Part IV, Chapter 21

Construction of the connectivity classes

In this chapter, we finish the construction of the connectivity classes which we characterize by means of an equivalence relation on the pairs in $\mathcal{T}_h \times \mathcal{N}$. We show that the resulting equivalence classes verify the two key assumptions (19.2) and (19.3) introduced in Chapter 19. Our starting point is to assume that the finite element at hand satisfies the two fundamental assumptions introduced in Chapter 20: the face unisolvence assumption (Assumption 20.12) and the face matching assumption (Assumption 20.14). These two assumptions turn out to be sufficient to fully characterize the connectivity classes of Raviart–Thomas elements. For the other elements (Lagrange, canonical hybrid, and Nédélec) for which there are degrees of freedom (dofs) attached to geometric entities of smaller dimension, we have to consider two additional abstract assumptions, the M-unisolvence assumption (Assumption 21.9) and the M-matching assumption (Assumption 21.10), which we show hold true for these elements. At the end of the chapter we propose enumeration techniques that facilitate the practical construction of the map χ_{lr} introduced in Assumption 20.14. This map is a key tool for the construction of the connectivity array `j_dof`. We assume in the entire chapter that the reference cell is either a simplex or a cuboid, we assume that $d = 3$, and we continue to use the notation introduced in Chapters 19 and 20.

21.1 Connectivity classes

In this section, we describe a way to build the connectivity classes that makes the two key assumptions from Chapter 19 hold true. This is done by constructing an equivalence relation on the set $\mathcal{T}_h \times \mathcal{N}$.

21.1.1 Geometric entities and macroelements

We start by introducing the geometric objects to which we will attach the dofs. Let $\mathcal{T}_h$ be a matching mesh and let $\mathcal{V}_h$, $\mathcal{E}_h$, and $\mathcal{F}_h$ be the sets collecting, respectively, the vertices, edges, and faces in the mesh $\mathcal{T}_h$ as defined in §8.2.

Definition 21.1 (Geometric entity). *Let $\mathcal{T}_h$ be a matching mesh. We call M* geometric entity *if M is a vertex $\boldsymbol{z} \in \mathcal{V}_h$, an edge $E \in \mathcal{E}_h$, a face $F \in \mathcal{F}_h$, or a cell $K \in \mathcal{T}_h$.*

Definition 21.2 (Macroelement). *Let M be a geometric entity. We associate with M the following subsets of $\mathcal{T}_h$ and D:*

$$\mathcal{T}_M := \{K \in \mathcal{T}_h \mid M \subset K\} \subset \mathcal{T}_h, \tag{21.1a}$$

$$D_M := \mathrm{int}(\{\boldsymbol{x} \in D \mid \exists K \in \mathcal{T}_M, \boldsymbol{x} \in K\}) \subset D. \tag{21.1b}$$

The set D_M is called macroelement *associated with the geometric entity M.*

Notice that the notion of macroelement is trivial for a mesh cell since in this case $\mathcal{T}_K := \{K\}$ and $D_K := \mathrm{int}(K)$. This notion is also very simple for a mesh face, since if $F \in \mathcal{F}_h^{\circ}$, then $\mathcal{T}_F := \{K_l, K_r\}$ where $F := \partial K_l \cap \partial K_r$ (so that $\mathrm{card}(\mathcal{T}_F) = 2$), whereas if $F \in \mathcal{F}_h^{\partial}$, then $\mathcal{T}_F := \{K_l\}$ where $F := \partial K_l \cap \partial D$ (so that $\mathrm{card}(\mathcal{T}_F) = 1$). For a vertex $\boldsymbol{z} \in \mathcal{V}_h$ or an edge $E \in \mathcal{E}_h$, there are in general more than two cells in $\mathcal{T}_{\boldsymbol{z}}$ and $\mathcal{T}_E$, and $\mathrm{card}(\mathcal{T}_{\boldsymbol{z}})$ and $\mathrm{card}(\mathcal{T}_E)$ are not known a priori. Figure 21.1 illustrates these concepts for a triangular mesh. Notice that if the geometric entity M is s.t. $\mathrm{card}(\mathcal{T}_M) \geq 2$, then M is a face, an edge, or a vertex. Hence, $\mathcal{T}_M$ can also be characterized as follows when $\mathrm{card}(\mathcal{T}_M) \geq 2$:

$$\mathcal{T}_M = \{K \in \mathcal{T}_h \mid M \subset \partial K\} \subset \mathcal{T}_h. \tag{21.2}$$

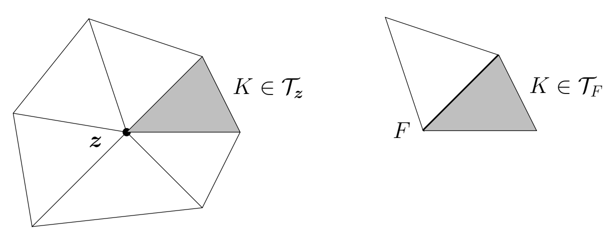

Fig. 21.1 Left: mesh vertex $\boldsymbol{z} \in \mathcal{V}_h$, macroelement $\mathcal{T}_{\boldsymbol{z}}$ composed of six mesh cells with one cell $K \in \mathcal{T}_{\boldsymbol{z}}$ highlighted in gray. Right: mesh face $F \in \mathcal{F}_h$, macroelement $\mathcal{T}_F$ composed of two mesh cells with one cell $K \in \mathcal{T}_F$ highlighted in gray. Note that the subsets $D_{\boldsymbol{z}}$ and D_F are connected.

Definition 21.3 (M-path). *Let M be a geometric entity. A collection of cells $(K_0, \ldots, K_L)$ in $\mathcal{T}_M$ is called M*-path *if either $L = 0$ or the following* *holds true for all* $l \in \{1{:}L\}$*:* $F_l := \partial K_{l-1} \cap \partial K_l \in \mathcal{F}_h^\circ$*. We say that* L *is the* length *of the* M*-path and that the* M*-path connects* K_0 *with* K_L*.*

Lemma 21.4 (*M*-path). *Let* M *be a geometric entity. Assume* $\mathrm{card}(\mathcal{T}_M) \geq 2$*. Then for every pair* (K, K') *of distinct cells in* $\mathcal{T}_M$*, there exists an* M*-path of length* $L \geq 1$ *connecting* K *with* K'*, and we have* $M \subset \bigcap_{l\in\{1:L\}} F_l$*.*

Proof. The subset D_M is connected since D is a Lipschitz domain. This implies the existence of the M-path. Finally, since $\mathrm{card}(\mathcal{T}_M) \geq 2$, (21.2) holds true, and since $K_{l-1}, K_l \in \mathcal{T}_M$ for all $l \in \{1{:}L\}$, we have $M \subset \partial K_{l-1}$ and $M \subset \partial K_l$. Hence, $M \subset F_l$ for all $l \in \{1{:}L\}$. □

It will be useful to describe geometric entities as an intersection of faces.

Lemma 21.5 (Geometric entity as intersection of faces). *Let* $K \in \mathcal{T}_h$ *be a mesh cell. The following holds true:* (i) *Let* $\mathcal{G} \subset \mathcal{F}_K$ *be a nonempty collection of faces of* K*. Then* $M := \bigcap_{F\in\mathcal{G}} F$ *is always a geometric entity when* $M \neq \emptyset$*.* (ii) *Let* M *be a geometric entity that is not a cell. Then there is a unique subset* $\mathcal{G}_{K,M} \subset \mathcal{F}_K$ *s.t.* $M = \bigcap_{F\in\mathcal{G}_{K,M}} F$*.*

Proof. (i) $\bigcap_{F\in\mathcal{G}} F$ is always a geometric entity when it is nonempty because K is a polyhedron.
(ii) Whether K is a simplex or a cuboid, if $M := \bigcap_{F\in\mathcal{G}} F$ is nonempty, then M is a vertex, an edge, or a face of K, and there cannot be any other possibility. If M is a vertex, there can only be exactly d faces s.t. $M = \bigcap_{F\in\mathcal{G}} F$. If M is an edge, there can only be exactly 2 faces s.t. $M = \bigcap_{F\in\mathcal{G}} F$. If M is a face, $\mathcal{G}$ contains only one face. □

Remark 21.6 (Prisms). The proof of Lemma 21.5 shows that for the statement (ii) to hold true when $d = 3$, every vertex has to be shared by exactly d faces. In addition to the tetrahedron and the hexahedron, another polyhedron having this property is the prism with triangular basis. □

21.1.2 The two key assumptions

Let us briefly motivate what we want to do. Our goal is to partition the set $\mathcal{N}$ according to the nature of the dofs and to use the same partition on every mesh cell. Let $K \in \mathcal{T}_h$. We say that i is an *internal dof* if there is no face $F \in \mathcal{F}_K$ s.t. $i \in \mathcal{N}_{K,F}$, and we write $i \in \mathcal{N}^\circ$. We say that i is a *boundary dof* if there is at least one face $F \in \mathcal{F}_K$ s.t. $i \in \mathcal{N}_{K,F}$, and we write $i \in \mathcal{N}^\partial$. A first natural partition of the dofs is thus $\mathcal{N} = \mathcal{N}^\circ \cup \mathcal{N}^\partial$. If all the subsets $\mathcal{N}_{K,F}$ are mutually disjoint, as it happens for the Raviart–Thomas elements, the collection of boundary dofs is further partitioned as $\mathcal{N}^\partial = \bigcup_{F\in\mathcal{F}_K} \mathcal{N}_{K,F}$. The situation is more intricate when the subsets $\mathcal{N}_{K,F}$ are not mutually disjoint since in this case we need to consider the intersections $\bigcap_{F\in\mathcal{G}} \mathcal{N}_{K,F}$ for the nonempty subsets $\mathcal{G} \subset \mathcal{F}_K$, and we are only interested in the subsets $\mathcal{G} \subset \mathcal{F}_K$ s.t. the above intersection is nonempty. The following lemma shows that for the finite elements considered in this book, the set $\bigcap_{F\in\mathcal{G}} F$ is nonempty if the set $\bigcap_{F\in\mathcal{G}} \mathcal{N}_{K,F}$ is nonempty.

Lemma 21.7 (Intersection of boundary dofs). *Let $K \in \mathcal{T}_h$ be a simplex or a cuboid. If K is a simplex, assume that there is no local shape function that has a nonzero γ-trace on all the faces of K. If K is a cuboid, assume that there is no local shape function that has a nonzero γ-trace on two opposite faces of K. Then for every nonempty set $\mathcal{G} \subset \mathcal{F}_K$, if $\bigcap_{F\in\mathcal{G}} \mathcal{N}_{K,F}$ is nonempty, then $\bigcap_{F\in\mathcal{G}} F$ is nonempty as well.*

Proof. Let us reason by contradiction and assume that $\bigcap_{F\in\mathcal{G}} F = \emptyset$. If K is a simplex, this implies that $\mathcal{G} = \mathcal{F}_K$, whereas if K is a cuboid, this implies that $\mathcal{G}$ contains two opposite faces. Recalling that $i \in \mathcal{N}_{K,F}$ iff $\gamma_{K,F}(\theta_{K,i}) \neq 0$, we infer from our assumption on the shape functions that $\bigcap_{F\in\mathcal{G}} \mathcal{N}_{K,F}$ is empty. This concludes the proof. □

All the simplicial finite elements considered in this book satisfy the assumption of Lemma 21.7 since the γ-trace of every shape function vanishes on at least one face. All the cuboidal finite elements considered in this book also satisfy the assumption of Lemma 21.7 since there is no shape function that has a nonzero γ-trace on two opposite faces.

Lemma 21.7 combined with Lemma 21.5 allows us to identify the geometric entities that are different from K with those nonempty subsets $\mathcal{G} \subset \mathcal{F}_K$ such that $\bigcap_{F\in\mathcal{G}} \mathcal{N}_{K,F}$ is nonempty. This leads to the following definition.

Definition 21.8 ($\mathcal{M}_h$). *We denote by $\mathcal{M}_h$ the collection of all the geometric entities M s.t. for all $K \in \mathcal{T}_M$, the unique nonempty subset $\mathcal{G}_{K,M} \subset \mathcal{F}_K$ satisfying $M = \bigcap_{F\in\mathcal{G}_{K,M}} F$ is s.t.*

$$\mathcal{N}_{K,M} := \bigcap_{F\in\mathcal{G}_{K,M}} \mathcal{N}_{K,F} \neq \emptyset. \tag{21.3}$$

We say that the finite element has face dofs *if $\mathcal{F}_h \subset \mathcal{M}_h$,* edge dofs *if $\mathcal{E}_h \subset \mathcal{M}_h$, and* vertex dofs *if $\mathcal{V}_h \subset \mathcal{M}_h$.*

Since $\mathcal{N}_{K,F}$ is nonempty for all $K \in \mathcal{T}_h$ and all $F \in \mathcal{F}_K$ (see Assumption 20.12 on face unisolvence), all the mesh faces are in $\mathcal{M}_h$, i.e., $\mathcal{F}_h \subset \mathcal{M}_h$. This means that all the finite elements considered in this book have face dofs. We will see in the next section that $\mathcal{M}_h = \mathcal{V}_h \cup \mathcal{E}_h \cup \mathcal{F}_h$ for the Lagrange elements and the canonical hybrid element, $\mathcal{M}_h = \mathcal{E}_h \cup \mathcal{F}_h$ for Nédélec elements, and $\mathcal{M}_h = \mathcal{F}_h$ for Raviart–Thomas elements.

We can now state the two key assumptions regarding the structure of the dofs that will help us identify the connectivity classes.

Assumption 21.9 (M-unisolvence). *For every geometric entity $M \in \mathcal{M}_h$ and every cell $K \in \mathcal{T}_M$ (i.e., $M \subset \partial K$), the following holds true:* (i) *There is a linear map $\gamma_{K,M}$ s.t. for every face $F \in \mathcal{G}_{K,M}$, we have* $\ker(\gamma_{K,F}) \subset \ker(\gamma_{K,M})$. (ii) *For all $i \in \mathcal{N}_{K,M}$, there is a linear form $\sigma_{K,M,i}$ s.t. $\sigma_{K,i} = \sigma_{K,M,i} \circ \gamma_{K,M}$.* (iii) *The triple $(M, P_{K,M}, \Sigma_{K,M})$ is a finite element where $P_{K,M} := \gamma_{K,M}(P_K)$ and $\Sigma_{K,M} := \{\sigma_{K,M,i}\}_{i\in\mathcal{N}_{K,M}}$.*

Assumption 21.10 (M-matching). *The following holds true for every interface $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ and every geometric entity $M \in \mathcal{M}_h$ s.t. $M \subset F$ (so that $K_l, K_r \in \mathcal{T}_M$ and $F \in \mathcal{G}_{K_l,M} \cap \mathcal{G}_{K_r,M}$):* (i) $P_{K_l,M} = P_{K_r,M}$. (ii) *The map χ_{lr} introduced in Assumption 20.14 is such that $\chi_{lr}(\mathcal{N}_{K_l,M}) = \mathcal{N}_{K_r,M}$, and the map $\chi_{lr,M} : \mathcal{N}_{K_l,M} \to \mathcal{N}_{K_r,M}$ defined by $\chi_{lr,M} := \chi_{F|\mathcal{N}_{K_l,M}}$ is s.t.*

$$\sigma_{K_l,M,i_l} = \sigma_{K_r,M,\chi_{lr,M}(i_l)}, \qquad \forall i_l \in \mathcal{N}_{K_l,M}, \tag{21.4}$$

i.e., $\Sigma_{K_l,M} = \Sigma_{K_r,M}$ and $\chi_{lr,M} : \mathcal{N}_{K_l,M} \to \mathcal{N}_{K_r,M}$ is bijective.

The definition of $\chi_{lr,M}$ in Assumption 21.10 is meaningful because $\mathcal{N}_{K_l,M} \subset \mathcal{N}_{K_l,F}$ and $\mathcal{N}_{K_r,M} \subset \mathcal{N}_{K_r,F}$ owing to (21.3). When the geometric entity M is a face, Assumption 21.9 and Assumption 21.10 are identical to Assumption 20.12 (face unisolvence) and Assumption 20.14 (face matching).

Given an M-path (see Definition 21.3) of length $L \geq 1$, we define the map $\chi_{F_l}^\epsilon$ for all $l \in \{1{:}L\}$ by setting $\chi_{F_l}^\epsilon := \chi_{F_l}$ if $\boldsymbol{n}_{F_l}$ points from K_{l-1} to K_l and $\chi_{F_l}^\epsilon := \chi_{F_l}^{-1}$ otherwise, where $\boldsymbol{n}_{F_l}$ is the unit normal vector orienting F_l.

Lemma 21.11 (Path independence). *Let $M \in \mathcal{M}_h$. Let K, K' be two cells in $\mathcal{T}_M$ (possibly identical) connected by an M-path of length $L \geq 1$, say $(K =: K_0, \ldots, K_L := K')$. Then for all $i \in \mathcal{N}_{K,M}$, the index $\chi_{F_L}^\epsilon \circ \ldots \circ \chi_{F_1}^\epsilon(i)$ with $F_l := \partial K_{l-1} \cap \partial K_l$, $\forall l \in \{1{:}L\}$, is independent of the M-path.*

Proof. Let $M \in \mathcal{M}_h$ be a geometric entity and let K, K' be two cells in $\mathcal{T}_M$. Let $(K =: K_{\beta,0} \ldots, K_{\beta,L_\beta} := K')$, $\forall \beta \in \{1,2\}$, be two M-paths in $\mathcal{T}_M$ connecting K to K', with $F_{\beta,l} := \partial K_{\beta,l-1} \cap \partial K_{\beta,l}$ for all $l \in \{1{:}L_\beta\}$. Let $i_1' := \chi_{F_{1,L_1}}^\epsilon \circ \ldots \circ \chi_{F_{1,1}}^\epsilon(i)$ and $i_2' := \chi_{F_{2,L_2}}^\epsilon \circ \ldots \circ \chi_{F_{2,1}}^\epsilon(i)$. Assumption 21.10 implies that $\sigma_{K,M,i} = \sigma_{K_{1,1},M,\chi_{F_{1,1}}^\epsilon(i)} = \ldots = \sigma_{K',M,i_1'}$ and $\sigma_{K,M,i} = \sigma_{K_{2,1},M,\chi_{F_{2,1}}^\epsilon(i)} = \ldots = \sigma_{K',M,i_2'}$. Hence, $\sigma_{K',M,i_1'} = \sigma_{K',M,i_2'}$. But, by Assumption 21.9, $(M, P_{K',M}, \Sigma_{K',M})$ is a finite element. Hence, $\sigma_{K',M,i_1'} = \sigma_{K',M,i_2'}$ iff $i_1' = i_2'$. ☐

21.1.3 Connectivity classes as equivalence classes

For all $(K,i) \in \mathcal{T}_h \times \mathcal{N}$, we introduce the smallest geometric entity associated with the dof $\sigma_{K,i}$. This object is the last brick we need to define the equivalence relation mentioned at the beginning of the chapter.

Lemma 21.12 ($\mathcal{M}_{K,i}$). *Let $K \in \mathcal{T}_h$ and i be a boundary dof. Then the following set is nonempty and is a member of $\mathcal{M}_h$:*

$$\mathcal{M}_{K,i} := \bigcap_{\{M \in \mathcal{M}_h \mid i \in \mathcal{N}_{K,M}\}} M. \tag{21.5}$$

Proof. The subset $\mathcal{G}_{K,i} := \{F \in \mathcal{F}_K \mid i \in \mathcal{N}_{K,F}\}$ is nonempty since i is a boundary dof. Then the set $\widetilde{\mathcal{M}}_{K,i} := \bigcap_{F \in \mathcal{G}_{K,i}} F$ is nonempty owing to Lemma 21.7 since $i \in \bigcap_{F \in \mathcal{G}_{K,i}} \mathcal{N}_{K,F}$, and it is a geometric entity owing to

Lemma 21.5. The rest of the proof consists of showing that $\widetilde{\mathcal{M}}_{K,i} = \mathcal{M}_{K,i}$. Since $\mathcal{G}_{K,i} \subset \{M \in \mathcal{M}_h \mid i \in \mathcal{N}_{K,M}\}$, we have $\mathcal{M}_{K,i} \subset \widetilde{\mathcal{M}}_{K,i}$. To prove the converse inclusion, let us consider M in the set $\{M \in \mathcal{M}_h \mid i \in \mathcal{N}_{K,M}\}$. By Lemma 21.5, there is $\emptyset \neq \mathcal{G}_{K,M} \subset \mathcal{F}_K$ s.t. $M = \bigcap_{F\in\mathcal{G}_{K,M}} F$, and the definition (21.3) of $\mathcal{N}_{K,M}$ implies that $i \in \bigcap_{F\in\mathcal{G}_{K,M}} \mathcal{N}_{K,F}$. Hence, for all $F \in \mathcal{G}_{K,M}$, we have $i \in \mathcal{N}_{K,F}$, which means that $\mathcal{G}_{K,M} \subset \mathcal{G}_{K,i}$, and this in turn yields $\widetilde{\mathcal{M}}_{K,i} = \bigcap_{F\in\mathcal{G}_{K,i}} F \subset \bigcap_{F\in\mathcal{G}_{K,M}} F = M$. Since the geometric entity M is arbitrary in $\{M \in \mathcal{M}_h \mid i \in \mathcal{N}_{K,M}\}$, we conclude that $\widetilde{\mathcal{M}}_{K,i} \subset \mathcal{M}_{K,i}$. □

We now partition the product set $\mathcal{T}_h \times \mathcal{N}$ into equivalence classes.

Definition 21.13 (Binary relation $\mathcal{R}$). *We say that $(K,i)\,\mathcal{R}\,(K',i')$ if and only if either $(K,i) = (K',i')$, or $K \neq K'$, $\mathcal{M}_{K,i} = \mathcal{M}_{K',i'} := M$, and given an M-path connecting K to K' in $\mathcal{T}_M$, say $(K = K_0, \ldots, K_L = K')$, with $F_l := \partial K_{l-1} \cap \partial K_l$, $\forall l \in \{1{:}L\}$, we have $i' = \chi^{\epsilon}_{F_L} \circ \ldots \circ \chi^{\epsilon}_{F_1}(i)$.*

This definition makes sense when $K \neq K'$ since in this case M cannot be equal to either K or K', and since $M \subset K \cap K'$, the cells K and K' are in $\mathcal{T}_M$. Owing to Lemma 21.4, K and K' can be connected by an M-path, and owing to Lemma 21.11, the index $\chi^{\epsilon}_{F_L} \circ \ldots \circ \chi^{\epsilon}_{F_1}(i)$ is independent of the M-path that is chosen to connect K to K'.

Lemma 21.14 (Equivalence relation). *Let Assumptions* 21.9 *and* 21.10 *hold true. Then the binary relation $\mathcal{R}$ is an equivalence relation.*

Proof. $\mathcal{R}$ is by definition reflexive. By enumerating the cells in the M-path in reverse order, we infer that $\mathcal{R}$ is symmetric. Finally, let us prove that $\mathcal{R}$ is transitive. Let $(K,i)\,\mathcal{R}\,(K',i')$ and $(K',i')\,\mathcal{R}\,(K'',i'')$. Then $\mathcal{M}_{K,i} = \mathcal{M}_{K',i'} = \mathcal{M}_{K'',i''} := M$. If $(K,i) = (K',i')$ or $(K',i') = (K'',i'')$, there is nothing to prove. Otherwise, we have $K \neq K'$ and $K' \neq K''$. Let $(K =: K_{1,0} \ldots, K_{1,L_1} := K')$, $(K' =: K_{2,0} \ldots, K_{2,L_2} := K'')$ be two M-paths, respectively, connecting K to K' and K' to K''. Let us set $F_{\beta,l} := \partial K_{\beta,l-1} \cap \partial K_{\beta,l}$ for all $l \in \{1{:}L_\beta\}$ and all $\beta \in \{1,2\}$. Then $(K =: K_{1,0} \ldots, K_{1,L_1} = K_{2,0} \ldots, K_{2,L_2} := K'')$ is an M-path and $i'' = \chi^{\epsilon}_{F_{2,L_2}} \circ \ldots \circ \chi^{\epsilon}_{F_{2,1}}(i') = \chi^{\epsilon}_{F_{2,L_2}} \circ \ldots \circ \chi^{\epsilon}_{F_{2,1}} \circ \chi^{\epsilon}_{F_{1,L_1}} \circ \ldots \circ \chi^{\epsilon}_{F_{1,1}}(i)$. If $K \neq K''$, this argument proves that $(K,i)\mathcal{R}(K'',i'')$. If $K = K''$, Assumption 21.10 implies that $\sigma_{K,M,i} = \sigma_{K'',M,i''} = \sigma_{K,M,i''}$, which is possible only if $i = i''$ owing to Assumption 21.9. Hence, we have again $(K,i)\mathcal{R}(K'',i'')$. □

Let $\mathcal{A}_h$ be the set of the equivalence classes induced by $\mathcal{R}$ over $\mathcal{T}_h \times \mathcal{N}$. Let us now consider any map $\mathtt{j\_dof} : \mathcal{T}_h \times \mathcal{N} \to \mathcal{A}_h$ such that

$$[\,\mathtt{j\_dof}(K,i) = \mathtt{j\_dof}(K',i')\,] \iff [\,(K,i)\mathcal{R}(K',i')\,]. \tag{21.6}$$

Letting I be the cardinality of $\mathcal{A}_h$, there are $I!$ ways to define j_dof. Whichever choice that is made to define j_dof, let us now prove that the two assumptions (19.2) and (19.3) made in Chapter 19 hold true. Recall that these are the two structural conditions that we required from `j_dof` in Chapter 19 to construct the conforming subspace $P_k^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$.

Lemma 21.15 (Equivalence relation at interfaces). *Let $F \in \mathcal{F}_h^\circ$ with $F := \partial K_l \cap \partial K_r$ and let χ_{lr} be the map introduced in Assumption* 20.14. *The following holds true for all $i_l \in \mathcal{N}_{K_l,F}$:* (i) $\mathcal{M}_{K_l,i_l} = \mathcal{M}_{K_r,\chi_{lr}(i_l)}$; (ii) $\mathtt{j\_dof}(K_l,i_l) = \mathtt{j\_dof}(K_r,\chi_{lr}(i_l))$.

Proof. Since $\chi_{lr}(\mathcal{N}_{K_l,M}) = \mathcal{N}_{K_r,M}$ owing to the M-matching assumption, we have

$$\begin{aligned}\{M \in \mathcal{M}_h \mid \chi_{lr}(i_l) \in \mathcal{N}_{K_r,M}\} &= \{M \in \mathcal{M}_h \mid \chi_{lr}(i_l) \in \chi_{lr}(\mathcal{N}_{K_l,M})\}\\ &= \{M \in \mathcal{M}_h \mid i_l \in \mathcal{N}_{K_l,M}\}.\end{aligned}$$

Owing to the identity (21.5), we infer that $\mathcal{M}_{K_l,i_l} = \mathcal{M}_{K_r,\chi_{lr}(i_l)}$. The second claim follows readily because $\mathcal{M}_{K_l,i_l} = \mathcal{M}_{K_r,\chi_{lr}(i_l)}$ and the two distinct cells K_l and K_r can be connected by an M-path of length 1 crossing F in such a way that (trivially) $\chi_{lr}(i_l) = \chi_{lr}(i_l)$. This proves that $(K_l,i_l)\mathcal{R}(K_r,\chi_{lr}(i_l))$, i.e., we have $\mathtt{j\_dof}(K_l,i_l) = \mathtt{j\_dof}(K_r,\chi_{lr}(i_l))$ owing to (21.6). □

Let $a \in \mathcal{A}_h$ with representative (K,i). Let us set $M := \mathcal{M}_{K,i}$ and $\chi_{K,K,M}(i) := i$. For all $K' \in \mathcal{T}_M$ such that $K \neq K'$, let us set $\chi_{K,K',M}(i) := \chi_{F_L}^{\epsilon} \circ \ldots \circ \chi_{F_1}^{\epsilon}(i)$, where $(K =: K_0, \ldots K_L := K')$ is any M-path connecting K to K'. Lemma 21.11 together with Item (i) from Lemma 21.15 gives the following characterization of the connectivity class a:

$$a = \bigcup_{K' \in \mathcal{T}_M} \{(K', \chi_{K,K',M}(i))\}. \tag{21.7}$$

We conclude by stating the main result of this section.

Theorem 21.16 (Verification of the assumptions from Chapter 19). *Let Assumptions* 21.9 *and* 21.10 *hold true. Let* `j_dof` *be defined in* (21.6). *Then Assumptions* (19.2) *and* (19.3) *hold true.*

Proof. Let us start with (19.3) which is easier to verify. By definition, we have $(K,i)\mathcal{R}(K,i')$ iff $i = i'$, that is, $\mathtt{j\_dof}(K,i) = \mathtt{j\_dof}(K,i')$ implies that $i = i'$. Let us now prove (19.2) for all $v_h \in P_k^{\mathrm{x,b}}(\mathcal{T}_h;\mathbb{R}^q)$. Let us start with the implication $\Longrightarrow$ in (19.2), i.e., we assume that $v_h \in P_k^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$. Let $(K,i),(K',i')$ be two pairs in the same connectivity class and let $M := \mathcal{M}_{K,i} = \mathcal{M}_{K',i'}$. We want to show that $\sigma_{K,i}(v_{h|K}) = \sigma_{K',i'}(v_{h|K'})$. Since this claim is obvious if $K = K'$, we assume that $K \neq K'$ and we consider an M-path connecting K to K' in $\mathcal{T}_M$, say $(K =: K_0 \ldots, K_L := K')$ and $F_l := \partial K_{l-1} \cap \partial K_l$, $\forall l \in \{1{:}L\}$. Repeated applications of the implication $\Longrightarrow$ from Lemma 20.15 show that since $[\![v_h]\!]_{F_l}^{\mathrm{x}} = 0$ for all $l \in \{1{:}L\}$, we have $\sigma_{K,i}(v_{h|K}) = \sigma_{K',\chi_{F_L}^{\epsilon}\circ\ldots\circ\chi_{F_1}^{\epsilon}(i)}(v_{h|K'}) = \sigma_{K',i'}(v_{h|K'})$, which is the desired result. Let us now prove the other implication $\Longleftarrow$ in (19.2). Let us consider $v_h \in P_k^{\mathrm{x,b}}(\mathcal{T}_h;\mathbb{R}^q)$ and let $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ be a mesh interface. For all $i_l \in \mathcal{N}_{K_l,F}$, we have $\mathtt{j\_dof}(K_l,i_l) = \mathtt{j\_dof}(K_r,\chi_{lr}(i_l))$ owing to Lemma 21.15. By assumption, we also have $\sigma_{K_l,i_l}(v_{h|K_l}) = \sigma_{K_r,\chi_{lr}(i_l)}(v_{h|K_r})$ for all $i_l \in \mathcal{N}_{K_l,F}$. Owing to the implication $\Longleftarrow$ from Lemma 20.15, we infer that $[\![v_h]\!]_F^{\mathrm{x}} = 0$. Since this result holds true for all $F \in \mathcal{F}_h^\circ$, we conclude that $v_h \in P_k^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$. $\square$

21.2 Verification of the assumptions

The goal of this section is to verify that Assumptions 21.9 and 21.10 are indeed satisfied by the Lagrange, canonical hybrid, Nédélec, and Raviart–Thomas elements. We assume that $d = 3$.

21.2.1 Lagrange and canonical hybrid elements

For the Lagrange elements there are four types of geometric entities: cells, faces, edges, and vertices. We have to verify Assumptions 21.9 and 21.10 for the vertices and the edges.

Assume first that M is a vertex, say $M := \{\boldsymbol{z}\}$. For all $K \in \mathcal{T}_{\boldsymbol{z}}$, let $\boldsymbol{a}_{K,i}$ be the unique vertex in K such that $\boldsymbol{a}_{K,i} = \boldsymbol{z}$ and let us set $\gamma_{K,\boldsymbol{z}}(p) := p(\boldsymbol{a}_{K,i})$ for all $p \in P_K$. Clearly $\ker(\gamma_{K,F}) \subset \ker(\gamma_{K,\boldsymbol{z}})$ for all $F \in \mathcal{F}_K$. Then $P_{K,\boldsymbol{z}} := \gamma_{K,\boldsymbol{z}}(P_K) = \mathbb{R}$ because $p(\boldsymbol{a}_{K,i}) = p(\boldsymbol{z}_i)$ spans $\mathbb{R}$ when p spans P_K. Furthermore, setting $\sigma_{K,\boldsymbol{z},i}(x) := x$ for all $x \in \mathbb{R}$, we have $\sigma_{K,i}(p) = p(\boldsymbol{a}_{K,i}) = \sigma_{K,\boldsymbol{z},i}(p(\boldsymbol{a}_{K,i})) = (\sigma_{K,\boldsymbol{z},i} \circ \gamma_{K,\boldsymbol{z}})(p)$. We observe that $P_{K,\boldsymbol{z}}$ and $\Sigma_{K,\boldsymbol{z}} := \{\sigma_{K,\boldsymbol{z},i}\}$ do not depend on K and that $(\boldsymbol{z}, P_{K,\boldsymbol{z}}, \Sigma_{K,\boldsymbol{z}})$ is a finite element.

Assume now that $M := E$ is an edge of K, and let us set $\widehat{E} := \boldsymbol{T}_K^{-1}(E)$. We define $\gamma_{K,E}(p) := p_{|E}$ for all $p \in P_K$. Hence, $\ker(\gamma_{K,F}) \subset \ker(\gamma_{K,E})$ for all $F \in \mathcal{F}_K$. Moreover, $\gamma_{K,E}(p) = \widehat{p} \circ \boldsymbol{T}_{K|\widehat{E}}^{-1} = \widehat{p} \circ \boldsymbol{T}_{\widehat{E}} \circ \boldsymbol{T}_{\widehat{E}}^{-1} \circ \boldsymbol{T}_{K|\widehat{E}}^{-1}$, where $\boldsymbol{T}_{\widehat{E}} : \widehat{S}^1 \to \widehat{E}$ is any bijective affine mapping between the unit segment in $\mathbb{R}$ and the reference edge $\widehat{E}$. By proceeding as in the proof of Lemma 7.10, we conclude that $P_{K,E} := \gamma_{K,E}(P_K) = \mathbb{P}_{k,1} \circ \boldsymbol{T}_{K,E}^{-1}$ with $\boldsymbol{T}_{K,E} := \boldsymbol{T}_{K|\widehat{E}} \circ \boldsymbol{T}_{\widehat{E}}$. By proceeding as in the proof of Lemma 20.6, we conclude that $P_{K_l,E} = P_{K_r,E}$ for all $K_l, K_r \in \mathcal{T}_E$ with a common interface. For every Lagrange node $\boldsymbol{a}_{K,i}$ located on E, we define $\sigma_{K,E,i}(p) := p(\boldsymbol{a}_{K,i})$ for all $p \in P_{K,E}$, and we denote by $\Sigma_{K,M}$ the collection of these dofs. All the Lagrange finite elements considered in this book are such that $(E, P_{K,E}, \Sigma_{K,E})$ is a finite element.

In conclusion, we have verified that Assumption 21.9 and Item (i) of Assumption 21.10 hold true, whether M is a vertex or an edge. It remains to verify that one can construct a map $\chi_{lr} : \mathcal{N}_{K_l,F} \to \mathcal{N}_{K_r,F}$ s.t. Item (ii) of Assumption 21.10 also holds true. This construction is done in §21.3.

Similar arguments as above can be invoked for the canonical hybrid element. We invite the reader to verify that Assumption 21.9 and Item (i) of Assumption 21.10 hold true for the canonical hybrid element, whether M is a vertex or an edge.

21.2.2 Nédélec elements

We invite the reader to verify that Assumption 21.9 and Item (i) of Assumption 21.10 hold true for the edge dofs of the $\mathbb{N}_{k,d}$. It remains to verify that one can construct a map $\chi_{lr} : \mathcal{N}_{K_l,F} \to \mathcal{N}_{K_r,F}$ s.t. Item(ii) of Assumption 21.10 also holds true. This construction is done in §21.3.

21.2.3 Raviart–Thomas elements

There is nothing to prove for these elements since Assumption 21.9 is identical to Assumption 20.12 and Assumption 21.10 is identical to Assumption 20.14, and we have already verified in §20.4.1 that Assumption 20.12 and Assumption 20.14 are met by the Raviart–Thomas elements.

21.3 Practical construction

In this section, we investigate systematic ways to construct the maps χ_{lr} and `j_dof`. The construction of χ_{lr} is done in such a way that Item (ii) of Assumption 21.10 holds true. As before, the reference cell $\widehat{K}$ can be either a simplex or a cuboid in $\mathbb{R}^d$, $d \in \{2, 3\}$.

21.3.1 Enumeration of the geometric entities in $\widehat{K}$

The construction of χ_{lr} is greatly simplified by adopting reasonable enumeration conventions on the reference cell $\widehat{K}$ and by using the orientation of the mesh. We start by enumerating the geometric entities in $\widehat{K}$. We first enumerate the n_{cv} vertices, say from 1 to n_{cv}, as in Table 10.1 in §10.2. We start with the origin of $\widehat{K}$, say $\widehat{\boldsymbol{z}}_1 := \mathbf{0}$, then we enumerate d vertices in such a way that the orientation of the basis $(\widehat{\boldsymbol{z}}_2 - \widehat{\boldsymbol{z}}_1, \ldots, \widehat{\boldsymbol{z}}_{d+1} - \widehat{\boldsymbol{z}}_1)$ is the same as that of the ambient space $\mathbb{R}^d$ (assumed to be based on the right-hand rule). There is no other vertex to enumerate if $\widehat{K}$ is the unit simplex. If $\widehat{K}$ is the unit square, the last vertex is assigned number 4, and if $\widehat{K}$ is the unit cube, the last vertex of the face containing $\{\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_3\}$ is assigned number 5, then we set $\widehat{\boldsymbol{z}}_6 := \widehat{\boldsymbol{z}}_2 + \boldsymbol{e}_z$, $\widehat{\boldsymbol{z}}_7 := \widehat{\boldsymbol{z}}_3 + \boldsymbol{e}_z$, and $\widehat{\boldsymbol{z}}_8 := \widehat{\boldsymbol{z}}_5 + \boldsymbol{e}_z$; see Figure 21.2 and Figure 10.2.

We now enumerate the edges of $\widehat{K}$ from 1 to n_{ce} and the faces of $\widehat{K}$ from 1 to n_{cf}. The way the enumeration is done does not really matter for our purpose, but to be complete, we now propose one possible enumeration

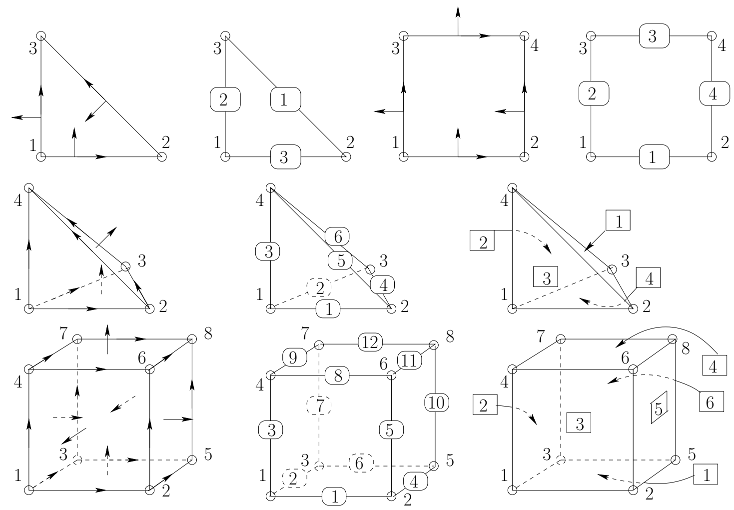

Fig. 21.2 Orientation of the edges and faces and enumeration of the vertices, edges, and faces of the reference cell in dimensions two and three. In dimension two, edges and faces coincide as geometric entities but they are oriented differently: an edge is oriented by a tangent vector and a face by a normal vector.

technique in Figure 21.2 and Table 21.1. The convention adopted in Table 21.1 is that $\widehat{E} = (\widehat{\boldsymbol{z}}_p, \widehat{\boldsymbol{z}}_q)$, $p < q$, means that $\widehat{E}$ passes through the two vertices $\widehat{\boldsymbol{z}}_p$, $\widehat{\boldsymbol{z}}_q$, and the edge is oriented by setting $\widehat{\boldsymbol{\tau}}_E := (\widehat{\boldsymbol{z}}_q - \widehat{\boldsymbol{z}}_p)/\|\widehat{\boldsymbol{z}}_q - \widehat{\boldsymbol{z}}_p\|_{\ell^2}$. The point $\widehat{\boldsymbol{z}}_p$ is called origin of the oriented edge $\widehat{E}$. The notation $\widehat{F} = (\widehat{\boldsymbol{z}}_p, \widehat{\boldsymbol{z}}_q, \widehat{\boldsymbol{z}}_r)$, $p < q < r$, means that $\widehat{F}$ passes though the three vertices $\widehat{\boldsymbol{z}}_p$, $\widehat{\boldsymbol{z}}_q$, $\widehat{\boldsymbol{z}}_r$, and that the unit normal $\widehat{\boldsymbol{n}}_{\widehat{F}}$ orienting $\widehat{F}$ is such that $(\widehat{\boldsymbol{z}}_q - \widehat{\boldsymbol{z}}_p, \widehat{\boldsymbol{z}}_r - \widehat{\boldsymbol{z}}_p, \widehat{\boldsymbol{n}}_{\widehat{F}})$ is a right-hand basis, i.e., $\widehat{\boldsymbol{n}}_{\widehat{F}} = ((\widehat{\boldsymbol{z}}_q - \widehat{\boldsymbol{z}}_p)\times(\widehat{\boldsymbol{z}}_r - \widehat{\boldsymbol{z}}_p))/\|(\widehat{\boldsymbol{z}}_q - \widehat{\boldsymbol{z}}_p)\times(\widehat{\boldsymbol{z}}_r - \widehat{\boldsymbol{z}}_p)\|_{\ell^2}$ (see (10.9)). The vertex $\widehat{\boldsymbol{z}}_p$ is called origin of the oriented face $\widehat{F}$. Note that for both the reference simplex and the reference cuboid, the orientation of the geometric entities is done by using the increasing vertex-index enumeration technique explained in §10.4.

Let now K be a cell in a mesh $\mathcal{T}_h$. Let $\boldsymbol{z}$, E, F be a vertex, an edge, and a face of K, respectively. We are going to say in the rest of this section that the local index of $\boldsymbol{z}$, E, F in K is, respectively, p, q, r if there is a vertex $\widehat{\boldsymbol{z}}_p$, $p \in \{1{:}n_{\mathrm{cv}}\}$, an edge $\widehat{E}_q$, $q \in \{1{:}n_{\mathrm{ce}}\}$, and a face $\widehat{F}_r$, $r \in \{1{:}n_{\mathrm{cf}}\}$, such that $\boldsymbol{z} = \boldsymbol{T}_K(\widehat{\boldsymbol{z}}_p)$, $E = \boldsymbol{T}_K(\widehat{E}_q)$, and $F = \boldsymbol{T}_K(\widehat{F}_r)$.

21.3.2 Example of a construction of χ_{lr} and `j_dof`

We now present an example of practical construction of the maps χ_{lr} and `j_dof`. One important advantage of the proposed enumeration is that it can be implemented in parallel since for each cell K of index $m \in \{1{:}N_{\mathrm{c}}\}$, the

2D simplex	V	$\widehat{\boldsymbol{z}}_1 = (0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0)$, $\widehat{\boldsymbol{z}}_3 = (0,1)$
	E	$\widehat{E}_1 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_3)$, $\widehat{E}_2 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_3)$, $\widehat{E}_2 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2)$
3D simplex	V	$\widehat{\boldsymbol{z}}_1 = (0,0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0,0)$, $\widehat{\boldsymbol{z}}_3 = (0,1,0)$, $\widehat{\boldsymbol{z}}_4 = (0,0,1)$
	E	$\widehat{E}_1 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2)$, $\widehat{E}_2 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_3)$, $\widehat{E}_3 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_4)$
		$\widehat{E}_4 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_3)$, $\widehat{E}_5 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_4)$, $\widehat{E}_6 = (\widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_4)$
	F	$\widehat{F}_1 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_4)$, $\widehat{F}_2 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_4)$
		$\widehat{F}_3 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_4)$, $\widehat{F}_4 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_3)$
2D square	V	$\widehat{\boldsymbol{z}}_1 = (0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0)$, $\widehat{\boldsymbol{z}}_3 = (0,1)$, $\widehat{\boldsymbol{z}}_4 = (1,1)$
	E	$\widehat{E}_1 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2)$, $\widehat{E}_2 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_3)$, $\widehat{E}_3 = (\widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_4)$, $\widehat{E}_4 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_4)$
3D cube	V	$\widehat{\boldsymbol{z}}_1 = (0,0,0)$, $\widehat{\boldsymbol{z}}_2 = (1,0,0)$, $\widehat{\boldsymbol{z}}_3 = (0,1,0)$, $\widehat{\boldsymbol{z}}_4 = (0,0,1)$
		$\widehat{\boldsymbol{z}}_5 = (1,1,0)$, $\widehat{\boldsymbol{z}}_6 = (1,0,1)$, $\widehat{\boldsymbol{z}}_7 = (0,1,1)$, $\widehat{\boldsymbol{z}}_8 = (1,1,1)$
	E	$\widehat{E}_1 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2)$, $\widehat{E}_2 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_3)$, $\widehat{E}_3 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_4)$, $\widehat{E}_4 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_5)$
		$\widehat{E}_5 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_6)$, $\widehat{E}_6 = (\widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_5)$, $\widehat{E}_7 = (\widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_7)$, $\widehat{E}_8 = (\widehat{\boldsymbol{z}}_4, \widehat{\boldsymbol{z}}_6)$
		$\widehat{E}_9 = (\widehat{\boldsymbol{z}}_4, \widehat{\boldsymbol{z}}_7)$, $\widehat{E}_{10} = (\widehat{\boldsymbol{z}}_5, \widehat{\boldsymbol{z}}_8)$, $\widehat{E}_{11} = (\widehat{\boldsymbol{z}}_6, \widehat{\boldsymbol{z}}_8)$, $\widehat{E}_{12} = (\widehat{\boldsymbol{z}}_7, \widehat{\boldsymbol{z}}_8)$
	F	$\widehat{F}_1 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_3)$, $\widehat{F}_2 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_4)$, $\widehat{F}_3 = (\widehat{\boldsymbol{z}}_1, \widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_4)$
		$\widehat{F}_4 = (\widehat{\boldsymbol{z}}_4, \widehat{\boldsymbol{z}}_6, \widehat{\boldsymbol{z}}_7)$, $\widehat{F}_5 = (\widehat{\boldsymbol{z}}_2, \widehat{\boldsymbol{z}}_5, \widehat{\boldsymbol{z}}_6)$, $\widehat{F}_6 = (\widehat{\boldsymbol{z}}_3, \widehat{\boldsymbol{z}}_5, \widehat{\boldsymbol{z}}_7)$

Table 21.1 Enumeration and orientation of the vertices, edges, and faces in simplices and cuboids in dimensions two and three.

proposed enumeration technique only requires to have access to local information like $\texttt{j\_cv}(m, 1{:}n_{\mathrm{cv}})$, $\texttt{j\_ce}(m, 1{:}n_{\mathrm{ce}})$, $\texttt{j\_cf}(m, 1{:}n_{\mathrm{cf}})$, which is usually provided by mesh generators. Recall that $\texttt{j\_cf}(m, i)$ is the global index of the i-th vertex of the m-th cell, $\texttt{j\_ce}(m, e)$ is the global index of the e-th edge of the m-th cell, and $\texttt{j\_cf}(m, f)$ is the global index of the f-th face of the m-th cell.

Enumeration of the vertex dofs. Let us assume that there are $n_{\mathrm{sh}}^{\mathrm{v}}$ dofs per vertex. For scalar-valued Lagrange elements or the scalar-valued canonical hybrid element, we have $n_{\mathrm{sh}}^{\mathrm{v}} := 1$. We adopt the convention $n_{\mathrm{sh}}^{\mathrm{v}} := 0$ for $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ elements. Given a mesh cell K, we enumerate the local dofs in K as follows. Letting $n \in \{1{:}n_{\mathrm{sh}}^{\mathrm{v}}\}$, $v \in \{1{:}n_{\mathrm{cv}}\}$, the n-th dof attached to the v-th vertex is assigned the index $i := (v-1)n_{\mathrm{sh}}^{\mathrm{v}} + n$.

Let us now define $\texttt{j\_dof}$ and, given an interface $F := \partial K_l \cap \partial K_r$, let us define χ_{lr}. Let $\boldsymbol{z}$ be vertex of the face F. Let $v_l, v_r \in \{1{:}n_{\mathrm{cv}}\}$ be the local index of $\boldsymbol{z}$ in K_l, K_r, respectively, and let m_l, m_r be the indices of K_l, K_r in $\mathcal{T}_h$, respectively. Hence, $\texttt{j\_cv}(m_l, v_l) = \texttt{j\_cv}(m_r, v_r)$. Let $i_{l0} := (v_l - 1)n_{\mathrm{sh}}^{\mathrm{v}}$ and $i_{r0} := (v_r - 1)n_{\mathrm{sh}}^{\mathrm{v}}$. Then upon setting $\chi_{lr}(i_{l0} + i) := i_{r0} + i$ for all $i \in \{1{:}n_{\mathrm{sh}}^{\mathrm{v}}\}$, we observe that χ_{lr} maps a vertex dof of K_l to a vertex dof of K_r and by construction the vertex associated with $i_{l0} + i$ (with index $\texttt{j\_cv}(m_l, v_l)$) is the same as that associated with $i_{r0} + i$ (with index $\texttt{j\_cv}(m_r, v_r)$). Finally, $\texttt{j\_dof}$ is obtained by setting

$$i := (v-1)n_{\mathrm{sh}}^{\mathrm{v}} + n, \tag{21.8a}$$

$$\mathtt{j\_dof}(m,i) := (\mathtt{j\_cv}(m,v)-1)n_{\mathrm{sh}}^{\mathrm{v}} + n, \tag{21.8b}$$

for all $n \in \{1{:}n_{\mathrm{sh}}^{\mathrm{v}}\}$ and all $v \in \{1{:}n_{\mathrm{cv}}\}$. This defines $n_{\mathrm{sh}}^{\mathrm{v}} N_{\mathrm{v}}$ equivalence classes enumerated from 1 to $n_{\mathrm{sh}}^{\mathrm{v}} N_{\mathrm{v}}$.

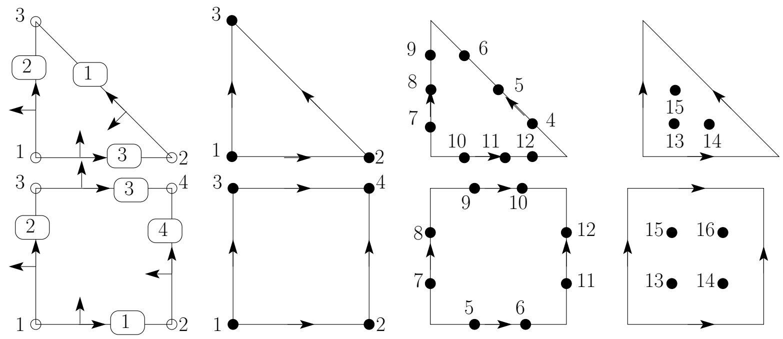

Fig. 21.3 Enumeration of geometric entities and dofs for triangles (top) and squares (bottom). Orientation of edges and faces, enumeration of vertices and faces (leftmost panels), enumeration of vertex dofs (center left panels), enumeration of edge dofs for $\mathbb{P}_{4,2}$ and $\mathbb{Q}_{3,2}$ elements (center right panels), enumeration of volume dofs (rightmost panels).

Enumeration of the edge dofs. Let $n_{\mathrm{sh}}^{\mathrm{e}}$ be the number of dofs per edge. For $\mathbb{P}_{k+1,d}$ and $\mathbb{Q}_{k+1,d}$ scalar-valued elements (Lagrange or canonical hybrid) and for $\mathbf{N}_{k,d}$ Nédélec elements, we have $n_{\mathrm{sh}}^{\mathrm{e}} = \dim(\mathbb{P}_{k,1})$ with $k \geq 0$. Let us now adopt a strategy to enumerate the edge dofs in K that allows us to generate χ_{lr} with information associated with the edges only. Let $E := (\boldsymbol{z}_p, \boldsymbol{z}_q)$ be an oriented edge of K with origin $\boldsymbol{z}_p$, $p, q \in \{1{:}N_{\mathrm{v}}\}$. Let $e \in \{1{:}n_{\mathrm{ce}}\}$ be the local index of E in K. Setting $i_0 := n_{\mathrm{cv}} n_{\mathrm{sh}}^{\mathrm{v}} + (e-1)n_{\mathrm{sh}}^{\mathrm{e}}$, we enumerate the dofs associated with E from i_0+1 to $i_0+n_{\mathrm{sh}}^{\mathrm{e}}$ by moving along E from $\boldsymbol{z}_p$ to $\boldsymbol{z}_q$. Since the orientation of the mesh is generation-compatible (see Definition 10.3), the orientation of the edge is unchanged by the geometric mapping $\boldsymbol{T}_K$ for all $K \in \mathcal{T}_E$. This implies that no matter which edge $\widehat{E}$ of $\widehat{K}$ is mapped to E, the edge dofs $\{\sigma_{K,E,i}\}_{i\in\{1:n_{\mathrm{ce}}\}}$ are always listed in the same order as those in $\{\widehat{\sigma}_{\widehat{K},\widehat{E},i}\}_{i\in\{1:n_{\mathrm{ce}}\}}$ because the edge dofs are invariant under any vertex permutation (see Assumption 20.7 and Item (iii) in Lemma 20.22). The proposed enumeration is illustrated in the two panels in the third column of Figure 21.3 for the $\mathbb{P}_{4,2}$ and $\mathbb{Q}_{3,2}$ Lagrange elements, in the left panel of Figure 21.4 for the $\mathbf{N}_{2,3}$ Nédélec element, in Figure 21.5 for the $\mathbb{P}_{3,3}$ Lagrange element, and in Figure 21.6 for the $\mathbb{Q}_{3,3}$ Lagrange element.

Let us now define j_dof and, given an interface $F := \partial K_l \cap \partial K_r$, let us define χ_{lr}. Let E be an edge of the face F. Let $e_l, e_r \in \{1{:}n_{\mathrm{ce}}\}$ be the local index of E in K_l, K_r, respectively, and let m_l, m_r be the index of K_l, K_r in $\mathcal{T}_h$,

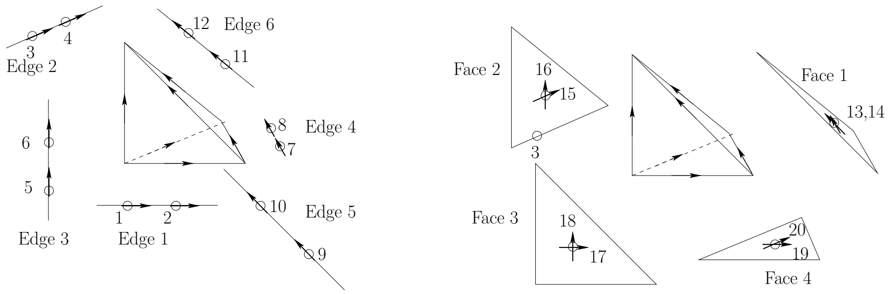

Fig. 21.4 Enumeration of dofs for the $\mathbb{N}_{2,3}$ element. Left: edge dofs. Right: face dofs.

respectively. Hence, $\texttt{j\_ce}(m_l, e_l) = \texttt{j\_ce}(m_r, e_r)$. Let $i_{l0} := n_{\mathrm{cv}} n^{\mathrm{v}}_{\mathrm{sh}} + (e_l - 1) n^{\mathrm{e}}_{\mathrm{sh}}$ and $i_{r0} := n_{\mathrm{cv}} n^{\mathrm{v}}_{\mathrm{sh}} + (e_r - 1) n^{\mathrm{e}}_{\mathrm{sh}}$. Then setting $\chi_{lr}(i_{l0} + i) := i_{r0} + i$ for all $i \in \{1{:}n^{\mathrm{e}}_{\mathrm{sh}}\}$, we observe that χ_{lr} maps an edge dof of K_l to an edge dof of K_r and by construction the edge associated with $i_{l0}+i$ (with index $\texttt{j\_ce}(m_l, e_l)$) is the same as that associated with $i_{r0}+i$ (with index $\texttt{j\_ce}(m_r, e_r)$). Concerning `j_dof`, since all the vertex dofs have already been enumerated using (21.8), we continue with the edge dofs by setting

$$i := n_{\mathrm{cv}} n^{\mathrm{v}}_{\mathrm{sh}} + (e-1) n^{\mathrm{e}}_{\mathrm{sh}} + n, \tag{21.9a}$$

$$\texttt{j\_dof}(m, i) := n^{\mathrm{v}}_{\mathrm{sh}} N_{\mathrm{v}} + (\texttt{j\_ce}(m, e) - 1) n^{\mathrm{e}}_{\mathrm{sh}} + n, \tag{21.9b}$$

for all $n \in \{1{:}n^{\mathrm{e}}_{\mathrm{sh}}\}$ and all $e \in \{1{:}n_{\mathrm{ce}}\}$. This defines $n^{\mathrm{e}}_{\mathrm{sh}} N_{\mathrm{e}}$ equivalence classes enumerated from $n^{\mathrm{v}}_{\mathrm{sh}} N_{\mathrm{v}} + 1$ to $n^{\mathrm{v}}_{\mathrm{sh}} N_{\mathrm{v}} + n^{\mathrm{e}}_{\mathrm{sh}} N_{\mathrm{e}}$.

Enumeration of the face dofs. Let us proceed with the enumeration of the face dofs in dimension 3. Let F be a face of K. Let $\boldsymbol{z}_p$ be the origin of F. Let $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ be the two unit vectors orienting the edges starting from $\boldsymbol{z}_p$ (recall that $\boldsymbol{n}_F$ has been defined s.t. $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{n}_F)$ has the same orientation as the right-hand basis in $\mathbb{R}^3$ (see (10.9)). Let $f \in \{1{:}n_{\mathrm{cf}}\}$ be the local index of F in K. The face dofs on F are enumerated from $i_0 + 1$ to $i_0 + n^{\mathrm{f}}_{\mathrm{sh}}$, where $i_0 := n_{\mathrm{cv}} n^{\mathrm{v}}_{\mathrm{sh}} + n_{\mathrm{ce}} n^{\mathrm{e}}_{\mathrm{sh}} + (f-1) n^{\mathrm{f}}_{\mathrm{sh}}$. When the dofs in F are attached to nodes located in F, as for Lagrange elements, one possible enumeration technique is to look at F with the vector $\boldsymbol{\tau}_1$ horizontal, the origin of F on the left, $\boldsymbol{\tau}_2$ pointing upward, and $\boldsymbol{n}_F$ pointing towards us. Then one enumerates the dofs on F by moving across F from left to right and bottom to top. The proposed enumeration is illustrated in Figure 21.5 for the $\mathbb{P}_{3,3}$ Lagrange element (where there is 1 face dof) and in Figure 21.6 for the $\mathbb{Q}_{3,3}$ Lagrange element (where there are 4 face dofs). For the Nédélec and Raviart–Thomas elements, the enumeration of the face dofs can be performed by enumerating the modal basis associated with these dofs just like above. For the Nédélec elements, one has two dofs for each modal basis function, say one associated with $\boldsymbol{\tau}_1$ and one associated with $\boldsymbol{\tau}_2$. One first enumerates the dof associated with $\boldsymbol{\tau}_1$, then the dof associated with $\boldsymbol{\tau}_2$. An example is shown in the right panel of Figure 21.4.

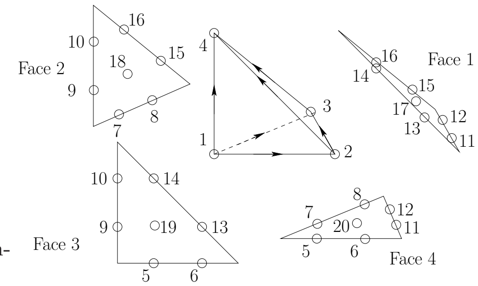

Fig. 21.5 Enumeration of dofs in dimension three for the $\mathbb{P}_{3,3}$ element.

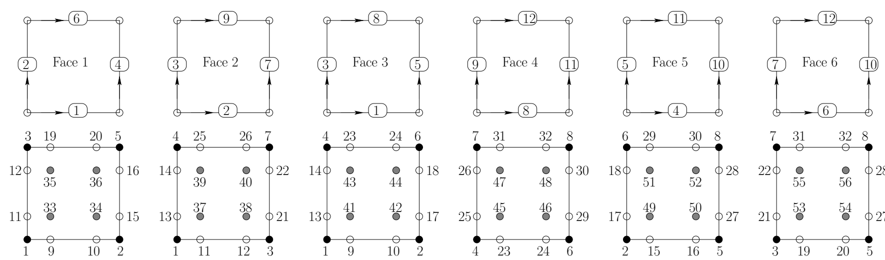

Fig. 21.6 Enumeration of $\mathbb{Q}_{3,3}$ dofs in a cube. The enumeration of the edges and faces is shown in the top panels. The enumeration of the dofs is shown in the bottom panels for the 6 faces of the cube. The vertex dofs are shown in black, the edge dofs are shown in white, and the face dofs are shown in gray. The remaining 8 volume dofs are hidden.

Assume now that $F := \partial K_l \cap \partial K_r$. Let $f_l, f_r \in \{1{:}n_{\rm cf}\}$ be the index of F in K_l, K_r, and let m_l, m_r be the indices of K_l, K_r in $\mathcal{T}_h$, i.e., $\texttt{j\_cf}(m_l, f_l) = \texttt{j\_cf}(m_r, f_r)$. Let $i_{l0} := n_{\rm cv}n_{\rm sh}^{\rm v} + n_{\rm ce}n_{\rm sh}^{\rm e} + (f_l - 1)n_{\rm sh}^{\rm f}$ and $i_{r0} := n_{\rm cv}n_{\rm sh}^{\rm v} + n_{\rm ce}n_{\rm sh}^{\rm e} + (f_r - 1)n_{\rm sh}^{\rm f}$. Then we set $\chi_{lr}(i_{l0} + i) := i_{r0} + i$ for all $i \in \{1{:}n_{\rm sh}^{\rm f}\}$. Concerning $\texttt{j\_dof}$, since all the vertex and edge dofs have already been enumerated using (21.8) and (21.9), we continue with the face dofs by setting

$$i := n_{\rm cv}n_{\rm sh}^{\rm v} + n_{\rm ce}n_{\rm sh}^{\rm e} + (f-1)n_{\rm sh}^{\rm f} + n, \tag{21.10a}$$

$$\texttt{j\_dof}(m,i) := n_{\rm sh}^{\rm v}N_{\rm v} + n_{\rm sh}^{\rm e}N_{\rm e} + (\texttt{j\_cf}(m,f) - 1)n_{\rm sh}^{\rm f} + n, \tag{21.10b}$$

for all $n \in \{1{:}n_{\rm sh}^{\rm f}\}$ and all $f \in \{1{:}n_{\rm cf}\}$. This defines $n_{\rm sh}^{\rm f}N_{\rm f}$ equivalence classes enumerated from $n_{\rm sh}^{\rm v}N_{\rm v} + n_{\rm sh}^{\rm e}N_{\rm e} + 1$ to $n_{\rm sh}^{\rm v}N_{\rm v} + n_{\rm sh}^{\rm e}N_{\rm e} + n_{\rm sh}^{\rm f}N_{\rm f}$. An example using the proposed enumeration for the $\mathbb{P}_{3,3}$ element is shown in Figure 21.5. An example of enumeration for the $\mathbb{Q}_{3,3}$ element is shown in Figure 21.6.

Enumeration of the volume dofs. The way the enumeration of the volume dofs is done does not matter, but to be consistent with the above definitions, one can proceed as follows. For Lagrange elements, one starts with the dof that is the closest to the origin of K and traverse the volume dofs by using the orientation of K. In dimension two, for instance, one can proceed as above since K can be viewed as a two-dimensional face, as illustrated in the rightmost panels in Figure 21.3 for the $\mathbb{P}_{4,2}$ and $\mathbb{Q}_{3,2}$ Lagrange elements. In dimension three, one can traverse all the volume dofs by moving first along the direction $\boldsymbol{\tau}_1$, then along the direction $\boldsymbol{\tau}_2$, and finally along the direction $\boldsymbol{\tau}_3$. For Nédélec and Raviart–Thomas elements, one uses the enumeration of the modal basis functions defining the volume dofs. For these elements one has 3 dofs for each modal basis function (in dimension 3), say one associated with each direction $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3$. For each modal function one first enumerates the dof associated with $\boldsymbol{\tau}_1$, then the dof associated with $\boldsymbol{\tau}_2$, and one finishes with the dof associated with $\boldsymbol{\tau}_3$, then one moves to the next modal function. The connectivity array can now be completed by setting

$$i := n_{\mathrm{cv}} n_{\mathrm{sh}}^{\mathrm{v}} + n_{\mathrm{ce}} n_{\mathrm{sh}}^{\mathrm{e}} + n_{\mathrm{cf}} n_{\mathrm{sh}}^{\mathrm{f}} + n, \tag{21.11a}$$

$$\texttt{j\_dof}(m,i) := n_{\mathrm{sh}}^{\mathrm{v}} N_{\mathrm{v}} + n_{\mathrm{sh}}^{\mathrm{e}} N_{\mathrm{e}} + n_{\mathrm{sh}}^{\mathrm{f}} N_{\mathrm{f}} + (m-1) n_{\mathrm{sh}}^{\mathrm{v}} + n, \tag{21.11b}$$

for all $n \in \{1{:}n_{\mathrm{sh}}^{\mathrm{c}}\}$ and all $m \in \{1{:}N_{\mathrm{c}}\}$.

Exercises

Exercise 21.1 (Mesh orientation, $\mathcal{N}_{K,F}$, χ_{lr}). Consider the mesh $\mathcal{T}_h$ shown in Exercise 19.1. (i) Orient the mesh by using the increasing vertex-index enumeration technique. (ii) Consider the corresponding space $P_2^{\mathrm{g}}(\mathcal{T}_h)$. Use the enumeration convention adopted in this chapter for the dofs. Find the two cells K_l, K_r for the second face of the cell 5 and for the first face of the cell 3. (iii) Let F be the second face of the cell 5. Identify $\mathcal{N}_{5,F}$, $\texttt{j\_dof}(5,\mathcal{N}_{5,F})$, and the map χ_{lr}. (iv) Let F' be the first face of the cell 3. Identify $\mathcal{N}_{3,F'}$, $\texttt{j\_dof}(3,\mathcal{N}_{3,F'})$, and the map χ_{lr}.

Exercise 21.2 (M-dofs). Let $K \in \mathcal{T}_h$, let $F \in \mathcal{F}_K$, and let $M \in \mathcal{M}_h$ be a geometric entity s.t. $M \subset F$. Prove that $\mathcal{N}_{K,M} \subset \mathcal{N}_{K,F}$.

Exercise 21.3 ($\mathbb{Q}_{k,3}$ dofs). Determine $n_{\mathrm{sh}}^{\mathrm{v}}, n_{\mathrm{sh}}^{\mathrm{e}}, n_{\mathrm{sh}}^{\mathrm{f}}, n_{\mathrm{sh}}^{\mathrm{c}}$ for scalar-valued $\mathbb{Q}_{k,3}$ Lagrange elements.