Part V, Chapter 24

Weak formulation of model problems

In Part V, composed of Chapters 24 and 25, we introduce the notion of weak formulations and state two well-posedness results: the Lax–Milgram lemma and the more fundamental Banach–Nečas–Babuška theorem. Weak formulations are useful for building finite element approximations to partial differential equations (PDEs). This chapter presents a step-by-step derivation of weak formulations. We start by considering a few simple PDEs posed over a bounded subset D of $\mathbb{R}^d$. Our goal is to reformulate these problems in weak form using the important notion of *test functions*. We show by examples that there are many ways to write weak formulations. Choosing one can be guided, e.g., by the smoothness of the data and the quantities of interest (e.g., the solution or its gradient). The reader who is not familiar with functional analysis arguments is invited to review the four chapters composing Part I before reading Part V.

24.1 A second-order PDE

Let D be a Lipschitz domain in $\mathbb{R}^d$ (see §3.1) and consider a function $f : D \to \mathbb{R}$. The problem we want to solve consists of seeking a function $u : D \to \mathbb{R}$ with some appropriate smoothness yet to be clearly defined such that

$$-\Delta u = f \text{ in } D \qquad u = 0 \text{ on } \partial D, \tag{24.1}$$

where the Laplace operator is defined by $\Delta u := \nabla{\cdot}(\nabla u)$. In Cartesian coordinates, we have $\Delta u := \sum_{i\in\{1:d\}} \frac{\partial^2 u}{\partial x_i^2}$.

The PDE $-\Delta u = f$ in D is called *Poisson equation* (and *Laplace equation* when $f = 0$). The Laplace operator is ubiquitous in physics since it is the prototypical operator modelling diffusion processes. Applications include heat transfer (where u is the temperature and f the heat source), mass transfer (where u is the concentration of a species and f the mass source), porous media flow (where u is the hydraulic head and f the mass source), electrostatics (where u is the electrostatic potential and f the charge density), and static equilibria of membranes (where u is the transverse membrane displacement and f the transverse load).

The condition enforced on ∂D in (24.1) is called *boundary condition*. A condition prescribing the value of the solution at the boundary is called *Dirichlet condition*, and when the prescribed value is zero, the condition is called *homogeneous Dirichlet condition*. In the context of the above models, the Dirichlet condition means that the temperature (the concentration, the hydraulic head, the electrostatic potential, or the transverse membrane displacement) is prescribed at the boundary. Other boundary conditions can be prescribed for the Poisson equation, as reviewed in Chapter 31 in the more general context of second-order elliptic PDEs.

To sum up, (24.1) is the Poisson equation (or problem) with a homogeneous Dirichlet condition. We now present three weak formulations of (24.1).

24.1.1 First weak formulation

We derive a weak formulation of (24.1) by proceeding informally. Consider an arbitrary test function $\varphi \in C_0^\infty(D)$, where $C_0^\infty(D)$ is the space of infinitely differentiable functions compactly supported in D. As a first step, we multiply the PDE in (24.1) by φ and integrate over D to obtain

$$-\int_D (\Delta u)\varphi \, \mathrm{d}x = \int_D f\varphi \, \mathrm{d}x. \tag{24.2}$$

Equation (24.2) is equivalent to the PDE in (24.1) if Δu is smooth enough (e.g., integrable over D). Indeed, if an integrable function g satisfies $\int_D g\varphi \, \mathrm{d}x = 0$ for all $\varphi \in C_0^\infty(D)$, Theorem 1.32 implies that $g = 0$ a.e. in D.

As a second step, we use the *divergence formula* stating that for any smooth vector-valued function $\boldsymbol{\Phi}$,

$$\int_D \nabla\cdot\boldsymbol{\Phi} \, \mathrm{d}x = \int_{\partial D} \boldsymbol{\Phi}\cdot\boldsymbol{n} \, \mathrm{d}s, \tag{24.3}$$

where $\boldsymbol{n}$ is the outward unit normal to D. We apply this formula to the function $\boldsymbol{\Phi} := w\nabla v$, where v and w are two scalar-valued smooth functions. Since $\nabla\cdot\boldsymbol{\Phi} = \nabla w\cdot\nabla v + w\Delta v$, we infer that

$$-\int_D (\Delta v)w \, \mathrm{d}x = \int_D \nabla v\cdot\nabla w \, \mathrm{d}x - \int_{\partial D} (\boldsymbol{n}\cdot\nabla v)w \, \mathrm{d}s. \tag{24.4}$$

This is *Green's formula*, which is a very useful tool to derive weak formulations of PDEs involving the Laplace operator. This formula is valid for instance if $v \in C^2(D) \cap C^1(\overline{D})$ and $w \in C^1(D) \cap C^0(\overline{D})$, and it can be extended to functions in the usual Sobolev spaces. In particular, it remains valid for all $v \in H^2(D)$ and all $w \in H^1(D)$. We apply Green's formula to the functions

$v := u$ and $w := \varphi$, assuming enough smoothness for u. Since φ vanishes at the boundary, we transform (24.2) into

$$\int_D \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_D f \varphi \, \mathrm{d}x, \qquad \forall \varphi \in C_0^\infty(D). \tag{24.5}$$

We now recast (24.5) into a functional framework. Let us take $f \in L^2(D)$. We observe that a natural solution space is

$$H^1(D) := \{v \in L^2(D) \mid \nabla v \in \boldsymbol{L}^2(D)\}. \tag{24.6}$$

Recall from Proposition 2.9 that $H^1(D)$ is a Hilbert space when equipped with the inner product $(u,v)_{H^1(D)} := \int_D uv \, \mathrm{d}x + \ell_D^2 \int_D \nabla u \cdot \nabla v \, \mathrm{d}x$ with associated norm $\|v\|_{H^1(D)} := (\int_D v^2 \, \mathrm{d}x + \ell_D^2 \int_D \|\nabla v\|_{\ell^2}^2 \, \mathrm{d}x)^{\frac{1}{2}}$, where $\|\cdot\|_{\ell^2}$ denotes the Euclidean norm in $\mathbb{R}^d$ and ℓ_D is a length scale associated with the domain D, e.g., $\ell_D := \mathrm{diam}(D)$ (one can take $\ell_D := 1$ when working in nondimensional form). In order to account for the boundary condition in (24.1), we consider the subspace spanned by those functions in $H^1(D)$ that vanish at the boundary. It turns out that this space is $H_0^1(D)$; see Theorem 3.10. Finally, we can extend the space of the test functions in (24.5) to the closure of $C_0^\infty(D)$ in $H^1(D)$, which is by definition $H_0^1(D)$ (see Definition 3.9). To see this, we consider any test function $w \in H_0^1(D)$, observe that there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_0^\infty(D)$ converging to w in $H_0^1(D)$, and pass to the limit in (24.5) with φ_n used as the test function. To sum up, a weak formulation of the Poisson equation with homogeneous Dirichlet condition is as follows:

$$\begin{cases} \text{Find } u \in V := H_0^1(D) \text{ such that} \\ \int_D \nabla u \cdot \nabla w \, \mathrm{d}x = \int_D f w \, \mathrm{d}x, \quad \forall w \in V. \end{cases} \tag{24.7}$$

A function u solving (24.7) is called *weak solution* to (24.1).

We now investigate whether a solution to (24.7) (i.e., a weak solution to (24.1)) satisfies the PDE and the boundary condition in (24.1). Similarly to Definition 2.3, we say that a vector-valued field $\boldsymbol{\sigma} \in \boldsymbol{L}_{\mathrm{loc}}^1(D) := L_{\mathrm{loc}}^1(D;\mathbb{R}^d)$ has a weak divergence $\psi \in L_{\mathrm{loc}}^1(D)$ if

$$\int_D \boldsymbol{\sigma} \cdot \nabla \varphi \, \mathrm{d}x = -\int_D \psi \varphi \, \mathrm{d}x, \qquad \forall \varphi \in C_0^\infty(D), \tag{24.8}$$

and we write $\nabla \cdot \boldsymbol{\sigma} := \psi$. The argument of Lemma 2.4 shows that the weak divergence of a vector-valued field, if it exists, is uniquely defined.

Proposition 24.1 (Weak solution). *Assume that u solves* (24.7) *with $f \in L^2(D)$. Then $-\nabla u$ has a weak divergence equal to f, the PDE in* (24.1) *is satisfied a.e. in D, and the boundary condition a.e. in ∂D.*

Proof. Let u be a weak solution. Then $\nabla u \in \boldsymbol{L}^2(D) \subset \boldsymbol{L}_{\mathrm{loc}}^1(D)$. Taking as a test function in (24.7) an arbitrary function $\varphi \in C_0^\infty(D) \subset H_0^1(D)$ and observing that $f \in L^2(D) \subset L^1_{\text{loc}}(D)$, we infer from the definition (24.8) of the weak divergence that the vector-valued field $\boldsymbol{\sigma} := -\nabla u$ has a weak divergence equal to f. Hence, the PDE is satisfied in the sense that $-\nabla{\cdot}(\nabla u) = f$ in $L^2(D)$, i.e., both functions are equal a.e. in D. Since $u \in H^1_0(D)$, u vanishes a.e. in ∂D owing to the trace theorem (Theorem 3.10). □

The crucial advantage of the weak formulation (24.7) with respect to the original formulation (24.1) is that, as we will see in the next chapter, there exist powerful tools that allow us to assert the existence and uniqueness of weak solutions. It is noteworthy that uniqueness is not a trivial property in spaces larger than $H^1(D)$, and existence is nontrivial in spaces smaller than $H^1(D)$. For instance, one can construct domains in which uniqueness does not hold in $L^2(D)$, and existence does not hold in $H^2(D)$; see Exercise 24.2.

24.1.2 Second weak formulation

To derive our second formulation, we introduce the vector-valued function $\boldsymbol{\sigma} := -\nabla u$. To avoid notational collisions, we use the letter p instead of u to denote the scalar-valued unknown function, and we use the symbol u to denote the pair $(\boldsymbol{\sigma}, p)$. In many applications, p plays the role of a potential and $\boldsymbol{\sigma}$ plays the role of a (diffusive) flux. More generally, p is called *primal variable* and $\boldsymbol{\sigma}$ *dual variable*.

Since $\boldsymbol{\sigma} = -\nabla p$ and $-\Delta p = f$, we obtain $\nabla{\cdot}\boldsymbol{\sigma} = f$. Therefore, the model problem is now written as follows:

$$\boldsymbol{\sigma} + \nabla p = 0 \text{ in } D, \qquad \nabla{\cdot}\boldsymbol{\sigma} = f \text{ in } D, \qquad p = 0 \text{ on } \partial D. \tag{24.9}$$

This is the *mixed formulation* of the original problem (24.1). The PDEs in (24.9) are often called *Darcy's equations* (in the context of porous media flows, p is the hydraulic head and $\boldsymbol{\sigma}$ the filtration velocity).

We multiply the first PDE in (24.9) by a vector-valued test function $\boldsymbol{\tau}$ and integrate over D to obtain

$$\int_D \boldsymbol{\sigma}{\cdot}\boldsymbol{\tau}\,\mathrm{d}x + \int_D \nabla p{\cdot}\boldsymbol{\tau}\,\mathrm{d}x = 0. \tag{24.10}$$

We multiply the second PDE in (24.9) by a scalar-valued test function q and integrate over D to obtain

$$\int_D (\nabla{\cdot}\boldsymbol{\sigma}) q\,\mathrm{d}x = \int_D f q\,\mathrm{d}x. \tag{24.11}$$

No integration by parts is performed in this approach.

We now specify a functional framework. We consider $H^1(D)$ as the solution space for p (so that $\nabla p \in \boldsymbol{L}^2(D)$ and $p \in L^2(D)$), and $\boldsymbol{H}(\text{div}; D)$ as the solution space for $\boldsymbol{\sigma}$ with $\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\text{div};D)} := (\|\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2 \|\nabla{\cdot}\boldsymbol{\sigma}\|^2_{L^2(D)})^{\frac{1}{2}}$ (recall that ℓ_D is a characteristic length associated with D, e.g., $\ell_D := \text{diam}(D)$).

Moreover, we enforce the boundary condition explicitly by restricting p to be in the space $H^1_0(D)$. With this setting, the test function $\boldsymbol{\tau}$ can be taken in $\boldsymbol{L}^2(D)$ and the test function q in $L^2(D)$. To sum up, a second weak formulation is as follows:

$$\begin{cases} \text{Find } u := (\boldsymbol{\sigma}, p) \in V \text{ such that} \\ \int_D (\boldsymbol{\sigma}{\cdot}\boldsymbol{\tau} + \nabla p{\cdot}\boldsymbol{\tau} + (\nabla{\cdot}\boldsymbol{\sigma})q)\,\mathrm{d}x = \int_D fq\,\mathrm{d}x, \quad \forall w := (\boldsymbol{\tau}, q) \in W, \end{cases} \tag{24.12}$$

with the functional spaces $V := \boldsymbol{H}(\mathrm{div}; D) \times H^1_0(D)$ and $W := \boldsymbol{L}^2(D) \times L^2(D)$. Note that the space where the solution is expected to be (trial space) differs from the space where the test functions are taken (test space).

Proposition 24.2 (Weak solution). *Assume that u solves* (24.12) *with $f \in L^2(D)$. Then the PDEs in* (24.9) *are satisfied a.e. in D, and the boundary condition a.e. in ∂D.*

Proof. Left as an exercise. □

24.1.3 Third weak formulation

We start with the mixed formulation (24.9), and we now perform an integration by parts on the term involving $\nabla{\cdot}\boldsymbol{\sigma}$. Proceeding informally, we obtain

$$-\int_D \boldsymbol{\sigma}{\cdot}\nabla q\,\mathrm{d}x + \int_{\partial D} (\boldsymbol{n}{\cdot}\boldsymbol{\sigma})q\,\mathrm{d}s = \int_D fq\,\mathrm{d}x. \tag{24.13}$$

We take the test function q in $H^1(D)$ for the first integral to make sense. Moreover, to eliminate the boundary integral, we restrict q to be in the space $H^1_0(D)$. Now the dual variable $\boldsymbol{\sigma}$ can be taken in $\boldsymbol{L}^2(D)$. To sum up, a third weak formulation is as follows:

$$\begin{cases} \text{Find } u := (\boldsymbol{\sigma}, p) \in V \text{ such that} \\ \int_D (\boldsymbol{\sigma}{\cdot}\boldsymbol{\tau} + \nabla p{\cdot}\boldsymbol{\tau} + \boldsymbol{\sigma}{\cdot}\nabla q)\,\mathrm{d}x = -\int_D fq\,\mathrm{d}x, \quad \forall w := (\boldsymbol{\tau}, q) \in V, \end{cases} \tag{24.14}$$

with the same functional space $V := \boldsymbol{L}^2(D) \times H^1_0(D)$ for the trial and test spaces. The change of sign on the right-hand side has been introduced to make the left-hand side symmetric with respect to $(\boldsymbol{\sigma}, p)$ and $(\boldsymbol{\tau}, q)$.

Proposition 24.3. *Let u solve* (24.14) *with $f \in L^2(D)$. Then the PDEs in* (24.9) *are satisfied a.e. in D, and the boundary condition a.e. in ∂D.*

Proof. Left as an exercise. □

24.2 A first-order PDE

For simplicity, we consider a one-dimensional model problem (a more general setting is covered in Chapter 56). Let $D := (0, 1)$ and let $f : D \to \mathbb{R}$ be a smooth function. The problem we want to solve consists of seeking a function $u : D \to \mathbb{R}$ such that

$$u' = f \quad \text{in } D, \qquad u(0) = 0. \tag{24.15}$$

Proceeding informally, the solution to this problem is the function defined as follows:

$$u(x) := \int_0^x f(t)\,\mathrm{d}t, \qquad \forall x \in D. \tag{24.16}$$

To give a precise mathematical meaning to this statement, we assume that $f \in L^1(D)$, and we introduce the Sobolev space (see Definition 2.8)

$$W^{1,1}(D) := \{v \in L^1(D) \mid v' \in L^1(D)\}, \tag{24.17}$$

where as usual we interpret the derivatives in the weak sense.

Lemma 24.4 (Solution in $W^{1,1}(D)$). *If $f \in L^1(D)$, the problem* (24.15) *has a unique solution in $W^{1,1}(D)$ which is given by* (24.16)*.*

Proof. Let u be defined in (24.16).
(1) Let us first show that $u \in C^0(\overline{D})$ (recall that $\overline{D} = [0,1]$). Let $x \in \overline{D}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to x in $\overline{D}$. This gives

$$u(x) - u(x_n) = \int_0^x f(t)\,\mathrm{d}t - \int_0^{x_n} f(t)\,\mathrm{d}t = \int_{x_n}^x f(t)\,\mathrm{d}t = \int_D \mathbb{1}_{[x_n,x]}(t) f(t)\,\mathrm{d}t,$$

where $\mathbb{1}_{[x_n,x]}$ is the indicator function of the interval $[x_n, x]$. Since $\mathbb{1}_{[x_n,x]} f \to 0$ and $|\mathbb{1}_{[x_n,x]} f| \le |f|$ a.e. in D, Lebesgue's dominated convergence theorem (Theorem 1.23) implies that $u(x_n) \to u(x)$. This shows that $u \in C^0(\overline{D})$. Hence, the boundary condition $u(0) = 0$ is meaningful.
(2) Let us now prove that $u' = f$ a.e. in D. One can verify (see Exercise 24.7) that

$$\int_0^1 \left(\int_0^x f(t)\,\mathrm{d}t \right) \varphi'(x)\,\mathrm{d}x = -\int_0^1 f(x)\varphi(x)\,\mathrm{d}x, \quad \forall \varphi \in C_0^\infty(D). \tag{24.18}$$

Since the left-hand side is equal to $\int_0^1 u(x)\varphi'(x)\,\mathrm{d}x$ and $f \in L^1(D) \subset L^1_{\mathrm{loc}}(D)$, we infer that u has a weak derivative in $L^1_{\mathrm{loc}}(D)$ equal to f. This implies that the PDE in (24.15) is satisfied a.e. in D.
(3) Uniqueness of the solution is a consequence of Lemma 2.11 since the difference of two weak solutions is constant on D (since it has zero weak derivative) and vanishes at $x = 0$. □

We now present two possible mathematical settings for the weak formulation of the problem (24.15).

24.2.1 Formulation in $L^1(D)$

Since $f \in L^1(D)$ and $u \in W^{1,1}(D)$ with $u(0) = 0$, a first weak formulation is obtained by just multiplying the PDE in (24.15) by a test function w and integrating over D:

$$\int_D u' w \,\mathrm{d}t = \int_D f w \,\mathrm{d}t. \tag{24.19}$$

This equality is meaningful for all $w \in W^{(\infty)} := L^\infty(D)$. Moreover, the boundary condition $u(0) = 0$ can be explicitly enforced by considering the solution space $V^{(1)} := \{v \in W^{1,1}(D) \mid v(0) = 0\}$. Thus, a first weak formulation of (24.15) is as follows:

$$\begin{cases} \text{Find } u \in V^{(1)} \text{ such that} \\ \int_D u' w \,\mathrm{d}t = \int_D f w \,\mathrm{d}t, \quad \forall w \in W^{(\infty)}. \end{cases} \tag{24.20}$$

Remark 24.5 (Literature). Solving first-order PDEs using L^1-based formulations has been introduced by Lavery [276, 277]; see also Guermond [227], Guermond and Popov [228], and the references therein. □

24.2.2 Formulation in $L^2(D)$

Although the weak formulation (24.20) gives a well-posed problem (as we shall see in §25.4.2), the dominant viewpoint in the literature consists of using L^2-based formulations. This leads us to consider a second weak formulation where the source term f has slightly more smoothness, i.e., $f \in L^2(D)$ instead of just $f \in L^1(D)$, thereby allowing us to work in a Hilbertian setting. Since $L^2(D) \subset L^1(D)$, we have $f \in L^1(D)$, and we can still consider the function u defined in (24.16). This function turns out to be in $H^1(D)$ if $f \in L^2(D)$. Indeed, the Cauchy–Schwarz inequality and Fubini's theorem imply that

$$\begin{aligned}\int_0^1 |u(x)|^2 \,\mathrm{d}x &= \int_0^1 \left| \int_0^x f(t) \,\mathrm{d}t \right|^2 \mathrm{d}x \le \int_0^1 \left(\int_0^x |f(t)|^2 \,\mathrm{d}t \right) x \,\mathrm{d}x \\ &= \int_0^1 \left(\int_t^1 \mathrm{d}x \right) |f(t)|^2 dt = \int_0^1 (1-t)|f(t)|^2 dt \le \int_0^1 |f(t)|^2 dt,\end{aligned}$$

which shows that $\|u\|_{L^2(D)} \le \|f\|_{L^2(D)}$. Moreover, $\|u'\|_{L^2(D)} = \|f\|_{L^2(D)}$. Hence, $u \in H^1(D)$. We can then restrict the test functions to the Hilbert space $W^{(2)} := L^2(D)$ and use the Hilbert space $V^{(2)} := \{v \in H^1(D) \mid v(0) = 0\}$ as the solution space. Thus, a second weak formulation of (24.20), provided $f \in L^2(D)$, is as follows:

$$\begin{cases} \text{Find } u \in V^{(2)} \text{ such that} \\ \int_D u' w \,\mathrm{d}t = \int_D f w \,\mathrm{d}t, \quad \forall w \in W^{(2)}. \end{cases} \tag{24.21}$$

The main change with respect to (24.20) is in the trial and test spaces.

24.3 A complex-valued model problem

Some model problems are formulated using complex-valued functions. A salient example is Maxwell's equations in the time-harmonic regime; see §43.1. For simplicity, let us consider here the PDE

$$\mathrm{i}u - \nu \Delta u = f \quad \text{in } D, \tag{24.22}$$

with $u : D \to \mathbb{C}$, $f : D \to \mathbb{C}$, $\mathrm{i}^2 = -1$, and a real number $\nu > 0$. To fix the ideas, we enforce a homogeneous Dirichlet condition on u at the boundary.

When working with complex-valued functions, one uses the complex conjugate of the test function in the weak problem, i.e., the starting point of the weak formulation is the identity

$$\int_D \mathrm{i}u\overline{w}\,\mathrm{d}x + \nu \int_D \nabla u{\cdot}\nabla\overline{w}\,\mathrm{d}x = \int_D f\overline{w}\,\mathrm{d}x. \tag{24.23}$$

One can then proceed as in §24.1.1 (for instance). The functional setting uses the functional space $V := H^1_0(D;\mathbb{C})$, and the weak formulation is as follows:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \int_D \mathrm{i}u\overline{w}\,\mathrm{d}x + \nu \int_D \nabla u{\cdot}\nabla\overline{w}\,\mathrm{d}x = \int_D f\overline{w}\,\mathrm{d}x, \quad \forall w \in V. \end{cases} \tag{24.24}$$

Proposition 24.1 is readily adapted to this setting.

The reason for using the complex conjugate of test functions is that it allows us to infer positivity properties on the real and imaginary parts of the quantity $a(u,w) := \int_D \mathrm{i}u\overline{w}\,\mathrm{d}x + \nu \int_D \nabla u{\cdot}\nabla\overline{w}\,\mathrm{d}x$ by taking $w := u$ as the test function. Indeed, we obtain

$$a(u,u) = \mathrm{i}\int_D |u|^2\,\mathrm{d}x + \nu\int_D \|\nabla u\|^2_{\ell^2(\mathbb{C}^d)}\,\mathrm{d}x = \mathrm{i}\|u\|^2_{L^2(D;\mathbb{C})} + \nu\|\nabla u\|^2_{L^2(D;\mathbb{C}^d)}.$$

This means that $\Re(a(u,u)) = \nu\|\nabla u\|^2_{L^2(D;\mathbb{C}^d)}$ and $\Im(a(u,u)) = \|u\|^2_{L^2(D;\mathbb{C})}$. These results imply that

$$\Re(e^{-\mathrm{i}\frac{\pi}{4}}a(u,u)) \geq \frac{1}{\sqrt{2}}\min(1,\nu\ell_D^{-2})\|u\|^2_{H^1(D;\mathbb{C})}, \tag{24.25}$$

where we recall that the Hilbert space $L^2(D;\mathbb{C})$ is equipped with the inner product $(v,w)_{L^2(D)} := \int_D v\overline{w}\,\mathrm{d}x$ and the Hilbert space $H^1(D;\mathbb{C})$ is equipped with the inner product $(v,w)_{H^1(D)} := \int_D v\overline{w}\,\mathrm{d}x + \ell_D^2\int_D \nabla v{\cdot}\nabla\overline{w}\,\mathrm{d}x$, where ℓ_D is a characteristic length associated with D, e.g., $\ell_D := \mathrm{diam}(D)$.

24.4 Toward an abstract model problem

We conclude this chapter by casting all of the above weak formulations into a unified setting. We consider complex-valued functions since it is in general simpler to go from complex to real numbers than the other way around. Whenever relevant, we indicate the (minor) changes to apply in this situation (apart from replacing $\mathbb{C}$ by $\mathbb{R}$).

The above weak formulations fit into the following abstract model problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,w) = \ell(w), \quad \forall w \in W, \end{cases} \tag{24.26}$$

with maps $a : V \times W \to \mathbb{C}$ and $\ell : W \to \mathbb{C}$, where V, W are complex vector spaces whose elements are functions defined on D. V is called *trial space* or *solution space*, and W is called *test space*. Members of V are called *trial functions* and members of W are called *test functions*. The maps a and ℓ are called *forms* since their codomain is $\mathbb{C}$ (or $\mathbb{R}$ in the real case).

Recall that a map $A : V \to \mathbb{C}$ is said to be *linear* if $A(v_1 + v_2) = A(v_1) + A(v_2)$ for all $v_1, v_2 \in V$ and $A(\lambda v) = \lambda A(v)$ for all $\lambda \in \mathbb{C}$ and all $v \in V$, whereas a map $B : W \to \mathbb{C}$ is said to be *antilinear* if $B(w_1 + w_2) = B(w_1) + B(w_2)$ for all $w_1, w_2 \in W$ and $B(\lambda w) = \overline{\lambda} B(w)$ for all $\lambda \in \mathbb{C}$ and all $w \in W$. Then ℓ in (24.26) is an *antilinear form*, whereas a is a *sesquilinear form* (that is, the map $a(\cdot, w)$ is linear for all $w \in W$, and the map $a(v, \cdot)$ is antilinear for all $v \in V$). In the real case, ℓ is a *linear form* and a is a *bilinear form* (that is, it is linear in each of its arguments).

Remark 24.6 (Linearity). The linearity of a w.r.t. to its first argument is a consequence of the linearity of the problem, whereas the (anti)linearity of a w.r.t. its second argument results from the weak formulation. □

Remark 24.7 (Bilinearity). Bilinear forms and linear forms on $V \times W$ are different objects. For instance, the action of a linear form on $(v, 0) \in V \times W$ is not necessarily zero, whereas $a(v, 0) = 0$ if a is a bilinear form. □

Remark 24.8 (Test functions). The role of the test functions in the weak formulations (24.20) and (24.26) are somewhat different. Since $L^\infty(D)$ is the dual space of $L^1(D)$ (the reverse is not true), the test functions $w \in L^\infty(D)$ in (24.20) act on the function $f \in L^1(D)$. Hence, in principle it should be more appropriate to write $w(\ell)$ instead of $\ell(w)$ in (24.26). Although this alternative viewpoint is not often considered in the literature, it actually allows for a more general setting regarding well-posedness. We return to this point in §25.3.2. This distinction is not relevant for model problems set in a Hilbertian framework. □

Exercises

Exercise 24.1 (Forms). Let $D := (0,1)$. Which of these maps are linear or bilinear forms on $L^2(D)\times L^2(D)$: $a_1(f,g) := \int_D (f+g+1)\,\mathrm{d}x$, $a_2(f,g) := \int_D x(f-g)\,\mathrm{d}x$, $a_3(f,g) := \int_D (1+x^2)fg\,\mathrm{d}x$, $a_4(f,g) := \int_D (f+g)^2\,\mathrm{d}x$?

Exercise 24.2 ((Non)-uniqueness). Consider the domain D in $\mathbb{R}^2$ whose definition in polar coordinates is $D := \{(r,\theta) \mid r \in (0,1),\, \theta \in (\frac{\pi}{\alpha}, 0)\}$ with $\alpha \in (-1, -\frac{1}{2})$. Let $\partial D_1 := \{(r,\theta) \mid r = 1,\, \theta \in (\frac{\pi}{\alpha}, 0)\}$ and $\partial D_2 := \partial D \backslash \partial D_1$. Consider the PDE $-\Delta u = 0$ in D with the Dirichlet conditions $u = \sin(\alpha\theta)$ on ∂D_1 and $u = 0$ on ∂D_2. (i) Let $\varphi_1 := r^\alpha \sin(\alpha\theta)$ and $\varphi_2 := r^{-\alpha}\sin(\alpha\theta)$. Prove that φ_1 and φ_2 solve the above problem. (*Hint*: in polar coordinates $\Delta\varphi = \frac{1}{r}\partial_r(r\partial_r\varphi) + \frac{1}{r^2}\partial_{\theta\theta}\varphi$.) (ii) Prove that φ_1 and φ_2 are in $L^2(D)$ if $\alpha \in (-1, -\frac{1}{2})$. (iii) Consider the problem of seeking $u \in H^1(D)$ s.t. $u = \sin(\alpha\theta)$ on ∂D_1, $u = 0$ on ∂D_2, and $\int_D \nabla u \cdot \nabla v = 0$ for all $v \in H_0^1(D)$. Prove that φ_2 solves this problem, but φ_1 does not. Comment.

Exercise 24.3 (Poisson in 1D). Let $D := (0,1)$ and $f(x) := \frac{1}{x(1-x)}$. Consider the PDE $-\partial_x((1+\sin(x)^2)\partial_x u) = f$ in D with the Dirichlet conditions $u(0) = u(1) = 0$. Write a weak formulation of this problem with both trial and test spaces equal to $H_0^1(D)$ and show that the linear form on the right-hand side is bounded on $H_0^1(D)$. (*Hint*: notice that $f(x) = \frac{1}{x} + \frac{1}{1-x}$.)

Exercise 24.4 (Weak formulations). Prove Propositions 24.2 and 24.3.

Exercise 24.5 (Darcy). (i) Derive another variation on (24.12) and (24.14) with the functional spaces $V = W := \boldsymbol{H}(\mathrm{div}; D)\times L^2(D)$. (*Hint*: use Theorem 4.15.) (ii) Derive yet another variation with the functional spaces $V := \boldsymbol{L}^2(D)\times L^2(D)$ and $W := \boldsymbol{H}(\mathrm{div}; D)\times H_0^1(D)$.

Exercise 24.6 (Variational formulation). Prove that u solves (24.7) if and only if u minimizes over $H_0^1(D)$ the energy functional

$$\mathfrak{E}(v) := \frac{1}{2}\int_D |\nabla v|^2\,\mathrm{d}x - \int_D fv\,\mathrm{d}x.$$

(*Hint*: show first that $\mathfrak{E}(v+tw) = \mathfrak{E}(v) + t\left\{\int_D \nabla v\cdot\nabla w\,\mathrm{d}x - \int_D fw\,\mathrm{d}x\right\} + \frac{1}{2}t^2\int_D |\nabla w|^2\,\mathrm{d}x$ for all $v, w \in H_0^1(D)$ and all $t \in \mathbb{R}$.)

Exercise 24.7 (Derivative of primitive). Prove (24.18). (*Hint*: use Theorem 1.38 and Lebesgue's dominated convergence theorem.)

Exercise 24.8 (Biharmonic problem). Let D be an open, bounded, set in $\mathbb{R}^d$ with smooth boundary. Derive a weak formulation for the biharmonic problem

$$\Delta(\Delta u) = f \text{ in } D, \qquad u = \partial_n u = 0 \text{ on } \partial D,$$

with $f \in L^2(D)$. (*Hint*: use Theorem 3.16.)