\mathrm{g}}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{P}_{\kappa,0}^{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\nabla\times} & \boldsymbol{P}_{\kappa,0}^{\mathrm{d}}(\mathcal{T}_h) & \xrightarrow{\nabla\cdot} & P_{\kappa,0}^{\mathrm{b}}(\mathcal{T}_h) \end{array} \tag{19.43}$$

Proof. Observe that the tangential boundary trace of ∇f is zero if $\gamma^{\mathrm{g}}(f) = 0$ and that the normal trace of $\nabla\times\boldsymbol{g}$ is zero if $\gamma^{\mathrm{c}}(\boldsymbol{g}) = \boldsymbol{0}$. □

Remark 19.15 (Extensions). The above argumentation can be adapted to enforce a zero trace on a part of the boundary that corresponds to a strict subset of the boundary faces in $\mathcal{F}_h^{\partial}$. The details are left to the reader. Furthermore, the commuting diagram (19.43) can be rewritten by using the spaces $V_0^{\mathrm{g}}(D)$, $\boldsymbol{V}_0^{\mathrm{c}}(D)+\nabla V_0^{\mathrm{g}}(D)$, $\boldsymbol{V}_0^{\mathrm{d}}(D)+\nabla\times\boldsymbol{V}_0^{\mathrm{c}}(D)$, and $\boldsymbol{V}_0^{\mathrm{b}}(D)+\nabla\cdot\boldsymbol{V}_0^{\mathrm{d}}(D)$ instead of $\check{V}_0^{\mathrm{g}}(D)$, $\check{\boldsymbol{V}}_0^{\mathrm{c}}(D)$, $\check{\boldsymbol{V}}_0^{\mathrm{d}}(D)$, $V_0^{\mathrm{b}}(D)$. □

Exercises

Exercise 19.1 (Connectivity classes). Consider the mesh shown in Figure 19.4 and let $P_2^{\mathrm{g}}(\mathcal{T}_h)$ be the associated finite element space composed of continuous Lagrange $\mathbb{P}_2$ finite elements. Assume that the enumeration of the Lagrange nodes has been done with the increasing vertex-index technique (see (10.10)). (i) What is the domain and the codomain of `j_dof`? (ii) Identify `j_dof`$^{-1}(8)$ and `j_dof`$^{-1}(13)$. (iii) Identify $\mathcal{T}_6$ and $\mathcal{T}_{10}$.

Exercise 19.2 (Stiffness, mass, incidence matrices). Let $\{\lambda_n\}_{n\in\{1:N_{\mathrm{v}}\}}$ be the global shape functions in $P_1^{\mathrm{g}}(\mathcal{T}_h)$. Let $\{\boldsymbol{\theta}_m\}_{m\in\{1:N_{\mathrm{e}}\}}$ be the global shape functions in $\boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h)$. (i) Recall the incidence matrix $\mathcal{M}^{\mathrm{ev}} \in \mathbb{R}^{N_{\mathrm{e}}\times N_{\mathrm{v}}}$ defined in Remark 10.2. Prove that $\nabla\lambda_n = \sum_{m\in\{1:N_{\mathrm{e}}\}} \mathcal{M}_{mn}^{\mathrm{ev}}\boldsymbol{\theta}_m$ for all $n \in \{1:N_{\mathrm{v}}\}$. (*Hint*: compute $\sigma_m^{\mathrm{e}}(\nabla\lambda_n)$ where $\{\sigma_m^{\mathrm{e}}\}_{m\in\{1:N_{\mathrm{e}}\}}$ is the dual basis of $\{\boldsymbol{\theta}_m\}_{m\in\{1:N_{\mathrm{e}}\}}$, i.e., the associated dofs.) (ii) Let $\mathcal{A} \in \mathbb{R}^{N_{\mathrm{v}}\times N_{\mathrm{v}}}$ be the Courant stiffness matrix with entries $\mathcal{A}_{nn'} := \int_D \nabla\lambda_n\cdot\nabla\lambda_{n'}\,\mathrm{d}x$ for all $n, n' \in \{1:N_{\mathrm{v}}\}$,

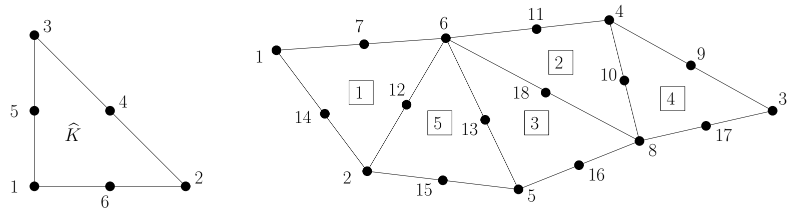

Fig. 19.4 Illustration for Exercise 19.1.

and let $\mathcal{N} \in \mathbb{R}^{N_e \times N_e}$ be the Nédélec mass matrix with entries $\mathcal{N}_{mm'} := \int_D \boldsymbol{\theta}_m \cdot \boldsymbol{\theta}_{m'} \, dx$ for all $m, m' \in \{1{:}N_e\}$. Prove that $\mathcal{A} = (\mathcal{M}^{\text{ev}})^{\mathsf{T}} \mathcal{N} \mathcal{M}^{\text{ev}}$.

Exercise 19.3 (Zero trace). (i) Show that $\varphi_a \in P^{\text{x}}_{k,0}(\mathcal{T}_h)$ for all $a \in \mathcal{A}^{\circ}_h$. (ii) Prove Proposition 19.13.

Exercise 19.4 (Approximability in L^p). Let $p \in [1, \infty)$. Prove that $\lim_{h\downarrow 0} \inf_{v_h \in P^{\text{g}}_k(\mathcal{T}_h)} \|v - v_h\|_{L^p(D)} = 0$ for all $v \in L^p(D)$. (*Hint*: by density.)

Exercise 19.5 (Hermite). Let $\mathcal{T}_h := \{[x_i, x_{i+1}]\}_{i \in \{0:I\}}$ be a mesh of the interval $D := (a, b)$. Recall the Hermite finite element from Exercise 5.4. Specify global shape functions $\{\varphi_{i,0}, \varphi_{i,1}\}_{i \in \{0:I+1\}}$ in $H_h := \{v_h \in C^1(\overline{D}) \mid \forall i \in \{0{:}I\},\ v_{h|[x_i, x_{i+1}]} \in \mathbb{P}_3\}$. (*Hint*: consider values of the function or of its derivative at the mesh nodes.) Can the bicubic Hermite rectangular finite element from Exercise 6.8 be used to enforce C^1-continuity for $d = 2$?