Part IV, Chapter 20

Face gluing

The goal of this chapter and the following one is to construct the connectivity array `j_dof` introduced in the previous chapter so that the two structural properties (19.2) and (19.3) hold true. In the present chapter, we focus on (19.2), and more specifically we are going to see how we can enforce the zero-jump condition $[\![v_h]\!]^{\mathrm{x}}_F = 0$ by means of the degrees of freedom (dofs) on the two mesh cells sharing the interface $F \in \mathcal{F}^{\circ}_h$ for v_h in the broken finite element space $P^{\mathrm{x,b}}_k(\mathcal{T}_h)$. In particular, we identify two key structural assumptions on the dofs of the finite element making this construction possible. The first assumption is called face unisolvence (see Assumption 20.1), and the second one is called face matching (see Assumption 20.3). We first introduce these ideas with Lagrange elements to make the argumentation easier to understand. Then we generalize the concepts to the Nédélec and the Raviart–Thomas finite elements in a unified setting that encompasses all the finite elements considered in the book. The two main results of this chapter are Lemma 20.4 for Lagrange elements and Lemma 20.15 for the general situation. In the entire chapter, D is a polyhedron in $\mathbb{R}^d$ and $\mathcal{T}_h$ is an oriented matching mesh covering D exactly (see Chapter 10 on mesh orientation).

20.1 The two gluing assumptions (Lagrange)

For Lagrange elements our aim is to construct the H^1-conforming subspace

$$P^{\mathrm{g}}_k(\mathcal{T}_h) := \{v_h \in P^{\mathrm{g,b}}_k(\mathcal{T}_h) \mid [\![v_h]\!]^{\mathrm{g}}_F = 0,\ \forall F \in \mathcal{F}^{\circ}_h\}, \tag{20.1}$$

where $P^{\mathrm{g,b}}_k(\mathcal{T}_h)$ is a broken finite element space and $[\![\cdot]\!]^{\mathrm{g}}_F := [\![\cdot]\!]_F$ is the jump operator across the mesh interfaces introduced in Definition 8.10. Recall that we have $P^{\mathrm{g}}_k(\mathcal{T}_h) = P^{\mathrm{g,b}}_k(\mathcal{T}_h) \cap H^1(D)$.

The Lagrange nodes of the reference cell $\widehat{K}$ are denoted by $\{\widehat{\boldsymbol{a}}_i\}_{i\in\mathcal{N}}$ so that the dofs $\widehat{\Sigma} := \{\widehat{\sigma}_i\}_{i\in\mathcal{N}}$ are s.t. $\widehat{\sigma}_i(\widehat{p}) := \widehat{p}(\widehat{\boldsymbol{a}}_i)$ for all $i \in \mathcal{N}$ and all $\widehat{p} \in \widehat{P}$. The

Lagrange nodes of $K \in \mathcal{T}_h$ are denoted by $\{\boldsymbol{a}_{K,i} := \boldsymbol{T}_K(\widehat{\boldsymbol{a}}_i)\}_{i\in\mathcal{N}}$, where $\boldsymbol{T}_K : \widehat{K} \to K$ is the geometric mapping. The dofs in K are s.t. $\sigma_{K,i}(p) = p(\boldsymbol{a}_{K,i})$ for all $i \in \mathcal{N}$ and all $p \in P_K$ with $P_K := (\psi_K^{\mathrm{g}})^{-1}(\widehat{P})$, where $\psi_K^{\mathrm{g}}(v) := v \circ \boldsymbol{T}_K$ is the pullback by the geometric mapping. We do not assume in this section that the geometric mapping $\boldsymbol{T}_K$ is affine.

We now formalize the structure of the reference element that will allow us to enforce the zero-jump condition in (20.1). We make two assumptions which we will show hold true in the next section for the simplicial and the tensor-product Lagrange elements. Our first key assumption is the following.

Assumption 20.1 (Face unisolvence). *Let $\widehat{F}$ be a face of $\widehat{K}$, i.e., $\widehat{F} \in \mathcal{F}_{\widehat{K}}$, and let $\mathcal{N}_{\widehat{K},\widehat{F}} \subset \mathcal{N}$ be the collection of the indices of the Lagrange nodes in $\widehat{K}$ located on $\widehat{F}$. We assume that*

$$\forall \widehat{p} \in \widehat{P}, \qquad [\,\widehat{\sigma}_i(\widehat{p}) = 0,\, \forall i \in \mathcal{N}_{\widehat{K},\widehat{F}}\,] \iff [\,\widehat{p}_{|\widehat{F}} = 0\,]. \tag{20.2}$$

Let K be a mesh cell and let F be a face of K., i.e., $F \in \mathcal{F}_K$. Let $\widehat{F}$ be the face of $\widehat{K}$ s.t. $\widehat{F} := \boldsymbol{T}_K^{-1}(F)$. Let $\mathcal{N}_{K,F} \subset \mathcal{N}$ be the collection of the indices of the Lagrange nodes in K located on F. The above definitions imply that

$$[\,i \in \mathcal{N}_{K,F}\,] \iff [\,\boldsymbol{a}_{K,i} \in F\,] \iff [\,\widehat{\boldsymbol{a}}_i \in \widehat{F}\,] \iff [\,i \in \mathcal{N}_{\widehat{K},\widehat{F}}\,], \tag{20.3}$$

that is, we have

$$\mathcal{N}_{K,F} = \mathcal{N}_{\widehat{K},\widehat{F}} = \mathcal{N}_{\widehat{K},\boldsymbol{T}_K^{-1}(F)}, \qquad \forall K \in \mathcal{T}_h,\ \forall F \in \mathcal{F}_K. \tag{20.4}$$

We define the trace space $P_{K,F} := \operatorname{span}\{\theta_{K,i|F}\}_{i\in\mathcal{N}_{K,F}}$, so that $P_{K,F} = \gamma_{K,F}^{\mathrm{g}}(P_K)$, where we recall that the trace map $\gamma_{K,F}^{\mathrm{g}}$ is defined by setting $\gamma_{K,F}^{\mathrm{g}}(v) := v_{|F}$ for all $v \in P_K$. We define the set of the dofs associated with the Lagrange nodes located on F, $\Sigma_{K,F} := \{\sigma_{K,F,i}\}_{i\in\mathcal{N}_{K,F}}$, by setting $\sigma_{K,F,i}(q) := q(\boldsymbol{a}_{K,i})$ for all $i \in \mathcal{N}_{K,F}$ and all $q \in P_{K,F}$. Notice that $\sigma_{K,F,i}$ acts on functions in $P_{K,F}$ (i.e., functions defined on F), whereas $\sigma_{K,i}$ acts on functions in P_K (i.e., functions defined on K).

Let us state an important consequence of Assumption 20.1.

Lemma 20.2 (Face element). *Let $K \in \mathcal{T}_h$ and $F \in \mathcal{F}_K$. Under Assumption 20.1, the triple $(F, P_{K,F}, \Sigma_{K,F})$ is a finite element.*

Proof. We use Remark 5.3 to prove unisolvence. Since we have

$$\sigma_{K,F,j}(\theta_{K,i|F}) = \theta_{K,i|F}(\boldsymbol{a}_{K,j}) = \theta_{K,i}(\boldsymbol{a}_{K,j}) = \delta_{ij},$$

for all $i, j \in \mathcal{N}_{K,F}$, we infer that the family $\{\theta_{K,i|F}\}_{i\in\mathcal{N}_{K,F}}$ is linearly independent, which implies that $\dim(P_{K,F}) = \operatorname{card}(\Sigma_{K,F})$. Let now $q \in P_{K,F}$ be s.t. $\sigma_{K,F,i}(q) = 0$ for all $i \in \mathcal{N}_{K,F}$. By definition of $P_{K,F}$ and P_K, there is $\widehat{p} \in \widehat{P}$ s.t. $q = (\widehat{p}\circ\boldsymbol{T}_K^{-1})_{|F}$. Hence, for all $i \in \mathcal{N}_{\widehat{K},\widehat{F}} = \mathcal{N}_{K,F}$, we have $\boldsymbol{a}_{K,i} \in F$ and

$$\widehat{\sigma}_i(\widehat{p}) = \widehat{p}(\widehat{\boldsymbol{a}}_i) = (\widehat{p} \circ T_K^{-1})(\boldsymbol{a}_{K,i}) = (\widehat{p} \circ T_K^{-1})_{|F}(\boldsymbol{a}_{K,i})$$
$$= q(\boldsymbol{a}_{K,i}) = \sigma_{K,F,i}(q) = 0.$$

Assumption 20.1 (face unisolvence) implies that $\widehat{p}_{|\widehat{F}} = 0$, so that $q = 0$. □

Recall that since the mesh is matching, any interface $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ is a face of K_l and a face of K_r, i.e., $F \in \mathcal{F}_{K_l} \cap \mathcal{F}_{K_r}$. Our second key assumption is formulated as follows.

Assumption 20.3 (Face matching). *For all $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$, we have* (i) $P_{K_l,F} = P_{K_r,F} =: P_F$ *and* (ii) $\Sigma_{K_l,F} = \Sigma_{K_r,F} =: \Sigma_F$*, i.e., there is a bijective map $\chi_{lr} : \mathcal{N}_{K_l,F} \to \mathcal{N}_{K_r,F}$ s.t. $\boldsymbol{a}_{K_l,i} = \boldsymbol{a}_{K_r,\chi_{lr}(i)}$ for all $i \in \mathcal{N}_{K_l,F}$.*

We are now in a position to state the main result of this section.

Lemma 20.4 (Zero-jump). *Let $v_h \in P_k^{\mathrm{g,b}}(\mathcal{T}_h)$ and $F \in \mathcal{F}_h^\circ$. Under Assumptions* 20.1 *and* 20.3*, the following equivalence holds true:*

$$[\, [\![v_h]\!]_F = 0 \,] \iff [\, v_{h|K_l}(\boldsymbol{a}_{K_l,i}) = v_{h|K_r}(\boldsymbol{a}_{K_r,\chi_{lr}(i)}), \forall i \in \mathcal{N}_{K_l,F} \,]. \quad (20.5)$$

Proof. Let $v_h \in P_k^{\mathrm{g,b}}(\mathcal{T}_h)$ and $F \in \mathcal{F}_h^\circ$. Let v_l be the restriction of $v_{h|K_l}$ to F, and let v_r be the restriction of $v_{h|K_r}$ to F. Since $v_h \in P_k^{\mathrm{g,b}}(\mathcal{T}_h)$, we have $v_l \in P_{K_l,F}$ and $v_r \in P_{K_r,F}$. Owing to Assumption 20.3, we also have $v_r \in P_{K_l,F}$, i.e., $[\![v_h]\!]_F = v_l - v_r \in P_{K_l,F}$. Since $(F, P_{K_l,F}, \Sigma_{K_l,F})$ is a finite element owing to Lemma 20.2 (which follows from Assumption 20.1), we infer that $[\![v_h]\!]_F = v_l - v_r = 0$ iff $(v_l - v_r)(\boldsymbol{a}_{K_l,i}) = 0$ for all $i \in \mathcal{N}_{K_l,F}$. But $v_l(\boldsymbol{a}_{K_l,i}) = v_{h|K_l}(\boldsymbol{a}_{K_l,i})$ and, owing to Assumption 20.3, we also have $v_r(\boldsymbol{a}_{K_l,i}) = v_{h|K_r}(\boldsymbol{a}_{K_l,i}) = v_{h|K_r}(\boldsymbol{a}_{K_r,\chi_{lr}(i)})$. This proves (20.5). □

20.2 Verification of the assumptions (Lagrange)

In this section, we verify Assumptions 20.1 and 20.3 for Lagrange $\mathbb{P}_{k,d}$ elements when $\widehat{K}$ is a simplex and for Lagrange $\mathbb{Q}_{k,d}$ elements when $\widehat{K}$ is a cuboid. Since these two assumptions trivially hold true when $d = 1$, we assume in this section that $d \geq 2$. We do not assume that the geometric mapping $\boldsymbol{T}_K : \widehat{K} \to K$ is affine.

20.2.1 Face unisolvence

Assumption 20.1 has been proved in Lemma 6.15 for Lagrange $\mathbb{Q}_{k,d}$ elements and in Lemma 7.13 for Lagrange $\mathbb{P}_{k,d}$ elements. Note that the face unisolvence assumption is not met for the Crouzeix–Raviart element.

20.2.2 The space $P_{K,F}$

Let us now identify the space $P_{K,F}$ for all $K \in \mathcal{T}_h$ and all $F \in \mathcal{F}_K$. Let us set $\widehat{F} := \boldsymbol{T}_K^{-1}(F)$. Then $\widehat{F} \in \mathcal{F}_{\widehat{K}}$, i.e., $\widehat{F}$ is a face of the reference cell $\widehat{K}$. Let $\widehat{F}^{d-1}$ be the unit simplex in $\mathbb{R}^{d-1}$ if $\widehat{K}$ is the unit simplex of $\mathbb{R}^d$ or let $\widehat{F}^{d-1}$ be the unit cuboid of $\mathbb{R}^{d-1}$ if $\widehat{K}$ is the unit cuboid of $\mathbb{R}^d$. Since both $\widehat{F}^{d-1}$ and $\widehat{F}$ are either $(d-1)$-dimensional simplices or cuboids, it is always possible to construct an affine bijective mapping $\boldsymbol{T}_{\widehat{F}}$ from $\widehat{F}^{d-1}$ to $\widehat{F}$. Let us denote

$$\boldsymbol{T}_{\widehat{F}} : \widehat{F}^{d-1} \to \widehat{F}, \qquad \boldsymbol{T}_{K,F} := \boldsymbol{T}_{K|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}} : \widehat{F}^{d-1} \to F. \tag{20.6}$$

Lemma 20.5 (Characterization of $P_{K,F}$). *Let $\widehat{K}$ be either a simplex or a cuboid. Then $P_{K,F} = \widehat{P}_k^{d-1} \circ \boldsymbol{T}_{K,F}^{-1}$ where $\widehat{P}_k^{d-1} := \mathbb{P}_{k,d-1}$ if $\widehat{K}$ is a simplex and $\widehat{P}_k^{d-1} := \mathbb{Q}_{k,d-1}$ if $\widehat{K}$ is a cuboid.*

Proof. Let $q \in P_{K,F}$. By definition of $P_{K,F}$, there is $\widehat{p} \in \widehat{P}$ s.t.

$$q = (\widehat{p} \circ \boldsymbol{T}_K^{-1})_{|F} = \widehat{p}_{|\widehat{F}} \circ \boldsymbol{T}_{K|F}^{-1} = (\widehat{p}_{|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}}) \circ (\boldsymbol{T}_{K|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}})^{-1} = (\widehat{p}_{|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}}) \circ \boldsymbol{T}_{K,F}^{-1}.$$

Since $\widehat{p}_{|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}} \in \widehat{P}_k^{d-1}$ (see Lemma 6.13 or Lemma 7.10 depending on the nature of $\widehat{F}$), we conclude that $q \in \widehat{P}_k^{d-1} \circ \boldsymbol{T}_{K,F}^{-1}$. This shows that $P_{K,F} \subset \widehat{P}_k^{d-1} \circ \boldsymbol{T}_{K,F}^{-1}$. The converse inclusion is proved by similar arguments. □

20.2.3 Face matching

We now establish that $P_{K_l,F} = P_{K_r,F}$ and $\Sigma_{K_l,F} = \Sigma_{K_r,F}$.

Lemma 20.6 (Face matching, (i)). *Assume that $\widehat{K}$ is either a simplex or a cuboid. Let $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^{\circ}$. Then $P_{K_l,F} = P_{K_r,F}$.*

Proof. Let us set $\widehat{F}_l := \boldsymbol{T}_{K_l}^{-1}(F)$ and $\widehat{F}_r := \boldsymbol{T}_{K_r}^{-1}(F)$. Since the mesh is matching, $\widehat{F}_l$ and $\widehat{F}_r$ are faces of $\widehat{K}$. By construction, the mapping

$$\boldsymbol{T}_{K_l|F}^{-1} \circ \boldsymbol{T}_{K_r|\widehat{F}_r} : \widehat{F}_r \to \widehat{F}_l$$

is bijective, and turns out to be affine even when the mappings $\boldsymbol{T}_{K_l}$ and $\boldsymbol{T}_{K_r}$ are nonaffine as shown in Exercise 20.1. Then the mapping $\boldsymbol{S}_{rl} : \widehat{F}^{d-1} \to \widehat{F}^{d-1}$ s.t.

$$\boldsymbol{S}_{rl} := \boldsymbol{T}_{K_l,F}^{-1} \circ \boldsymbol{T}_{K_r,F} = \boldsymbol{T}_{\widehat{F}_l}^{-1} \circ \boldsymbol{T}_{K_l|F}^{-1} \circ \boldsymbol{T}_{K_r|\widehat{F}_r} \circ \boldsymbol{T}_{\widehat{F}_r}$$

is affine (because the mappings $\boldsymbol{T}_{\widehat{F}_l}^{-1}$, $\boldsymbol{T}_{K_l|F}^{-1} \circ \boldsymbol{T}_{K_r|\widehat{F}_r}$, and $\boldsymbol{T}_{\widehat{F}_r}$ are affine) and bijective; see Figure 20.1. Since $\widehat{P}_k^{d-1} = \mathbb{P}_{k,d-1}$ or $\widehat{P}_k^{d-1} = \mathbb{Q}_{k,d-1}$ depending on the nature of $\widehat{K}$, we infer that $\widehat{P}_k^{d-1}$ is invariant under $\boldsymbol{S}_{rl}$, i.e., $\widehat{P}_k^{d-1}\circ\boldsymbol{S}_{rl} = \widehat{P}_k^{d-1}$. Using this property together with the identity $P_{K,F} = \widehat{P}_k^{d-1} \circ \boldsymbol{T}_{K,F}^{-1}$ proved in Lemma 20.5, we infer that

$$P_{K_l,F} = \widehat{P}_k^{d-1} \circ \boldsymbol{T}_{K_l,F}^{-1} = \widehat{P}_k^{d-1} \circ \boldsymbol{S}_{rl} \circ \boldsymbol{T}_{K_r,F}^{-1} = \widehat{P}_k^{d-1} \circ \boldsymbol{T}_{K_r,F}^{-1} = P_{K_r,F}. \qquad \square$$

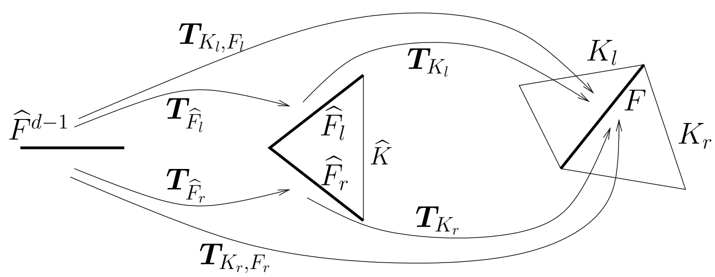

Fig. 20.1 Two-dimensional example ($d = 2$): geometric mappings associated with an interface F, the reference faces $\widehat{F}_l$ and $\widehat{F}_r$, and the unit segment $\widehat{F}^{d-1}$.

To establish that $\Sigma_{K_l,F} = \Sigma_{K_r,F}$ for a general set of Lagrange nodes in $\widehat{K}$, we formulate a symmetry assumption on the Lagrange nodes located on the faces of $\widehat{K}$. This assumption turns out to be sufficient in order to establish that $\Sigma_{K_l,F} = \Sigma_{K_r,F}$. Combined with the result from Lemma 20.6, this allows us to conclude that Assumption 20.3 (face matching) is indeed satisfied.

Assumption 20.7 (Invariance by vertex permutation). *We assume that there is a set* $\{\widehat{\boldsymbol{s}}_m\}_{m\in\mathcal{N}_{\widehat{F}^{d-1}}}$ *of Lagrange nodes in* $\widehat{F}^{d-1}$, *with* $\mathcal{N}_{\widehat{F}^{d-1}} := \{1{:}n^{\rm f}\}$ *for some integer* $n^{\rm f} \geq 1$, *s.t. the following holds true:* (i) *The set* $\{\widehat{\boldsymbol{s}}_m\}_{m\in\mathcal{N}_{\widehat{F}^{d-1}}}$ *is invariant under any vertex permutation of* $\widehat{F}^{d-1}$. (ii) *For every face* $\widehat{F}$ *of* $\widehat{K}$, $\{\boldsymbol{T}_{\widehat{F}}(\widehat{\boldsymbol{s}}_m)\}_{m\in\mathcal{N}_{\widehat{F}^{d-1}}}$ *are the Lagrange nodes on* $\widehat{F}$.

Assumption 20.7(i) means that for every affine bijective mapping $\boldsymbol{S} : \widehat{F}^{d-1} \to \widehat{F}^{d-1}$, there is a permutation $\chi_{\boldsymbol{S}}$ of $\mathcal{N}_{\widehat{F}^{d-1}}$ such that $\boldsymbol{S}(\widehat{\boldsymbol{s}}_m) = \widehat{\boldsymbol{s}}_{\chi_{\boldsymbol{S}}(m)}$ for all $m \in \mathcal{N}_{\widehat{F}^{d-1}}$. Assumption 20.7(ii) means that $\text{card}(\mathcal{N}_{\widehat{K},\widehat{F}}) = n^{\rm f}$ is independent of the face $\widehat{F}$ of $\widehat{K}$ and that, for every $\widehat{F} \in \mathcal{F}_{\widehat{K}}$, there is a bijective map $\mathtt{j}^{\mathtt{fc}}_{\widehat{F}} : \mathcal{N}_{\widehat{F}^{d-1}} \to \mathcal{N}_{\widehat{K},\widehat{F}}$ such that (see Figure 20.2)

$$\boldsymbol{T}_{\widehat{F}}(\widehat{\boldsymbol{s}}_m) = \widehat{\boldsymbol{a}}_{\mathtt{j}^{\mathtt{fc}}_{\widehat{F}}(m)}, \qquad \forall m \in \mathcal{N}_{\widehat{F}^{d-1}}. \tag{20.7}$$

Example 20.8 ($\mathbb{Q}_{k,d}$ Lagrange elements). After inspection of Proposition 6.14 on the reference cuboid $\widehat{K} := [0,1]^d$, we realize that Assumption 20.7 holds true for tensor-product Lagrange elements provided that for every $i \in \{1{:}d\}$, the set of points $\{a_{i,l}\}_{l\in\{0:k\}}$ is such that $a_{i,l} = \alpha_l$ for

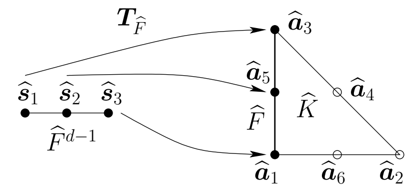

Fig. 20.2 Face (segment) $\widehat{F}^{d-1}$ with $n^{\mathrm{f}} := 3$ Lagrange nodes $\widehat{\boldsymbol{s}}_1, \widehat{\boldsymbol{s}}_2, \widehat{\boldsymbol{s}}_3$ mapped by $\boldsymbol{T}_{\widehat{F}}$ to the three Lagrange nodes on $\widehat{F}$. The enumeration of the Lagrange nodes of $\widehat{K}$ implies that $\mathcal{N}_{\widehat{K},\widehat{F}} = \{1, 3, 5\}$ and that $\mathtt{j}^{\mathtt{fc}}_{\widehat{F}}(1) = 3$, $\mathtt{j}^{\mathtt{fc}}_{\widehat{F}}(2) = 5$, $\mathtt{j}^{\mathtt{fc}}_{\widehat{F}}(3) = 1$.

every $l \in \{0{:}k\}$, where the points $0 = \alpha_0 < \ldots < \alpha_k = 1$ are all distinct in the interval $[0, 1]$ and satisfy the symmetry property $\alpha_l = 1 - \alpha_{k-l}$ for all $l \in \{0{:}\lfloor \frac{k}{2} \rfloor\}$. The Gauss–Lobatto nodes satisfy these assumptions (up to rescaling from $[-1, 1]$ to $[0, 1]$); see §6.2. □

Example 20.9 ($\mathbb{P}_{k,d}$ Lagrange elements). The simplicial Lagrange element described in Proposition 7.12 also satisfies the assumption on invariance by vertex permutation. In dimension two, for instance, the edge nodes are invariant under symmetry about the midpoint as shown in the left panel of Figure 20.3 (for $k = 2$). Note that it is possible to use a set of Lagrange nodes that is different from the one introduced in Proposition 7.12 provided the vertex permutation assumption holds true (in addition to the face unisolvence). For instance, one can use the Fekete points mentioned in Remark 7.14. □

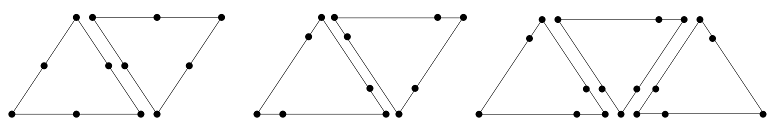

Fig. 20.3 $\mathbb{P}_{2,2}$ Lagrange element: two-dimensional example (left) and counterexample (center) for Assumption 20.3 (the triangles K_l and K_r are drawn slightly apart). In the rightmost panel, Assumption 20.3 is satisfied but not Assumption 20.7. This illustrates the fact that Assumption 20.7 is not needed to establish Assumption 20.3 if one enforces extra constraints on the way adjacent mesh cells come into contact.

Lemma 20.10 (Face matching, (ii)). *Assume that $\widehat{K}$ is either a simplex or a cuboid. Let $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^{\circ}$. Let Assumption 20.7 on invariance by vertex permutation be fulfilled. Then $\Sigma_{K_l,F} = \Sigma_{K_r,F}$.*

Proof. Let $i \in \mathcal{N}_{K_r,F} = \mathcal{N}_{\widehat{K},\widehat{F}_r}$ and let $\boldsymbol{a}_{K_r,i}$ be the corresponding Lagrange node of K_r located on F. Then $\boldsymbol{T}_{K_r}^{-1}(\boldsymbol{a}_{K_r,i}) = \widehat{\boldsymbol{a}}_i$ is a Lagrange node on $\widehat{F}_r$. Let $m \in \mathcal{N}_{\widehat{F}^{d-1}}$ be such that $i = \mathtt{j}^{\mathtt{fc}}_{\widehat{F}_r}(m)$, that is, $\widehat{\boldsymbol{a}}_i = \boldsymbol{T}_{\widehat{F}_r}(\widehat{\boldsymbol{s}}_m)$. Since we have established above that the mapping $\boldsymbol{S}_{rl} := \boldsymbol{T}_{\widehat{F}_l}^{-1} \circ \boldsymbol{T}_{K_l|F}^{-1} \circ \boldsymbol{T}_{K_r|\widehat{F}_r} \circ \boldsymbol{T}_{\widehat{F}_r}$ is affine, there is a permutation $\chi_{\boldsymbol{S}_{rl}} : \mathcal{N}_{\widehat{F}^{d-1}} \to \mathcal{N}_{\widehat{F}^{d-1}}$ such that $\boldsymbol{S}_{rl}(\widehat{\boldsymbol{s}}_m) = \widehat{\boldsymbol{s}}_{\chi_{\boldsymbol{S}_{rl}}(m)}$ for all $m \in \mathcal{N}_{\widehat{F}^{d-1}}$. Then the identity $\boldsymbol{S}_{rl}(\widehat{\boldsymbol{s}}_m) = \widehat{\boldsymbol{s}}_{\chi_{\boldsymbol{S}_{rl}}(m)}$ means that $(\boldsymbol{T}_{K_r|\widehat{F}_r} \circ \boldsymbol{T}_{\widehat{F}_r})(\widehat{\boldsymbol{s}}_m) = (\boldsymbol{T}_{K_l|\widehat{F}_l} \circ \boldsymbol{T}_{\widehat{F}_l})(\widehat{\boldsymbol{s}}_{\chi_{\boldsymbol{S}_{rl}}(m)})$, which can also be rewritten as $\boldsymbol{a}_{K_r,\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_r}(m)} = \boldsymbol{a}_{K_l,\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_l}(\chi_{\boldsymbol{S}_{rl}}(m))}$. Hence, we have

$$\sigma_{K_r,F,\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_r}(m)}(q) = q(\boldsymbol{a}_{K_r,\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_r}(m)}) = q(\boldsymbol{a}_{K_l,\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_l}(\chi_{\boldsymbol{S}_{rl}}(m))}) = \sigma_{K_l,F,\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_l}(\chi_{\boldsymbol{S}_{rl}}(m))}(q),$$

for all $q \in P_F$ and all $m \in \mathcal{N}_{\widehat{F}^{d-1}}$. This proves that $\Sigma_{K_l,F} = \Sigma_{K_r,F}$ since $\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_l} \circ \chi_{\boldsymbol{S}_{rl}} \circ (\mathtt{j}^{\mathtt{fc}}_{\widehat{F}_r})^{-1}$ is bijective. □

Remark 20.11 (Serendipity and prismatic elements). The reader is invited to verify that the face unisolvence assumption 20.1 holds true also for the serendipity elements described in §6.4.3 and for the prismatic elements described in Remark 7.16. The face matching assumption 20.3 holds true for the serendipity elements since the face dofs are the same as those of the corresponding $\mathbb{Q}_{k,d}$ element. The assumption 20.3 can also be shown to hold true for the prismatic elements provided the Lagrange nodes on the triangular faces and the Lagrange nodes on the quadrangular faces each satisfy the vertex permutation assumption. □

20.3 Generalization of the two gluing assumptions

In this section, we generalize the theory developed in §20.1 to enforce the jump condition $[\![v_h]\!]^{\mathrm{x}}_F = 0$ across all the mesh interfaces $F \in \mathcal{F}^{\circ}_h$ for $\mathrm{x} \in \{\mathrm{g},\mathrm{c},\mathrm{d}\}$ and $v_h \in P^{\mathrm{x},\mathrm{b}}_k(\mathcal{T}_h;\mathbb{R}^q)$. We are going to rephrase §20.1 in a slightly more abstract language. Recall from (18.8) that $[\![v_h]\!]^{\mathrm{x}}_F := \gamma^{\mathrm{x}}_{K_l,F}(v_{h|K_l}) - \gamma^{\mathrm{x}}_{K_r,F}(v_{h|K_r})$ with $F := \partial K_l \cap \partial K_r$ and the trace operator $\gamma^{\mathrm{x}}_{K,F}$ defined in (18.7) for every mesh cell $K \in \mathcal{T}_h$ and every face $F \in \mathcal{F}_K$ of K. We drop the superscript x whenever the context is unambiguous.

We start by identifying two structural properties of the finite element which we will call face unisolvence and face matching assumptions. We proceed in two steps. First, given a mesh cell $K \in \mathcal{T}_h$, we use the local finite element (K, P_K, Σ_K) with local shape functions $\{\theta_{K,i}\}_{i\in\mathcal{N}}$ and local dofs $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$, and invoke the face unisolvence assumption to construct a finite element attached to each face $F \in \mathcal{F}_K$. Then for every mesh interface $F := \partial K_l \cap \partial K_r \in \mathcal{F}^{\circ}_h$, we invoke the face matching assumption to make sure that the two face elements built on F from K_l and from K_r are identical (note that $F \in \mathcal{F}_{K_l} \cap \mathcal{F}_{K_r}$ since the mesh is matching). The theory is illustrated with various examples in §20.4. In this section (and the next one), we restrict the maps $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$ and $\gamma_{K,F}$ to P_K, so that the kernels of these maps are to be understood as subspaces of P_K (for simplicity, we keep the same notation for the restrictions). Our first key assumption is the following.

Assumption 20.12 (Face unisolvence). *For all $K \in \mathcal{T}_h$ and all $F \in \mathcal{F}_K$, there is a nonempty subset $\mathcal{N}_{K,F} \subset \mathcal{N}$ s.t.* $\ker(\gamma_{K,F}) = \bigcap_{i\in\mathcal{N}_{K,F}} \ker(\sigma_{K,i})$, *i.e., for all $p \in P_K$,*

$$[\,\sigma_{K,i}(p) = 0,\, \forall i \in \mathcal{N}_{K,F}\,] \iff [\,\gamma_{K,F}(p) = 0\,]. \tag{20.8}$$

Equivalently, we have $\ker(\gamma_{K,F}) = \text{span}\{\theta_{K,i}\}_{i\notin\mathcal{N}_{K,F}}$.

Let $\mathcal{N}_{K,F} \subset \mathcal{N}$ be defined according to Assumption 20.12. Let us define the corresponding trace space $P_{K,F}$ by setting

$$P_{K,F} := \gamma_{K,F}(P_K) = \text{span}\{\gamma_{K,F}(\theta_{K,i})\}_{i\in\mathcal{N}_{K,F}}. \tag{20.9}$$

Notice that $\gamma_{K,F}(\theta_{K,i}) \neq 0$ for all $i \in \mathcal{N}_{K,F}$ by construction. The inclusion $\ker(\gamma_{K,F}) \subset \ker(\sigma_{K,i})$ for all $i \in \mathcal{N}_{K,F}$ (which follows from Assumption 20.12) implies that there is a unique linear map $\sigma_{K,F,i} : P_{K,F} \to \mathbb{R}$ s.t. $\sigma_{K,i} = \sigma_{K,F,i} \circ \gamma_{K,F}$ (see Exercise 20.2). Finally, let us set

$$\Sigma_{K,F} := \{\sigma_{K,F,i}\}_{i\in\mathcal{N}_{K,F}}. \tag{20.10}$$

We can now state an important consequence of Assumption 20.12.

Lemma 20.13 (Face element). *Let* $K \in \mathcal{T}_h$ *and* $F \in \mathcal{F}_K$. *Under Assumption* 20.12*, the triple* $(F, P_{K,F}, \Sigma_{K,F})$ *is a finite element.*

Proof. We use Remark 5.3 to prove unisolvence. Since Assumption 20.12 means that $\ker(\gamma_{K,F}) = \text{span}\{\theta_{K,i}\}_{i\notin\mathcal{N}_{K,F}}$, we infer that $\dim(\ker(\gamma_{K,F})) = \text{card}(\mathcal{N}) - \text{card}(\mathcal{N}_{K,F})$. The rank nullity theorem implies that

$$\dim(P_{K,F}) = \dim(P_K) - \dim(\ker(\gamma_{K,F})) = \text{card}(\mathcal{N}_{K,F}) = \text{card}(\Sigma_{K,F}).$$

Let now $q \in P_{K,F}$ be s.t. $\sigma_{K,F,i}(q) = 0$ for all $i \in \mathcal{N}_{K,F}$. The definition of $P_{K,F}$ implies that there is $p \in P_K$ s.t. $q = \gamma_{K,F}(p)$. Hence, $\sigma_{K,i}(p) = \sigma_{K,F,i}(q) = 0$ for all $i \in \mathcal{N}_{K,F}$. In other words, $p \in \bigcap_{i\in\mathcal{N}_{K,F}} \ker(\sigma_{K,i})$. Hence, $p \in \ker(\gamma_{K,F})$. We conclude that $q = \gamma_{K,F}(p) = 0$. $\square$

Let $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ be the reference element and let ψ_K be the functional transformation that has been used to generate (K, P_K, Σ_K). Let $F \in \mathcal{F}_K$ and consider the face $\widehat{F} := \boldsymbol{T}_K^{-1}(F)$ of $\widehat{K}$. We are going to assume that for all $p \in P_K$, $\gamma_{K,F}(p) = 0$ iff $\gamma_{\widehat{K},\widehat{F}}(\widehat{p}) = 0$ with $\widehat{p} := \psi_K(p)$, i.e., we assume that

$$\ker(\gamma_{K,F}) = \ker(\gamma_{\widehat{K},\widehat{F}} \circ \psi_K). \tag{20.11}$$

This assumption holds true if ψ_K is the pullback by the geometric mapping $\boldsymbol{T}_K$ or one of the Piola transformations. Then Assumption 20.12 can be formulated on the reference element, and this assumption amounts to requiring that there exists a nonempty subset $\mathcal{N}_{\widehat{K},\widehat{F}} \subset \mathcal{N}$ s.t. $\bigcap_{i\in\mathcal{N}_{\widehat{K},\widehat{F}}} \ker(\widehat{\sigma}_i) = \ker(\gamma_{\widehat{K},\widehat{F}})$. Then we have

$$\mathcal{N}_{K,F} = \mathcal{N}_{\widehat{K},\widehat{F}} = \mathcal{N}_{\widehat{K},\boldsymbol{T}_K^{-1}(F)}, \qquad \forall K \in \mathcal{T}_h,\ \forall F \in \mathcal{F}_K. \tag{20.12}$$

Our second key assumption is the following.

Assumption 20.14 (Face matching). *For all* $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$*, we have* (i) $P_{K_l,F} = P_{K_r,F} =: P_F$ *and* (ii) $\Sigma_{K_l,F} = \Sigma_{K_r,F} =: \Sigma_F$*, i.e., there* *is a bijective map* $\chi_{lr} : \mathcal{N}_{K_l,F} \rightarrow \mathcal{N}_{K_r,F}$ *s.t.* $\sigma_{K_l,F,i} = \sigma_{K_r,F,\chi_{lr}(i)}$ *for all* $i \in \mathcal{N}_{K_l,F}$.

We are now in a position to state the main result of this section.

Lemma 20.15 (Zero γ-jump). *Let* $v_h \in P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$ *and* $F \in \mathcal{F}_h^{\circ}$. *Under Assumptions* 20.12 *and* 20.14*, the following equivalence holds true:*

$$[\,[\![v_h]\!]_F = 0\,] \iff [\,\sigma_{K_l,i}(v_{h|K_l}) = \sigma_{K_r,\chi_{lr}(i)}(v_{h|K_r}),\ \forall i \in \mathcal{N}_{K_l,F}\,]. \quad (20.13)$$

Proof. Since $v_h \in P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$, we have $v_{h|K_l} \in P_{K_l}$ and $v_{h|K_r} \in P_{K_r}$. Set $v_l := \gamma_{K_l,F}(v_{h|K_l})$ and $v_r := \gamma_{K_r,F}(v_{h|K_r})$, so that $[\![v_h]\!]_F = v_l - v_r$. Note that $v_l \in \gamma_{K_l,F}(P_{K_l}) = P_{K_l,F}$. Similarly, $v_r \in P_{K_r,F}$, and Assumption 20.14 implies that $v_r \in P_{K_l,F}$, i.e., $v_l - v_r \in P_{K_l,F}$. Since $(F, P_{K_l,F}, \Sigma_{K_l,F})$ is a finite element owing to Lemma 20.13 (which follows from Assumption 20.12), we infer that $[\![v_h]\!]_F = v_l - v_r = 0$ iff $\sigma_{K_l,F,i}(v_l - v_r) = 0$ for all $i \in \mathcal{N}_{K_l,F}$. To conclude the proof, we need to show that $\sigma_{K_l,F,i}(v_l - v_r) = \sigma_{K_l,i}(v_{h|K_l}) - \sigma_{K_r,\chi_{lr}(i)}(v_{h|K_r})$. On the one hand we have $\sigma_{K_l,F,i}(v_l) = \sigma_{K_l,F,i}(\gamma_{K_l,F}(v_{h|K_l})) = \sigma_{K_l,i}(v_{h|K_l})$, and on the other hand Assumption 20.14 implies that $\sigma_{K_l,F,i}(v_r) = \sigma_{K_r,F,\chi_{lr}(i)}(v_r) = \sigma_{K_r,F,\chi_{lr}(i)}(\gamma_{K_r,F}(v_{h|K_r})) = \sigma_{K_r,\chi_{lr}(i)}(v_{h|K_r})$. □

20.4 Verification of the two gluing assumptions

We now present examples of finite elements satisfying the two structural assumptions of §20.3. These assumptions have already been shown in §20.2 to hold true for Lagrange elements. In the present section, we focus on affine simplicial matching meshes and assume that the mesh is oriented in a generation-compatible way (see §10.2). We invite the reader to verify that these examples can be adapted to affine Cartesian meshes.

20.4.1 Raviart–Thomas elements

Let $k \geq 0$ and let us show that the $\mathbb{RT}_{k,d}$ Raviart–Thomas elements introduced in §14.3 can be used to build discrete functions with integrable divergence. Let $K \in \mathcal{T}_h$ and $F \in \mathcal{F}_K$. We consider the γ^{d}-trace defined by (18.7c), i.e., $\gamma^{\mathrm{d}}_{K,F}(\boldsymbol{v}) := \boldsymbol{v}_{|F}{\cdot}\boldsymbol{n}_F$ where $\boldsymbol{n}_F$ is the unit normal vector orienting F. Following §14.4, consider the face dofs $\sigma^{\mathrm{f}}_{F,m}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}{\cdot}\boldsymbol{\nu}_F)(\zeta_m \circ \boldsymbol{T}_{K,F}^{-1})\,\mathrm{d}s$, where $\boldsymbol{\nu}_F := |F|\boldsymbol{n}_F$, $\boldsymbol{T}_{K,F} := \boldsymbol{T}_{K|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}} : \widehat{S}^{d-1} \rightarrow F$, $\boldsymbol{T}_{\widehat{F}} : \widehat{S}^{d-1} \rightarrow \widehat{F}$ is an affine bijective mapping, $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ is a fixed basis of $\mathbb{P}_{k,d-1}$, and $n^{\mathrm{f}}_{\mathrm{sh}} := \dim(\mathbb{P}_{k,d-1})$ (see (14.12a)).

Lemma 20.16 (Face unisolvence). *Assumption* 20.12 *holds true with*

$$\mathcal{N}_{K,F} := \{i \in \mathcal{N} \mid \exists m(i) \in \{1{:}n^{\mathrm{f}}_{\mathrm{sh}}\},\ \sigma_{K,i} = \sigma^{\mathrm{f}}_{F,m(i)}\}, \quad (20.14)$$

i.e., $\mathcal{N}_{K,F}$ collects all the indices of the dofs involving an integral over F.

Proof. We first observe that the subset $\mathcal{N}_{K,F}$ is nonempty. Since $\gamma^{\mathrm{d}}_{K,F}(\boldsymbol{v}) = 0$ implies that $\boldsymbol{v}_{|F}\cdot\boldsymbol{n}_F = 0$ and since $\boldsymbol{n}_F$ and $\boldsymbol{\nu}_F$ are collinear, we infer that $\sigma_{K,i}(\boldsymbol{v}) = 0$ for all $i \in \mathcal{N}_{K,F}$ and all $\boldsymbol{v} \in \ker(\gamma^{\mathrm{d}}_{K,F})$, i.e., $\ker(\gamma^{\mathrm{d}}_{K,F}) \subset \bigcap_{i\in\mathcal{N}_{K,F}} \ker(\sigma_{K,i})$. The converse inclusion results from Lemma 14.14. Hence, Assumption 20.12 holds true. □

Lemma 20.17 ($P^{\mathrm{d}}_{K,F}$). *We have $P^{\mathrm{d}}_{K,F} := \gamma^{\mathrm{d}}_{K,F}(\mathbb{RT}_{k,d}) = \mathbb{P}_{k,d-1} \circ \boldsymbol{T}^{-1}_{K,F}$.*

Proof. We have $P^{\mathrm{d}}_{K,F} \subset \mathbb{P}_{k,d-1} \circ \boldsymbol{T}^{-1}_{K,F}$ owing to Lemma 14.7, and the equality follows by observing that $\dim(P^{\mathrm{d}}_{K,F}) = n^{\mathrm{f}}_{\mathrm{sh}} = \dim(\mathbb{P}_{k,d-1})$. □

Let us set $\mathcal{N}_{\widehat{S}^{d-1}} := \{1{:}n^{\mathrm{f}}_{\mathrm{sh}}\}$ and for all $\widehat{F} \in \mathcal{F}_{\widehat{K}}$, let us introduce the bijective map $\mathtt{j}^{\mathtt{sf}}_{\widehat{F}} : \mathcal{N}_{\widehat{S}^{d-1}} \to \mathcal{N}_{\widehat{K},\widehat{F}}$ defined by setting $\mathtt{j}^{\mathtt{sf}}_{\widehat{F}}(m) := i$ for all $m \in \mathcal{N}_{\widehat{S}^{d-1}}$, where i is s.t. $\widehat{\sigma}_i = \sigma^{\mathrm{f}}_{\widehat{F},m}$. Then Lemma 20.16 applied on the reference element means that $\mathcal{N}_{\widehat{K},\widehat{F}} = \mathtt{j}^{\mathtt{sf}}_{\widehat{F}}(\mathcal{N}_{\widehat{S}^{d-1}})$. Owing to (20.12), we infer that we have for all $K \in \mathcal{T}_h$ and all $F \in \mathcal{F}_K$,

$$\mathcal{N}_{K,F} = \mathcal{N}_{\widehat{K},\boldsymbol{T}_K^{-1}(F)} = \mathtt{j}^{\mathtt{sf}}_{\boldsymbol{T}_K^{-1}(F)}(\mathcal{N}_{\widehat{S}^{d-1}}). \tag{20.15}$$

Lemma 20.18 (Face matching). (i) *$(F, P^{\mathrm{d}}_{K,F}, \Sigma^{\mathrm{d}}_{K,F})$ is a modal scalar-valued finite element with $\sigma^{\mathrm{d}}_{K,F,i}(\phi) := \int_F (\zeta_m \circ \boldsymbol{T}^{-1}_{K,F})\phi\,\mathrm{d}s$, $i := \mathtt{j}^{\mathtt{sf}}_{\boldsymbol{T}_K^{-1}(F)}(m)$, for all $\phi \in P^{\mathrm{d}}_{K,F}$ and all $m \in \mathcal{N}_{\widehat{S}^{d-1}}$.* (ii) *For all $F := \partial K_l \cap \partial K_r \in \mathcal{F}^{\circ}_h$, we have $P^{\mathrm{d}}_{K_l,F} = P^{\mathrm{d}}_{K_r,F} =: P^{\mathrm{d}}_F$.* (iii) *$\Sigma^{\mathrm{d}}_{K_l,F} = \Sigma^{\mathrm{d}}_{K_r,F} =: \Sigma^{\mathrm{d}}_F$ if the basis $\{\zeta_m\}_{m\in\mathcal{N}_{\widehat{S}^{d-1}}}$ of $\mathbb{P}_{k,d-1}$ is invariant under any* vertex permutation *of $\widehat{S}^{d-1}$, i.e., for every affine bijective mapping $\boldsymbol{S} : \widehat{S}^{d-1} \to \widehat{S}^{d-1}$, there exists a permutation $\chi_{\boldsymbol{S}}$ of $\mathcal{N}_{\widehat{S}^{d-1}}$ such that $\zeta_m \circ \boldsymbol{S} = \zeta_{\chi_{\boldsymbol{S}}(m)}$ for all $m \in \mathcal{N}_{\widehat{S}^{d-1}}$.*

Proof. (i) The first claim is a consequence of Lemma 20.17 and of the definition of the face dofs of the $\mathbb{RT}_{k,d}$ element.
(ii) Let $F := \partial K_l \cap \partial K_r \in \mathcal{F}^{\circ}_h$, and set $\widehat{F}_l := \boldsymbol{T}^{-1}_{K_l}(F)$ and $\widehat{F}_r := \boldsymbol{T}^{-1}_{K_r}(F)$. Recalling that the mapping $\boldsymbol{S}_{rl} = \boldsymbol{T}^{-1}_{K_l,F} \circ \boldsymbol{T}_{K_r,F}$ is affine, as shown in Figure 20.1, we observe that

$$P^{\mathrm{d}}_{K_l,F} = \mathbb{P}_{k,d-1} \circ \boldsymbol{T}^{-1}_{K_l,F} = (\mathbb{P}_{k,d-1} \circ \boldsymbol{S}_{rl}) \circ \boldsymbol{T}^{-1}_{K_r,F} = \mathbb{P}_{k,d-1} \circ \boldsymbol{T}^{-1}_{K_r,F} = P^{\mathrm{d}}_{K_r,F},$$

as in the proof of Lemma 20.6.
(iii) Letting $\chi_{\boldsymbol{S}_{rl}}$ be the index permutation associated with the mapping $\boldsymbol{S}_{rl}$, the following holds true for all $m \in \mathcal{N}_{\widehat{S}^{d-1}}$:

$$\begin{aligned}\sigma^{\mathrm{d}}_{K_l,F,\mathtt{j}^{\mathtt{sf}}_{\widehat{F}_l}(m)}(\phi) &= \int_F (\zeta_m \circ \boldsymbol{T}^{-1}_{K_l,F})\phi\,\mathrm{d}s = \int_F (\zeta_m \circ \boldsymbol{S}_{rl} \circ \boldsymbol{T}^{-1}_{K_r,F})\phi\,\mathrm{d}s \\ &= \int_F (\zeta_{\chi_{\boldsymbol{S}_{rl}}(m)} \circ \boldsymbol{T}^{-1}_{K_r,F})\phi\,\mathrm{d}s = \sigma^{\mathrm{d}}_{K_r,F,\mathtt{j}^{\mathtt{sf}}_{\widehat{F}_r}(\chi_{\boldsymbol{S}_{rl}}(m))}(\phi),\end{aligned}$$

i.e., any dof $\sigma^{\rm d}_{K_l,F,i}$ in $\Sigma^{\rm d}_{K_l,F}$ is also in $\Sigma^{\rm d}_{K_r,F}$, and conversely. □

Remark 20.19 (Basis). Let us give two examples of a permutation-invariant basis of $\mathbb{P}_{k,d-1}$. Let $\{\widehat{\boldsymbol{s}}_0,\ldots,\widehat{\boldsymbol{s}}_{d-1}\}$ be the vertices of $\widehat{S}^{d-1}$. Let $\mathcal{A}_{k,d-1} := \{\alpha \in \mathbb{N}^{d-1} \mid |\alpha| \le k\}$ and consider the Lagrange nodes $\{\widehat{\boldsymbol{a}}_\alpha\}_{\alpha\in\mathcal{A}_{k,d-1}}$ defined by $\widehat{\boldsymbol{a}}_\alpha := \widehat{\boldsymbol{s}}_0 + \sum_{i\in\{1:d-1\}} \frac{\alpha_i}{k}(\widehat{\boldsymbol{s}}_i - \widehat{\boldsymbol{s}}_0)$. Then the Lagrange polynomials associated with $\{\widehat{\boldsymbol{a}}_\alpha\}_{\alpha\in\mathcal{A}_{k,d-1}}$ form a permutation-invariant basis of $\mathbb{P}_{k,d-1}$. Likewise the modal basis $\{\widehat{\lambda}_0^{\beta_0}\ldots\widehat{\lambda}_{d-1}^{\beta_{d-1}},\ \beta_0+\ldots+\beta_{d-1}=k\}$, where $(\widehat{\lambda}_0,\ldots,\widehat{\lambda}_{d-1})$ are the barycentric coordinates in $\widehat{S}^{d-1}$, is also a permutation-invariant basis of $\mathbb{P}_{k,d-1}$ (see Exercise 7.4(v)). □

20.4.2 Nédélec elements

Let $k\ge 0$ and let us show that the $\mathbb{N}_{k,d}$ Nédélec elements introduced in §15.3 can be used to build discrete functions with integrable curl. We assume that $d=3$ (the construction is analogous but simpler for $d=2$). Let $K\in\mathcal{T}_h$ and $F\in\mathcal{F}_K$. We consider the $\gamma^{\rm c}$-trace defined in (18.7b), i.e., $\gamma^{\rm c}_{K,F}(\boldsymbol{v}) := \boldsymbol{v}_{|F}\times\boldsymbol{n}_F$ where $\boldsymbol{n}_F$ is the unit normal vector orienting F. Proceeding as in §15.4, we consider the edge dofs $\sigma^{\rm e}_{E,m}(\boldsymbol{v}) := \frac{1}{|E|}\int_E(\boldsymbol{v}\cdot\boldsymbol{t}_E)(\mu_m\circ\boldsymbol{T}^{-1}_{K,E})\,{\rm d}l$, where $\boldsymbol{T}_{K,E} := \boldsymbol{T}_{K|\widehat{E}}\circ\boldsymbol{T}_{\widehat{E}} : \widehat{S}^1\to E$, $\boldsymbol{T}_{\widehat{E}} : \widehat{S}^1\to\widehat{E}$ is an affine bijective mapping, $\boldsymbol{t}_E$ is the edge vector orienting E, $\{\mu_m\}_{m\in\{1:n^{\rm e}_{\rm sh}\}}$ is a fixed basis of $\mathbb{P}_{k,1}$, and $n^{\rm e}_{\rm sh} := \dim(\mathbb{P}_{k,1})$. If $k\ge1$, we also consider the face dofs $\sigma^{\rm f}_{F,j,m}(\boldsymbol{v}) := \frac{1}{|F|}\int_F(\boldsymbol{v}\cdot\boldsymbol{t}_{F,j})(\zeta_m\circ\boldsymbol{T}^{-1}_{K,F})\,{\rm d}s$, where $\boldsymbol{T}_{K,F} := \boldsymbol{T}_{K|\widehat{F}}\circ\boldsymbol{T}_{\widehat{F}} : \widehat{S}^2\to F$, $\boldsymbol{T}_{\widehat{F}} : \widehat{S}^2\to\widehat{F}$ is an affine bijective mapping, $\{\boldsymbol{t}_{F,j}\}_{j\in\{1,2\}}$ are the two edge vectors orienting F, $\{\zeta_m\}_{m\in\{1:n^{\rm f}_{\rm sh}\}}$ is a fixed basis of $\mathbb{P}_{k-1,2}$, and $n^{\rm f}_{\rm sh} := \dim(\mathbb{P}_{k-1,2})$. For all $F\in\mathcal{F}_K$, let $\mathcal{E}_F$ be the collection of the three edges composing the boundary of F. Let

$$\mathcal{N}^{\rm e}_{K,F} := \{i\in\mathcal{N} \mid \exists(E(i),m(i))\in\mathcal{E}_F\times\{1{:}n^{\rm e}_{\rm sh}\},\ \sigma_{K,i}=\sigma^{\rm e}_{E(i),m(i)}\}$$

be the collection of the indices of the edge dofs associated with F and

$$\mathcal{N}^{\rm f}_{K,F} := \{i\in\mathcal{N} \mid \exists(j(i),m(i))\in\{1,2\}\times\{1{:}n^{\rm f}_{\rm sh}\},\sigma_{K,i}=\sigma^{\rm f}_{F,j(i),m(i)}\}$$

be the collection of the indices of the face dofs associated with F ($k\ge1$). We adopt the convention that $\mathcal{N}^{\rm f}_{K,F} := \emptyset$ if $k=0$.

Lemma 20.20 (Face unisolvence). *Assumption* 20.12 *holds true with the subset* $\mathcal{N}_{K,F} := \mathcal{N}^{\rm e}_{K,F}\cup\mathcal{N}^{\rm f}_{K,F}$.

Proof. We first observe that the subset $\mathcal{N}_{K,F}$ is nonempty. Let $\boldsymbol{v}\in\boldsymbol{P}^{\rm c}_K$ be such that $\gamma^{\rm c}_{K,F}(\boldsymbol{v})=\boldsymbol{0}$, i.e., $\boldsymbol{v}_{|F}\times\boldsymbol{n}_F=\boldsymbol{0}$. Then $\sigma_{K,i}(\boldsymbol{v})=0$ for all $i\in\mathcal{N}_{K,F}$, so that $\ker(\gamma_{K,F})\subset\bigcap_{i\in\mathcal{N}_{K,F}}\ker(\sigma_{K,i})$. The converse inclusion results from Lemma 15.15. □

Lemma 20.21 ($\boldsymbol{P}^{\rm c}_{K,F}$). $\boldsymbol{P}^{\rm c}_{K,F} := \gamma^{\rm c}_{K,F}(\mathbb{N}_{k,d}) = \mathbb{J}^{-\mathsf{T}}_{K,F}(\mathbb{N}_{k,2}\circ\boldsymbol{T}^{-1}_{K,F})\times\boldsymbol{n}_F$.

Proof. The inclusion $\boldsymbol{P}^{\mathrm{c}}_{K,F} \subset \mathbb{J}^{-\mathsf{T}}_{K,F}(\mathbb{N}_{k,2} \circ \boldsymbol{T}^{-1}_{K,F}) \times \boldsymbol{n}_F$ is shown as in the proof of Lemma 15.8. Equality follows by invoking a dimension argument, i.e., $\dim(\mathbb{J}^{-\mathsf{T}}_{K,F}(\mathbb{N}_{k,2} \circ \boldsymbol{T}^{-1}_{K,F}) \times \boldsymbol{n}_F) = \dim(\mathbb{N}_{k,2})$ and $\mathrm{card}(\mathcal{N}_{K,F}) = 2\dim(\mathbb{P}_{k-1,2}) + 3\dim(\mathbb{P}_{k,1}) = (k+1)(k+3) = \dim(\mathbb{N}_{k,2})$ owing to Lemma 15.7. □

Lemma 20.22 (Face matching). (i) *The triple $(F, \boldsymbol{P}^{\mathrm{c}}_{K,F}, \Sigma^{\mathrm{c}}_{K,F})$ is a two-dimensional Raviart–Thomas finite element with dofs*

$$\sigma^{\mathrm{c}}_{K,F,i}(\boldsymbol{\phi}) := \frac{1}{|E(i)|}\int_{E(i)} (\boldsymbol{\phi}\cdot\boldsymbol{t}^{\perp}_{E(i)})(\mu_{m(i)} \circ \boldsymbol{T}^{-1}_{K,E(i)})\,\mathrm{d}l, \quad \forall i \in \mathcal{N}^{\mathrm{e}}_{K,F}, \tag{20.16a}$$

$$\sigma^{\mathrm{c}}_{K,F,i}(\boldsymbol{\phi}) := \frac{1}{|F|}\int_{F} (\boldsymbol{\phi}\cdot\boldsymbol{t}^{\perp}_{F,j(i)})(\zeta_{m(i)} \circ \boldsymbol{T}^{-1}_{K,F})\,\mathrm{d}s, \quad \forall i \in \mathcal{N}^{\mathrm{f}}_{K,F}, \tag{20.16b}$$

for all $\boldsymbol{\phi} \in \boldsymbol{P}^{\mathrm{c}}_{K,F}$ and all $i \in \mathcal{N}_{K,F}$, with $\boldsymbol{t}^{\perp}_{E(i)} := \boldsymbol{t}_{E(i)} \times \boldsymbol{n}_F$ and $\boldsymbol{t}^{\perp}_{F,j(i)} := \boldsymbol{t}_{F,j(i)} \times \boldsymbol{n}_F$. (ii) *For all $F := \partial K_l \cap \partial K_r \in \mathcal{F}^{\circ}_h$, we have $\boldsymbol{P}^{\mathrm{c}}_{K_l,F} = \boldsymbol{P}^{\mathrm{c}}_{K_r,F} =: \boldsymbol{P}^{\mathrm{c}}_F$.* (iii) *We have $\Sigma^{\mathrm{c}}_{K_l,F} = \Sigma^{\mathrm{c}}_{K_r,F} =: \Sigma^{\mathrm{c}}_F$ if the chosen bases $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ and $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ are invariant under any vertex permutation of $\widehat{S}^2$ and $\widehat{S}^1$, respectively.*

Proof. The expressions in (20.16) follow from the definition of the edge and the face dofs of the $\mathbb{N}_{k,d}$ element and from the fact that $(\boldsymbol{n}_F \times (\boldsymbol{h} \times \boldsymbol{n}_F))\cdot\boldsymbol{t} = \boldsymbol{h}\cdot\boldsymbol{t}$ for all $\boldsymbol{h} \in \mathbb{R}^3$ and every vector $\boldsymbol{t}$ that is tangent to F. The rest of the proof is similar to that of Lemma 20.18. □

Remark 20.23 (Choice of basis). Examples of permutation-invariant bases of $\mathbb{P}_{k-1,2}$ and $\mathbb{P}_{k,1}$ are the nodal and the modal bases built by using either the Lagrange nodes in $\widehat{S}^2$ and $\widehat{S}^1$ or the barycentric coordinates in $\widehat{S}^2$ and $\widehat{S}^1$ as in Remark 20.19. □

20.4.3 Canonical hybrid elements

Let $k \geq 1$ and let us show that the canonical hybrid finite element introduced in §7.6 can be used to build discrete functions with integrable gradient. Assume $d = 3$ (the case $d = 2$ is similar). As for the Lagrange elements, we consider the γ^{g}-trace defined in (18.7a), i.e., $\gamma^{\mathrm{g}}_{K,F}(v) := v_{|F}$ for all $F \in \mathcal{F}_K$. Recall that the dofs of the canonical hybrid element are defined in (7.11). Let $\mathcal{N}_{K,F}$ be the collection of the dof indices of the following types: integrals over F of products with functions from the fixed basis $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ of $\mathbb{P}_{k-3,2}$ (if $k \geq 3$); integrals over the edges of F of products with functions from the fixed basis $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ of $\mathbb{P}_{k-2,1}$ (if $k \geq 2$); evaluation at the vertices of F. Note that $\mathrm{card}(\mathcal{N}_{K,F}) = 3 + 3n^{\mathrm{e}}_{\mathrm{sh}} + n^{\mathrm{f}}_{\mathrm{sh}}$ if $k \geq 3$. Assume that the basis $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ is invariant under every permutation of the vertices of the unit simplex $\widehat{S}^1$, and the basis $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ in invariant under every permutation of the vertices of the unit simplices $\widehat{S}^2$. Then one can prove that the canonical hybrid element satisfies the Assumptions 20.12 and 20.14; see Exercise 20.6.

Exercises

Exercise 20.1 (Affine mapping between faces). Let $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ and set $\widehat{F}_l := \boldsymbol{T}_{K_l}^{-1}(F)$ and $\widehat{F}_r := \boldsymbol{T}_{K_r}^{-1}(F)$. Prove that the mapping $\boldsymbol{T}_{rl} := \boldsymbol{T}_{K_l}^{-1} \circ \boldsymbol{T}_{K_r|\widehat{F}_r}$ is affine. (*Hint*: let $(\widehat{K}, \widehat{P}_{\text{geo}}, \widehat{\Sigma}_{\text{geo}})$ be the geometric reference Lagrange finite element. Observe that the two face finite elements $(\widehat{F}_l, \widehat{P}^{\text{g}}_{\text{geo},l}, \widehat{\Sigma}^{\text{g}}_{\text{geo},l})$ and $(\widehat{F}_r, \widehat{P}^{\text{g}}_{\text{geo},r}, \widehat{\Sigma}^{\text{g}}_{\text{geo},r})$ can be constructed from the same reference Lagrange finite element $(\widehat{F}^{d-1}, \widehat{P}^{d-1}_{\text{geo}}, \widehat{\Sigma}^{d-1}_{\text{geo}})$.)

Exercise 20.2 (Linear maps). Let E, F, G be finite-dimensional vector spaces, let $A \in \mathcal{L}(E;F)$ and let $T \in \mathcal{L}(E;G)$. Assume that $\ker(T) \subset \ker(A)$. Set $\tilde{G} := T(E)$. (i) Prove that there is $\tilde{A} \in \mathcal{L}(\tilde{G};F)$ s.t. $A = \tilde{A} \circ T$. (*Hint*: build a right inverse of T using a direct sum $E = E_1 \oplus E_2$ with $E_1 := \ker(T)$.) (ii) Show that $\tilde{A}$ is uniquely defined, i.e., does not depend on E_2.

Exercise 20.3 ($\gamma_{K,F}$ and $\mathcal{N}_{K,F}$). (i) Prove that $P_K = \sum_{F \in \mathcal{F}_K} \ker(\gamma^{\text{x}}_{K,F})$ (nondirect sum of vector spaces) if and only if there is $F \in \mathcal{F}_K$ s.t. $i \notin \mathcal{N}_{K,F}$ for all $i \in \mathcal{N}$. (ii) Let the face unisolvence assumption hold true. Let $\mathcal{F}(K,i) := \{F \in \mathcal{F}_K \mid \ker(\gamma_{K,F}) \subset \ker(\sigma_{K,i})\}$. Prove the following statements: (ii.a) $F \in \mathcal{F}(K,i)$ iff $i \in \mathcal{N}_{K,F}$; (ii.b) $F \in \mathcal{F}(K,i)$ iff $\gamma_{K,F}(\theta_{K,i}) \neq 0$ where $\theta_{K,i}$ is the local shape function associated with the dof i.

Exercise 20.4 (Reference face element). Let $\widehat{F}$ be any face of $\widehat{K}$. Let $\widehat{P}^{\text{x}} := \gamma^{\text{x}}_{\widehat{K},\widehat{F}}(\widehat{P})$ and let $\mathcal{N}_{\widehat{K},\widehat{F}}$ be the subset of $\mathcal{N}$ s.t. $\bigcap_{i \in \mathcal{N}_{\widehat{K},\widehat{F}}} \ker(\sigma_{\widehat{K},i}) = \ker(\gamma_{\widehat{K},\widehat{F}})$. Recall that this means that there exists $\widehat{\sigma}^{\text{x}}_{\widehat{F},i} : \widehat{P}_{\widehat{K},\widehat{F}} \to \mathbb{R}$ s.t. $\widehat{\sigma}_i = \widehat{\sigma}^{\text{x}}_{\widehat{F},i} \circ \gamma^{\text{x}}_{\widehat{K},\widehat{F}}$ for all $i \in \mathcal{N}_{\widehat{K},\widehat{F}}$. Assume that $\mathcal{N}_{\widehat{K},\widehat{F}}$ is nonempty, that the triple $\{\widehat{F}, \widehat{P}^{\text{x}}, \widehat{\Sigma}^{\text{x}}\}$ with $\widehat{\Sigma}^{\text{x}} := \{\widehat{\sigma}^{\text{x}}_{\widehat{F},i}\}_{i \in \mathcal{N}_{\widehat{K},\widehat{F}}}$ is a finite element, and that there is a linear bijective map $\psi_F : P^{\text{x}}_{K,F} \to \widehat{P}^{\text{x}}$ s.t. $\psi_F^{-1} \circ \gamma^{\text{x}}_{\widehat{K},\widehat{F}} = \gamma^{\text{x}}_{K,F} \circ \psi_K^{-1}$. Prove that Assumption 20.12 holds true and $\mathcal{N}_{K,F} = \mathcal{N}_{\widehat{K},\widehat{F}}$. (*Hint*: show that the finite element $\{F, P^{\text{x}}_{K,F}, \Sigma^{\text{x}}_{K,F}\}$ is generated from $\{\widehat{F}, \widehat{P}^{\text{x}}, \widehat{\Sigma}^{\text{x}}\}$ using the map ψ_F.)

Exercise 20.5 (Permutation invariance). Let $\widehat{S}^1 := [0,1]$ and consider the bases $\mathfrak{B}_1 := \{\mu_1(s) = 1-s, \mu_2(s) = s\}$ and $\mathfrak{B}_2 := \{\mu_1(s) = 1, \mu_2(s) = s\}$. Are these bases invariant under permutation of the vertices of $\widehat{S}^1$?

Exercise 20.6 (Canonical hybrid element, $d = 3$). Consider the assumptions made in §20.4.3. (i) Prove the face unisolvence assumption 20.12. (ii) Let $F \in \mathcal{F}_K$. Let $\boldsymbol{T}_{\widehat{F}} : \widehat{S}^2 \to \widehat{F}$ be an affine bijective mapping, and let $\boldsymbol{T}_{K,F} := \boldsymbol{T}_{K|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}} : \widehat{S}^2 \to F$. Verify that $P^{\text{g}}_{K,F} = \mathbb{P}_{k,d-1} \circ \boldsymbol{T}^{-1}_{K,F}$ and that $\{F, P^{\text{g}}_{K,F}, \Sigma^{\text{g}}_{K,F}\}$ is a two-dimensional canonical hybrid element. (iii) Prove that $P^{\text{g}}_{K_l,F} = P^{\text{g}}_{K_r,F} =: P^{\text{g}}_F$ and $\Sigma^{\text{g}}_{K_l,F} = \Sigma^{\text{g}}_{K_r,F} =: \Sigma^{\text{g}}_F$.

Exercise 20.7 ($P_{K,F}$). Let $\widehat{K}$ be the unit simplex in $\mathbb{R}^2$ and let $\{\widehat{F}_i\}_{i\in\{0:2\}}$ be the faces of $\widehat{K}$. Recall that for $\mathbb{P}_{k,d}$ scalar-valued elements, we have $P_{\widehat{K},\widehat{F}_i} := \gamma^{\mathrm{g}}_{\widehat{K},\widehat{F}_i}(\mathbb{P}_{k,d})$. (i) Compute a basis of $P_{\widehat{K},\widehat{F}_i}$ for all $i \in \{0:2\}$ assuming that $(\widehat{K},\widehat{P},\widehat{\Sigma})$ is the $\mathbb{P}_1$ Lagrange element. Is $(\widehat{F}_i, P_{\widehat{K},\widehat{F}_i}, \Sigma_{\widehat{K},\widehat{F}_i})$ a finite element? (ii) Compute a basis of $P_{\widehat{K},\widehat{F}_i}$ for all $i \in \{0:2\}$ assuming that $(\widehat{K},\widehat{P},\widehat{\Sigma})$ is the $\mathbb{P}_1$ Crouzeix–Raviart element. Is $(\widehat{F}_i, P_{\widehat{K},\widehat{F}_i}, \Sigma_{\widehat{K},\widehat{F}_i})$ a finite element?