Part VIII, Chapter 37

Nitsche's boundary penalty method

The main objective of this chapter is to present a technique to treat Dirichlet boundary conditions in a natural way using a penalty method. This technique is powerful and has many extensions. In particular, the idea is reused in the next chapter for discontinuous Galerkin methods. Another objective of this chapter is to illustrate the abstract error analysis of Chapter 27.

37.1 Main ideas and discrete problem

Let D be a Lipschitz domain in $\mathbb{R}^d$. We assume for simplicity that D is a polyhedron. Let $f \in L^2(D)$ be the source term, and let $g \in H^{\frac{1}{2}}(\partial D)$ be the Dirichlet boundary data. We consider the Poisson equation with Dirichlet conditions

$$-\Delta u = f \quad \text{in } D, \qquad \gamma^{\mathrm{g}}(u) = g \quad \text{on } \partial D, \tag{37.1}$$

where $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ is the trace map. Let $u_g \in H^1(D)$ be a lifting of g, i.e., $\gamma^{\mathrm{g}}(u_g) = g$ (recall that $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ is the trace map). We seek $u_0 \in H^1_0(D)$ s.t. $a(u_0, w) = \ell(w) - a(u_g, w)$ for all $w \in H^1_0(D)$, with

$$a(v, w) := \int_D \nabla v{\cdot}\nabla w \,\mathrm{d}x, \qquad \ell(w) := \int_D f w \,\mathrm{d}x. \tag{37.2}$$

This problem is well-posed in $H^1_0(D)$ owing to the Lax–Milgram lemma and the Poincaré–Steklov inequality in $H^1_0(D)$. Then the unique weak solution to (37.1) is $u := u_0 + u_g$ (see §31.2.2).

In this chapter, we take a route that is different from the above approach to construct an approximation of the solution. Instead of enforcing the Dirichlet boundary condition strongly, we are going to construct an H^1-conforming discretization of (37.1) that enforces this condition naturally. This means that we no longer require that the discrete test functions vanish at the boundary. The discrete counterpart of the bilinear form a must then be modified accordingly. To motivate the modification in question, let us proceed informally by assuming that all the functions we manipulate are sufficiently smooth. Testing (37.1) with a function w which we do not require to vanish at the boundary, the integration by parts formula (4.8b) gives

$$a(u,w) - \int_{\partial D} (\boldsymbol{n}\cdot\nabla u) w \,\mathrm{d}s = \ell(w). \tag{37.3}$$

The idea of Nitsche is to modify (37.3) by adding a term proportional to $\int_{\partial D} uw \,\mathrm{d}s$ on both sides of the above identity. This leads to

$$a(u,w) - \int_{\partial D} (\boldsymbol{n}\cdot\nabla u) w \,\mathrm{d}s + \varpi \int_{\partial D} uw \,\mathrm{d}s = \ell(w) + \varpi \int_{\partial D} gw \,\mathrm{d}s, \tag{37.4}$$

where the boundary value of u has been replaced by g in the boundary integral on the right-hand side. The yet unspecified parameter ϖ is assumed to be positive. Heuristically, if u satisfies (37.4) and if ϖ is large, one expects u to be close to g at the boundary. For this reason, ϖ is called *penalty parameter*.

The above ideas lead to an approximation method employing discrete trial and test spaces composed of functions that are not required to vanish at the boundary. Let $\mathcal{T}_h$ be a mesh from a shape-regular sequence of meshes so that each mesh covers D exactly. Let $\mathcal{F}_h^{\partial}$ be the collection of the boundary faces. Let $P_k^{\mathrm{g}}(\mathcal{T}_h)$ be the H^1-conforming finite element space of degree $k \geq 1$ based on $\mathcal{T}_h$; see (19.10). We consider the following discrete problem:

$$\begin{cases} \text{Find } u_h \in V_h := P_k^{\mathrm{g}}(\mathcal{T}_h) \text{ such that} \\ a_h(u_h, w_h) = \ell_h(w_h), \quad \forall w_h \in V_h, \end{cases} \tag{37.5}$$

where the discrete forms a_h and ℓ_h are inspired from (37.4):

$$a_h(v_h, w_h) := a(v_h, w_h) - \int_{\partial D} (\boldsymbol{n}\cdot\nabla v_h) w_h \,\mathrm{d}s + \sum_{F\in\mathcal{F}_h^{\partial}} \varpi(h_F) \int_F v_h w_h \,\mathrm{d}s,$$

$$\ell_h(w_h) := \ell(w_h) + \sum_{F\in\mathcal{F}_h^{\partial}} \varpi(h_F) \int_F g w_h \,\mathrm{d}s.$$

The second term in the definition of a_h is called *consistency term* (this term plays a key role when estimating the consistency error) and the third one is called *penalty term*. The penalty parameter $\varpi(h_F) > 0$, yet to be defined, depends on the diameter of the face F (or a uniformly equivalent local length scale). The stability analysis will reveal that $\varpi(h_F)$ should scale like h_F^{-1}. The approximation setting associated with Nitsche's boundary penalty method is nonconforming, i.e., $V_h \not\subset V := H_0^1(D)$, since functions in V_h may not vanish at the boundary, whereas functions in V do.

Remark 37.1 (Literature, extensions). The boundary penalty method has been introduced by Nitsche [314] to treat Dirichlet boundary conditions. It was extended in Juntunen and Stenberg [262] to Robin boundary conditions. We refer the reader to §41.3 where the more general PDE $-\nabla{\cdot}(\lambda\nabla u) = f$ with contrasted diffusivity λ is treated. □

37.2 Stability and well-posedness

The main objective of this section is to prove that the discrete bilinear form a_h is coercive on V_h if the penalty parameter is large enough. This is done by showing that the consistency term can be appropriately bounded. For all $F \in \mathcal{F}_h^\partial$, let us denote by K_l the unique mesh cell having F as a face, i.e., $F := \partial K_l \cap \partial D$. Let $\mathcal{T}_h^{\partial D}$ be the collection of the mesh cells having at least one boundary face, i.e., $\mathcal{T}_h^{\partial D} := \bigcup_{F\in\mathcal{F}_h^\partial}\{K_l\}$. (The set $\mathcal{T}_h^{\partial D}$ should not be confused with the larger set $\mathcal{T}_h^\partial$ defined in (22.28), which is the collection of the mesh cells touching the boundary.) Let n_∂ denote the maximum number of boundary faces that a mesh cell in $\mathcal{T}_h^{\partial D}$ can have, i.e., $n_\partial := \max_{K\in\mathcal{T}_h^{\partial D}} \operatorname{card}(\mathcal{F}_K \cap \mathcal{F}_h^\partial)$ ($n_\partial \le d$ for simplicial meshes). Owing to the regularity of the mesh sequence, the discrete trace inequality from Lemma 12.8 (with $p = q := 2$) implies that there is c_{dt} such that for all $v_h \in V_h$, all $F \in \mathcal{F}_h^\partial$, and all $h \in \mathcal{H}$,

$$\|\boldsymbol{n}{\cdot}\nabla v_h\|_{L^2(F)} \le c_{\mathrm{dt}} h_F^{-\frac{1}{2}} \|\nabla v_h\|_{\boldsymbol{L}^2(K_l)}. \tag{37.6}$$

Lemma 37.2 (Bound on consistency term). *The following holds true for all $v_h \in V_h$:*

$$\left|\int_{\partial D} (\boldsymbol{n}{\cdot}\nabla v_h) v_h \,\mathrm{d}s\right| \le n_\partial^{\frac{1}{2}} c_{\mathrm{dt}} \bigg(\sum_{K\in\mathcal{T}_h^{\partial D}} \|\nabla v_h\|^2_{\boldsymbol{L}^2(K)}\bigg)^{\frac{1}{2}} \bigg(\sum_{F\in\mathcal{F}_h^\partial} \frac{1}{h_F}\|v_h\|^2_{L^2(F)}\bigg)^{\frac{1}{2}}.$$

Proof. Let $v_h \in V_h$. Let $F \in \mathcal{F}_h^\partial$. Using the Cauchy–Schwarz inequality, bounding the normal component of the gradient by its Euclidean norm, and using the discrete trace inequality (37.6) componentwise, we infer that

$$\begin{aligned}\left|\int_{\partial D} (\boldsymbol{n}{\cdot}\nabla v_h) v_h \,\mathrm{d}s\right| &\le \bigg(\sum_{F\in\mathcal{F}_h^\partial} h_F \|\boldsymbol{n}{\cdot}\nabla v_h\|^2_{\boldsymbol{L}^2(F)}\bigg)^{\frac{1}{2}} \bigg(\sum_{F\in\mathcal{F}_h^\partial} \frac{1}{h_F}\|v_h\|^2_{L^2(F)}\bigg)^{\frac{1}{2}} \\ &\le c_{\mathrm{dt}} \bigg(\sum_{F\in\mathcal{F}_h^\partial} \|\nabla v_h\|^2_{\boldsymbol{L}^2(K_l)}\bigg)^{\frac{1}{2}} \bigg(\sum_{F\in\mathcal{F}_h^\partial} \frac{1}{h_F}\|v_h\|^2_{L^2(F)}\bigg)^{\frac{1}{2}}.\end{aligned}$$

Finally, we have $\sum_{F\in\mathcal{F}_h^\partial} \|{\cdot}\|^2_{\boldsymbol{L}^2(K_l)} = \sum_{K\in\mathcal{T}_h^{\partial D}} \operatorname{card}(\mathcal{F}_K \cap \mathcal{F}_h^\partial) \|{\cdot}\|^2_{\boldsymbol{L}^2(K)} \le n_\partial \sum_{K\in\mathcal{T}_h^{\partial D}} \|{\cdot}\|^2_{\boldsymbol{L}^2(K)}$. □

We equip the space V_h with the following norm:

$$\|v_h\|_{V_h}^2 := \|\nabla v_h\|_{\boldsymbol{L}^2(D)}^2 + |v_h|_\partial^2, \qquad |v_h|_\partial^2 := \sum_{F\in\mathcal{F}_h^\partial} \frac{1}{h_F}\|v_h\|_{L^2(F)}^2. \tag{37.7}$$

Note that $\|v_h\|_{V_h} = 0$ implies that v_h is constant on D and vanishes on ∂D, so that $v_h = 0$. Hence, $\|\cdot\|_{V_h}$ is a norm on V_h. Note also that the two terms composing the norm $\|\cdot\|_{V_h}$ are dimensionally consistent.

Lemma 37.3 (Coercivity, well-posedness). *Assume that the penalty parameter $\varpi(h_F)$ is defined s.t.*

$$\varpi(h_F) := \varpi_0 \frac{1}{h_F}, \qquad \forall F \in \mathcal{F}_h^\partial, \tag{37.8}$$

with $\varpi_0 > \frac{1}{4} n_\partial c_{\mathrm{dt}}^2$. (i) *The following coercivity property holds true:*

$$a_h(v_h, v_h) \geq \alpha \|v_h\|_{V_h}^2, \qquad \forall v_h \in V_h, \tag{37.9}$$

with $\alpha := \frac{\varpi_0 - \frac{1}{4} n_\partial c_{\mathrm{dt}}^2}{1+\varpi_0} > 0$, (ii) *The discrete problem* (37.5) *is well-posed.*

Proof. Let $v_h \in V_h$. We have

$$a_h(v_h, v_h) = \|\nabla v_h\|_{\boldsymbol{L}^2(D)}^2 - \int_{\partial D} (\boldsymbol{n}\cdot\nabla v_h) v_h \,\mathrm{d}s + \varpi_0 |v_h|_\partial^2.$$

Setting $z := (\sum_{K\in\mathcal{T}_h\setminus\mathcal{T}_h^{\partial D}} \|\nabla v_h\|_{\boldsymbol{L}^2(K)}^2)^{\frac{1}{2}}$, $x := (\sum_{K\in\mathcal{T}_h^{\partial D}} \|\nabla v_h\|_{\boldsymbol{L}^2(K)}^2)^{\frac{1}{2}}$, and $y := |v_h|_\partial$, and using Lemma 37.2, we infer that

$$a_h(v_h, v_h) \geq z^2 + (x^2 - n_\partial^{\frac{1}{2}} c_{\mathrm{dt}} xy + \varpi_0 y^2).$$

Coercivity follows from the inequality $x^2 - 2\beta xy + \varpi_0 y^2 \geq \frac{\varpi_0 - \beta^2}{1+\varpi_0}(x^2 + y^2)$ applied with $\beta := \frac{1}{2} n_\partial^{\frac{1}{2}} c_{\mathrm{dt}}$ (see Exercise 37.2) and since $\frac{\varpi_0-\beta^2}{1+\varpi_0} \leq \frac{\varpi_0}{1+\varpi_0} \leq 1$. Finally, well-posedness follows from the Lax–Milgram lemma. □

Remark 37.4 (Choice of penalty parameter). Ensuring the stability condition $\varpi_0 > \frac{1}{4} n_\partial c_{\mathrm{dt}}^2$ requires in practice to know a reasonable upper bound on the constant c_{dt}. The results of §12.2 show that c_{dt} scales like the polynomial degree k. More precisely, Lemma 12.10 shows that for simplices one can take $c_{\mathrm{dt}} := ((k+1)(k+d)/d)^{\frac{1}{2}}$ with $h_F := |K_l|/|F|$. □

37.3 Error analysis

In this section, we derive an energy error estimate, that is, we bound the error by using the coercivity norm and the abstract error estimate from Lemma 27.5. We also derive an improved L^2-error estimate by means of a duality argument.

37.3.1 Energy error estimate

We perform the error analysis under the assumption that the solution to (37.1) is in $H^{1+r}(D)$ with $r > \frac{1}{2}$, i.e., we set

$$V_{\text{S}} := H^{1+r}(D), \quad r > \frac{1}{2}. \tag{37.10}$$

The assumption $u \in V_{\text{S}}$ is reasonable in the setting of the Poisson equation with Dirichlet conditions in a Lipschitz polyhedron since it is consistent with the elliptic regularity theory (see Theorem 31.33). The important property that we use is that for any function $v \in V_{\text{S}}$, the normal derivative $\boldsymbol{n}\cdot\nabla v$ at the boundary is meaningful in $L^2(\partial D)$. We consider the space $V_\sharp := V_{\text{S}} + V_h$ equipped with the norm

$$\|v\|_{V_\sharp}^2 := \|\nabla v\|_{\boldsymbol{L}^2(D)}^2 + |v|_\partial^2 + \sum_{F\in\mathcal{F}_h^\partial} h_F \|\boldsymbol{n}\cdot\nabla v\|_{L^2(F)}^2, \tag{37.11}$$

with $|v|_\partial^2 := \sum_{F\in\mathcal{F}_h^\partial} \frac{1}{h_F}\|v\|_{L^2(F)}^2$. A discrete trace inequality shows that there is $c_\sharp$ s.t. $\|v_h\|_{V_\sharp} \le c_\sharp \|v_h\|_{V_h}$ for all $v_h \in V_h$ and all $h \in \mathcal{H}$, i.e., (27.5) holds true. Recall from Definition 27.3 that the consistency error is defined by setting $\langle \delta_h(v_h), w_h\rangle_{V_h',V_h} := \ell_h(w_h) - a_h(v_h, w_h)$ for all $v_h, w_h \in V_h$.

Lemma 37.5 (Consistency/boundedness). *Assume* (37.10)*. There is $\omega_\sharp$, uniform w.r.t. $u \in V_{\text{S}}$, s.t. for all $v_h \in V_h$ and all $h \in \mathcal{H}$,*

$$\|\delta_h(v_h)\|_{V_h'} \le \omega_\sharp \, \|u - v_h\|_{V_\sharp}. \tag{37.12}$$

Proof. Let $v_h, w_h \in V_h$. Since the normal derivative $\boldsymbol{n}\cdot\nabla u$ is meaningful at the boundary, using the PDE and the boundary condition in (37.1), we infer that

$$\begin{aligned}
\ell_h(w_h) &= \int_D -(\Delta u) w_h \,\mathrm{d}x + \sum_{F\in\mathcal{F}_h^\partial} \varpi_0 \frac{1}{h_F}\int_F g w_h \,\mathrm{d}s \\
&= \int_D \nabla u\cdot\nabla w_h \,\mathrm{d}x - \int_{\partial D} (\boldsymbol{n}\cdot\nabla u) w_h \,\mathrm{d}s + \sum_{F\in\mathcal{F}_h^\partial} \varpi_0 \frac{1}{h_F}\int_F u w_h \,\mathrm{d}s.
\end{aligned}$$

Letting $\eta := u - v_h$, this implies that

$$\begin{aligned}
\langle \delta_h(v_h), w_h\rangle_{V_h',V_h} = & \int_D \nabla\eta\cdot\nabla w_h \,\mathrm{d}x - \int_{\partial D} (\boldsymbol{n}\cdot\nabla\eta) w_h \,\mathrm{d}s \\
& + \sum_{F\in\mathcal{F}_h^\partial} \varpi_0 \frac{1}{h_F}\int_F \eta w_h \,\mathrm{d}s.
\end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain the estimate (37.12) with $\omega_\sharp := \max(1, \varpi_0)$. □

Theorem 37.6 (Convergence). *Let u solve* (37.1) *and let u_h solve* (37.5) *with the penalty parameter $\varpi_0 > \frac{1}{4}n_\partial c_{\mathrm{dt}}^2$. Assume* (37.10). (i) *There is c s.t. the following quasi-optimal error estimate holds true for all $h \in \mathcal{H}$:*

$$\|u - u_h\|_{V_\sharp} \le c \inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp}. \tag{37.13}$$

(ii) *Letting $t := \min(k, r)$, we have*

$$\|u - u_h\|_{V_\sharp} \le c \left(\sum_{K \in \mathcal{T}_h} h_K^{2t} |u|_{H^{1+t}(K)}^2 \right)^{\frac{1}{2}}. \tag{37.14}$$

Proof. (i) The estimate (37.13) follows from Lemma 27.5 combined with stability (Lemma 37.3) and consistency/boundedness (Lemma 37.5).
(ii) The proof of (37.14) is left as an exercise. □

37.3.2 L^2-norm estimate

We derive an improved error estimate of the form $\|u - u_h\|_{L^2(D)} \le ch^\gamma \ell_D^{1-\gamma} \|u - u_h\|_{V_\sharp}$ for some real number $\gamma > 0$, where ℓ_D is a length scale associated with D, e.g., $\ell_D := \mathrm{diam}(D)$. Proceeding as in §36.3.2, we invoke a *duality argument*. We consider the adjoint solution $\zeta_r \in V := H_0^1(D)$ for all $r \in L^2(D)$ such that

$$a(v, \zeta_r) = (v, r)_{L^2(D)}, \qquad \forall v \in V, \tag{37.15}$$

i.e., ζ_r solves $-\Delta \zeta_r = r$ in D and $\gamma^{\mathrm{g}}(\zeta_r) = 0$. (Note that we enforce a homogeneous Dirichlet condition on the adjoint solution.) Owing to the elliptic regularity theory (see §31.4), there is $s \in (0, 1]$ and a constant c_{smo} such that

$$\|\zeta_r\|_{H^{1+s}(D)} \le c_{\mathrm{smo}}\, \ell_D^2 \|r\|_{L^2(D)}, \qquad \forall r \in L^2(D). \tag{37.16}$$

In the present setting of the Poisson equation with Dirichlet conditions in a Lipschitz polyhedron, it is reasonable to assume that $s \in (\frac{1}{2}, 1]$.

Theorem 37.7 (L^2-estimate). *Let u solve* (37.1) *and let u_h solve* (37.5). *Assume that the elliptic regularity index satisfies $s \in (\frac{1}{2}, 1]$. There is c s.t. for all $h \in \mathcal{H}$,*

$$\|u - u_h\|_{L^2(D)} \le c\, h^{\frac{1}{2}} \ell_D^{\frac{1}{2}} \|u - u_h\|_{V_\sharp}. \tag{37.17}$$

Proof. Set $e := u - u_h$. We apply the abstract error estimate of Lemma 36.14 with $V_\sharp := V_{\mathrm{s}} + V_h$ as above, $Z_{\mathrm{s}} := H^{1+s}(D) \cap H_0^1(D)$, $Y_h := V_h \cap H_0^1(D)$, and $Z_\sharp := Z_{\mathrm{s}} + Y_h$ equipped with the H^1-seminorm. We consider the bilinear form $a_\sharp(v, w) := (\nabla v, \nabla w)_{\boldsymbol{L}^2(D)}$. Notice that $a_\sharp$ is bounded on $V_\sharp \times Z_\sharp$. Moreover, $a_\sharp(e, y_h) = 0$ for all $y_h \in Y_h$ since $Y_h \subset H_0^1(D)$, i.e., the Galerkin orthogonality property (36.28) holds true. Lemma 36.14 implies that

$$\|e\|_{L^2(D)} \le \left(\frac{\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'}}{\|e\|_{L^2(D)}} + \inf_{y_h \in Y_h} \frac{\|\nabla(\zeta_e - y_h)\|_{\boldsymbol{L}^2(D)}}{\|e\|_{L^2(D)}} \right) \|e\|_{V_\sharp},$$

where the first and the second term between parentheses are the *adjoint consistency error* and the interpolation error on the adjoint solution, respectively. Let us first bound the adjoint consistency error. Recall that $\delta^{\mathrm{adj}}(\zeta_e)$ is defined in such a way that the following identity holds true: For all $v \in V_\sharp$,

$$\langle \delta^{\mathrm{adj}}(\zeta_e), v\rangle_{V_\sharp', V_\sharp} = -(v, \Delta\zeta_e)_{L^2(D)} - a_\sharp(v, \zeta_e) = -(v, \boldsymbol{n}\cdot\nabla\zeta_e)_{L^2(\partial D)}.$$

The Cauchy–Schwarz inequality implies that

$$\begin{aligned} |\langle \delta^{\mathrm{adj}}(\zeta_e), v\rangle_{V_\sharp', V_\sharp}| &\le h^{\frac12}\|\nabla\zeta_e\|_{L^2(\partial D)}|v|_\partial \le h^{\frac12}\|\nabla\zeta_e\|_{L^2(\partial D)}\|v\|_{V_\sharp} \\ &\le c\, h^{\frac12}\ell_D^{-\frac32}\|\zeta_e\|_{H^{1+s}(D)}\|v\|_{V_\sharp}, \end{aligned}$$

since $s > \frac12$. Using (37.16), we infer that $\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'} \le ch^{\frac12}\ell_D^{\frac12}\|e\|_{L^2(D)}$. To bound the interpolation error on the adjoint solution, we consider the quasi-interpolation operator $\mathcal{I}_{h0}^{\mathrm{g,av}}$ from §22.4. Since $\mathcal{I}_{h0}^{\mathrm{g,av}}(\zeta_e) \in Y_h$, we deduce that

$$\begin{aligned} &\inf_{y_h\in Y_h}\|\nabla(\zeta_e - y_h)\|_{\boldsymbol{L}^2(D)} \le \|\nabla(\zeta_e - \mathcal{I}_{h0}^{\mathrm{g,av}}(\zeta_e))\|_{\boldsymbol{L}^2(D)} \\ &\le c\, h^s|\zeta_e|_{H^{1+s}(D)} \le c\, h^s\ell_D^{-1-s}\|\zeta_e\|_{H^{1+s}(D)} \le c\, c_{\mathrm{smo}}\, h^s\ell_D^{1-s}\|e\|_{L^2(D)}, \end{aligned}$$

where we used the approximation properties of $\mathcal{I}_{h0}^{\mathrm{g,av}}$ from Theorem 22.14 and the estimate (37.16). Since $s > \frac12$ and $h \le \ell_D$, we have $h^s\ell_D^{1-s} \le h^{\frac12}\ell_D^{\frac12}$, and this concludes the proof. □

37.3.3 Symmetrization

The estimate (37.17) is suboptimal by a factor $h^{s-\frac12}$, and this loss of optimality is caused by the adjoint consistency error which is only of order $h^{\frac12}$. This shortcoming can be avoided by symmetrizing a_h and modifying ℓ_h consistently. More precisely, we define

$$\begin{aligned} a_h^{\mathrm{sym}}(v_h, w_h) &:= a(v_h, w_h) - \int_{\partial D}(\boldsymbol{n}\cdot\nabla v_h)w_h\,\mathrm{d}s - \int_{\partial D} v_h(\boldsymbol{n}\cdot\nabla w_h)\,\mathrm{d}s \\ &\quad + \sum_{F\in\mathcal{F}_h^\partial}\varpi_0\frac{1}{h_F}\int_F v_h w_h\,\mathrm{d}s, \\ \ell_h^{\mathrm{sym}}(w_h) &:= \ell(w_h) - \int_{\partial D} g(\boldsymbol{n}\cdot\nabla w_h)\,\mathrm{d}s + \sum_{F\in\mathcal{F}_h^\partial}\varpi_0\frac{1}{h_F}\int_F g w_h\,\mathrm{d}s. \end{aligned}$$

Consider the following discrete problem:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a_h^{\mathrm{sym}}(u_h, w_h) = \ell_h^{\mathrm{sym}}(w_h), \quad \forall w_h \in V_h. \end{cases} \tag{37.18}$$

Adapting the proof of Lemma 37.3, one can show that the problem (37.18) is well-posed if one chooses the stabilization parameter s.t. $\varpi_0 > n_\partial c_{\rm dt}^2$.

Theorem 37.8 (L^2-estimate). *Let u solve* (37.1) *and let u_h solve* (37.18). *Assume $\varpi_0 > n_\partial c_{\rm dt}^2$ and that there is $s \in (\frac{1}{2}, 1]$ s.t. the adjoint solution satisfies the a priori estimate* (37.16). *There is c s.t. for all $h \in \mathcal{H}$,*

$$\|u - u_h\|_{L^2(D)} \le c\, h^s \ell_D^{1-s} \|u - u_h\|_{V_\sharp}. \tag{37.19}$$

Proof. We proceed as in the proof of Theorem 37.7 with the same spaces $V_\sharp$, $Z_\sharp$, and Y_h, but now we set $a_\sharp(v, w) := (\nabla v, \nabla w)_{\boldsymbol{L}^2(D)} - (v, \boldsymbol{n}\cdot\nabla w)_{L^2(\partial D)}$. We equip $Z_\sharp$ with the same norm as $V_\sharp$, so that $a_\sharp$ is bounded on $V_\sharp \times Z_\sharp$. The Galerkin orthogonality property still holds true for $a_\sharp$. Indeed, we have

$$\begin{aligned} a_\sharp(u, y_h) &= (f, y_h)_{L^2(D)} - (g, \boldsymbol{n}\cdot\nabla y_h)_{L^2(\partial D)} \\ &= \ell_h^{\rm sym}(y_h) = a_h^{\rm sym}(u_h, y_h) = a_\sharp(u_h, y_h), \qquad \forall y_h \in Y_h, \end{aligned}$$

since $\gamma^{\rm g}(u) = g$ and y_h vanishes on ∂D. Now the adjoint consistency error vanishes, and we still have $\|\zeta_e - \mathcal{I}_{h0}^{\rm g,av}(\zeta_e)\|_{Z_\sharp} \le c\, h^s |\zeta_e|_{H^{1+s}(D)}$. □

Exercises

Exercise 37.1 (Poincaré–Steklov). Let $\check{C}_{\rm PS}$ be defined in (31.23). Prove that $\check{C}_{\rm PS} \ell_D^{-1} \|v\|_{L^2(D)} \le (\|\nabla v\|_{\boldsymbol{L}^2(D)}^2 + |v|_\partial^2)^{\frac{1}{2}}$ for all $v \in H^1(D)$. (*Hint*: use $h \le \ell_D$ and (31.23).)

Exercise 37.2 (Quadratic inequality). Prove that $x^2 - 2\beta xy + \varpi_0 y^2 \ge \frac{\varpi_0 - \beta^2}{1+\varpi_0}(x^2 + y^2)$ for all real numbers x, y, $\varpi_0 \ge 0$ and $\beta \ge 0$.

Exercise 37.3 (Error estimate). Prove (37.14). (*Hint*: consider the quasi-interpolation operator from §22.3.)

Exercise 37.4 (Gradient). Let U be an open bounded set in $\mathbb{R}^d$, let $s \in (0, 1)$, and set $\boldsymbol{H}_{00}^s(U) := [\boldsymbol{L}^2(U), \boldsymbol{H}_0^1(U)]_{s,2}$. (i) Show that $\nabla : \boldsymbol{H}^{1-s}(U) \to (\boldsymbol{H}_{00}^s(U))'$ is bounded for all $s \in (0, 1)$. (*Hint*: use Theorems A.27 and A.30.) (ii) Assume that U is Lipschitz. Show that $\nabla : \boldsymbol{H}^{1-s}(U) \to \boldsymbol{H}^{-s}(U)$ is bounded for all $s \in (0, 1)$, $s \ne \frac{1}{2}$. (*Hint*: see (3.7), Theorem 3.19; see also Grisvard [223, Lem. 1.4.4.6].)

Exercise 37.5 (L^2-estimate). (i) Modify the proof of Theorem 37.7 by measuring the interpolation error on the adjoint solution with the operator $\mathcal{I}_h^{\rm g,av}$ instead of $\mathcal{I}_{h0}^{\rm g,av}$, i.e., use $Y_h := V_h$ instead of $Y_h := V_h \cap H_0^1(D)$. (*Hint*: set $a_\sharp(v, w) := (\nabla v, \nabla w)_{\boldsymbol{L}^2(D)} - (\boldsymbol{n}\cdot\nabla v, w)_{L^2(\partial D)} + \sum_{F \in \mathcal{F}_h^\partial} \varpi_0 \frac{1}{h_F}(v, w)_{L^2(F)}$.) (ii) Do the same for the proof of Theorem 37.8.