Part IX, Chapter 44

Maxwell's equations: control on the divergence

The analysis of Chapter 43 requires a coercivity property in $\boldsymbol{H}(\text{curl})$. There is, however, a loss of coercivity when the lower bound on the model parameter ν becomes very small. This situation occurs in the following two situations: (i) in the low frequency limit ($\omega \to 0$) when $\nu := \mathrm{i}\omega\mu$ as in the eddy current problem; (ii) if $\kappa \in \mathbb{R}$ and $\sigma \ll \omega\epsilon$ when $\nu := -\omega^2\epsilon + \mathrm{i}\omega\sigma$ as in the time-harmonic problem. We have also seen in Chapter 43 that a compactness property needs to be established to deduce an improved $\boldsymbol{L}^2$-error estimate by the duality argument. We show in this chapter that robust coercivity and compactness can be achieved by a weak control on the divergence of the discrete solution. The material of this chapter is based on [188].

44.1 Functional setting

In this section, we present the assumptions on the model problem and introduce a functional setting leading to a key smoothness result on the curl operator.

44.1.1 Model problem

We consider the model problem (43.9) on a Lipschitz domain D in $\mathbb{R}^3$. For simplicity, we restrict the scope to the homogeneous Dirichlet boundary condition $\boldsymbol{A}_{|\partial D}\times\boldsymbol{n} = \boldsymbol{0}$ (so that $\partial D_{\mathrm{d}} = \partial D$). The weak formulation is

$$\begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{V}_0 := \boldsymbol{H}_0(\text{curl}; D) \text{ such that} \\ a_{\nu,\kappa}(\boldsymbol{A}, \boldsymbol{b}) = \ell(\boldsymbol{b}), \quad \forall \boldsymbol{b} \in \boldsymbol{V}_0, \end{cases} \tag{44.1}$$

with $a_{\nu,\kappa}(\boldsymbol{a}, \boldsymbol{b}) := \int_D (\nu \boldsymbol{a}\cdot\overline{\boldsymbol{b}} + \kappa\nabla\times\boldsymbol{a}\cdot\nabla\times\overline{\boldsymbol{b}})\,\mathrm{d}x$ and $\ell(\boldsymbol{b}) := \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{b}}\,\mathrm{d}x$. We assume that $\boldsymbol{f} \in \boldsymbol{L}^2(D)$ and that $\nabla\cdot\boldsymbol{f} = 0$. The divergence-free condition on $\boldsymbol{f}$ implies the following important property on the solution $\boldsymbol{A}$:

$$\nabla\cdot(\nu \boldsymbol{A}) = 0. \tag{44.2}$$

Concerning the material properties ν and κ, we make the following assumptions: (i) Boundedness: $\nu, \kappa \in L^\infty(D;\mathbb{C})$ and we set $\nu_\sharp := \|\nu\|_{L^\infty(D;\mathbb{C})}$ and $\kappa_\sharp := \|\kappa\|_{L^\infty(D;\mathbb{C})}$. (ii) Rotated positivity: there are real numbers θ, $\nu_\flat > 0$, and $\kappa_\flat > 0$ s.t. (43.12) is satisfied, i.e.,

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re\big(e^{\mathrm{i}\theta}\nu(\boldsymbol{x})\big) \geq \nu_\flat, \qquad \operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re\big(e^{\mathrm{i}\theta}\kappa(\boldsymbol{x})\big) \geq \kappa_\flat. \tag{44.3}$$

We define the contrast factors $\nu_{\sharp/\flat} := \frac{\nu_\sharp}{\nu_\flat}$ and $\kappa_{\sharp/\flat} := \frac{\kappa_\sharp}{\kappa_\flat}$. We also define the magnetic Reynolds number $\gamma_{\nu,\kappa} := \nu_\sharp \ell_D^2 \kappa_\sharp^{-1}$. Several magnetic Reynolds numbers can be defined if the material is highly contrasted, but we will not explore this situation further. (iii) Piecewise smoothness: there is a partition of D into M disjoint Lipschitz polyhedra $\{D_m\}_{m\in\{1:M\}}$ s.t. $\nu_{|D_m}, \kappa_{|D_m} \in W^{1,\infty}(D_m)$ for all $m \in \{1:M\}$. The reader who is not comfortable with this assumption may think of ν, κ being constant without missing anything essential in the analysis.

44.1.2 A key smoothness result on the curl operator

Let us define the (complex-valued) functional spaces

$$M_0 := H_0^1(D), \qquad M_* := \{q \in H^1(D) \mid (q,1)_{L^2(D)} = 0\}, \tag{44.4}$$

as well as the following subspaces of $\boldsymbol{H}(\mathrm{curl};D)$:

$$\boldsymbol{X}_{0\nu} := \{\boldsymbol{b} \in \boldsymbol{H}_0(\mathrm{curl};D) \mid (\nu\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in M_0\}, \tag{44.5a}$$

$$\boldsymbol{X}_{*\kappa^{-1}} := \{\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl};D) \mid (\kappa^{-1}\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in M_*\}, \tag{44.5b}$$

where $(\cdot,\cdot)_{\boldsymbol{L}^2(D)}$ denotes the inner product in $\boldsymbol{L}^2(D)$. The main motivation for introducing the above subspaces is that $\boldsymbol{A} \in \boldsymbol{X}_{0\nu}$ owing to (44.2). Moreover, we will see below that $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{X}_{*\kappa^{-1}}$. Taking $m \in C_0^\infty(D)$ in (44.5a) shows that for all $\boldsymbol{b} \in \boldsymbol{X}_{0\nu}$, the field $\nu\boldsymbol{b}$ has a weak divergence in $L^2(D)$ and $\nabla\cdot(\nu\boldsymbol{b}) = 0$. Similarly, the definition (44.5b) implies that for all $\boldsymbol{b} \in \boldsymbol{X}_{*\kappa^{-1}}$, the field $\kappa^{-1}\boldsymbol{b}$ has a weak divergence in $L^2(D)$ and $\nabla\cdot(\kappa^{-1}\boldsymbol{b}) = 0$. Invoking the integration by parts formula (4.12) and the surjectivity of the trace map $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ then shows that $\gamma^{\mathrm{d}}(\kappa^{-1}\boldsymbol{b}) = 0$ for all $\boldsymbol{b} \in \boldsymbol{X}_{*\kappa^{-1}}$, where γ^{d} is the normal trace operator (recall that $\gamma^{\mathrm{d}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{n}$ if the field $\boldsymbol{v}$ is smooth).

Let us first state a simple result related to the Helmholtz decomposition of vector fields in $\boldsymbol{V}_0 := \boldsymbol{H}_0(\mathrm{curl};D)$ using the subspace $\boldsymbol{X}_{0\nu}$ (a similar result is available on $\boldsymbol{H}(\mathrm{curl};D)$ using the subspace $\boldsymbol{X}_{*\kappa^{-1}}$).

Lemma 44.1 (Helmholtz decomposition). *The following holds true:*

$$\boldsymbol{V}_0 = \boldsymbol{X}_{0\nu} \oplus \nabla M_0. \tag{44.6}$$

Proof. Let $\boldsymbol{b} \in \boldsymbol{V}_0$ and let $p \in M_0$ solve $(\nu\nabla p, \nabla q)_{\boldsymbol{L}^2(D)} = (\nu\boldsymbol{b}, \nabla q)_{\boldsymbol{L}^2(D)}$ for all $q \in M_0$. Our assumptions on ν imply that there is a unique solution to this problem. Then we set $\boldsymbol{v} := \boldsymbol{b} - \nabla p$ and observe that $\boldsymbol{v} \in \boldsymbol{X}_{0\nu}$. The sum is direct because if $\boldsymbol{0} = \boldsymbol{v} + \nabla p$, then the identity $\int_D \nu\nabla p{\cdot}\overline{\boldsymbol{v}}\,\mathrm{d}x = 0$, which holds true for all $p \in M_0$ and all $\boldsymbol{v} \in \boldsymbol{X}_{0\nu}$, implies that $\nabla p = \boldsymbol{0} = \boldsymbol{v}$. □

We can now state the main result of this section. This result extends Lemma 43.3 to heterogeneous domains. Given a smoothness index $s > 0$, we set $\|\boldsymbol{b}\|_{\boldsymbol{H}^s(D)} := (\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^{2s}|\boldsymbol{b}|^2_{\boldsymbol{H}^s(D)})^{\frac{1}{2}}$, where ℓ_D is some characteristic length of D, e.g., $\ell_D := \mathrm{diam}(D)$.

Lemma 44.2 (Regularity pickup). *Let D be a Lipschitz domain in $\mathbb{R}^3$.* (i) *Assume that the boundary ∂D is connected and that ν is piecewise smooth. There exist $s > 0$ and $\check{C} > 0$ (depending on D and the contrast factor $\nu_{\sharp/\flat}$ but not on $\nu_\flat$ alone) such that*

$$\check{C}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{H}^s(D)} \le \|\nabla{\times}\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\nu}. \tag{44.7}$$

(ii) *Assume that D is simply connected and that κ is piecewise smooth. There exist $s' > 0$ and $\check{C}' > 0$ (depending on D and the contrast factor $\kappa_{\sharp/\flat}$ but not on $\kappa_\flat$ alone) such that*

$$\check{C}'\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{H}^{s'}(D)} \le \|\nabla{\times}\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{b} \in \boldsymbol{X}_{*\kappa^{-1}}. \tag{44.8}$$

Proof. See Jochmann [259], Bonito et al. [70]. □

Remark 44.3 (Smoothness index). There are some situations where the smoothness indices s, s' can be larger than $\frac{1}{2}$. One example is that of isolated inclusions in an otherwise homogeneous material. We refer the reader to Ciarlet [121, §5.2] for further insight and examples. □

Lemma 44.2 has two important consequences. First, by restricting the smoothness index s to zero in (44.7), we obtain the following important stability result on the curl operator.

Lemma 44.4 (Poincaré–Steklov). *Assume that the boundary ∂D is connected and that ν is piecewise smooth. There is $\hat{C}_{\mathrm{PS}} > 0$ (depending on D and the contrast factor $\nu_{\sharp/\flat}$) such that the following Poincaré–Steklov inequality holds true:*

$$\hat{C}_{\mathrm{PS}}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} \le \|\nabla{\times}\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\nu}. \tag{44.9}$$

The bound (44.9) is what we need to establish a coercivity property on $\boldsymbol{X}_{0\nu}$ that is robust w.r.t. $\nu_\flat$. Indeed, we have

$$\begin{aligned}\Re\big(e^{\mathrm{i}\theta}a_{\nu,\kappa}(\boldsymbol{b},\boldsymbol{b})\big) &\geq \nu_\flat\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \kappa_\flat\|\nabla{\times}\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} \geq \kappa_\flat\|\nabla{\times}\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)}\\ &\geq \frac{1}{2}\kappa_\flat(\|\nabla{\times}\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \hat{C}^2_{\mathrm{PS}}\ell_D^{-2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)})\\ &\geq \frac{1}{2}\kappa_\flat\ell_D^{-2}\min(1,\hat{C}^2_{\mathrm{PS}})\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)},\end{aligned} \tag{44.10}$$

for all $\boldsymbol{b}\in\boldsymbol{X}_{0\nu}$, where we recall that $\boldsymbol{H}(\mathrm{curl};D)$ is equipped with the norm $\|\boldsymbol{b}\|_{\boldsymbol{H}(\mathrm{curl};D)} := (\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla{\times}\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)})^{\frac{1}{2}}$. This shows that the sesquilinear form $a_{\nu,\kappa}$ is coercive on $\boldsymbol{X}_{0\nu}$ with a coercivity constant depending on the contrast factor $\nu_{\sharp/\flat}$ but not on $\nu_\flat$ alone (whereas the coercivity constant on the larger space $\boldsymbol{V}_0$ is $\min(\nu_\flat,\ell_D^{-2}\kappa_\flat)$ (see (43.13a))).

Let us now examine the consequences of Lemma 44.2 on the Sobolev smoothness index of $\boldsymbol{A}$ and $\nabla{\times}\boldsymbol{A}$. Owing to (44.7), there is $s>0$ s.t. $\boldsymbol{A}\in\boldsymbol{H}^s(D)$. We will see in §44.3 that the embedding $\boldsymbol{H}^s(D)\hookrightarrow\boldsymbol{L}^2(D)$ is the compactness property that we need to apply the duality argument and derive an improved $\boldsymbol{L}^2$-error estimate. Furthermore, the field $\boldsymbol{R}:=\kappa\nabla{\times}\boldsymbol{A}$ is in $\boldsymbol{X}_{*\kappa^{-1}}$ (notice in particular that $\nabla{\times}\boldsymbol{R}=\boldsymbol{f}-\nu\boldsymbol{A}\in\boldsymbol{L}^2(D)$), so that we deduce from (44.8) that there is $s'>0$ s.t. $\boldsymbol{R}\in\boldsymbol{H}^{s'}(D)$. In addition, the material property κ being piecewise smooth, we infer that the following multiplier property holds true (see [259, Lem. 2] and [70, Prop. 2.1]): There exists $\tau>0$ and $C_{\kappa^{-1}}$ s.t.

$$|\kappa^{-1}\boldsymbol{\xi}|_{\boldsymbol{H}^{\tau'}(D)} \leq C_{\kappa^{-1}}|\boldsymbol{\xi}|_{\boldsymbol{H}^{\tau'}(D)}, \qquad \forall\boldsymbol{\xi}\in\boldsymbol{H}^{\tau}(D),\quad \forall\tau'\in[0,\tau]. \tag{44.11}$$

Letting $s'':=\min(s',\tau)>0$, we conclude that $\nabla{\times}\boldsymbol{A}\in\boldsymbol{H}^{s''}(D)$.

44.2 Coercivity revisited for edge elements

In this section, we revisit the $\boldsymbol{H}(\mathrm{curl})$-error analysis for the approximation of the weak problem (44.1) using Nédélec (or edge) elements (see Chapters 15 and 19). The key tool we are going to use is a discrete counterpart of the Poincaré–Steklov inequality (44.9). We consider a shape-regular sequence of affine meshes $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of D. We assume that D is a Lipschitz polyhedron and that each mesh covers D exactly.

44.2.1 Discrete Poincaré–Steklov inequality

Let $\boldsymbol{V}_{h0}$ be the $\boldsymbol{H}_0(\mathrm{curl})$-conforming space using Nédélec elements of order $k\geq 0$ defined by

$$\boldsymbol{V}_{h0} := \boldsymbol{P}^{\mathrm{c}}_{k,0}(\mathcal{T}_h) := \{\boldsymbol{b}_h\in\boldsymbol{P}^{\mathrm{c}}_k(\mathcal{T}_h) \mid \boldsymbol{b}_{h|\partial D}{\times}\boldsymbol{n}=\boldsymbol{0}\}. \tag{44.12}$$

Observe that the Dirichlet condition is enforced strongly in $\boldsymbol{V}_{h0}$. The discrete problem is formulated as follows:

$$\begin{cases} \text{Find } \boldsymbol{A}_h \in \boldsymbol{V}_{h0} \text{ such that} \\ a_{\nu,\kappa}(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h), \quad \forall \boldsymbol{b}_h \in \boldsymbol{V}_{h0}. \end{cases} \tag{44.13}$$

Since it is not reasonable to consider the space $\{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid \nabla\cdot(\nu \boldsymbol{b}_h) = 0\}$, because the normal component of $\nu \boldsymbol{b}_h$ may jump across the mesh interfaces, we are going to consider instead the subspace

$$\boldsymbol{X}_{h0\nu} := \{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid (\nu \boldsymbol{b}_h, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall m_h \in M_{h0}\}, \tag{44.14}$$

where $M_{h0} := P^{\mathrm{g}}_{k+1,0}(\mathcal{T}_h;\mathbb{C})$ is conforming in $H^1_0(D;\mathbb{C})$. Note that the polynomial degrees of the finite element spaces M_{h0} and $\boldsymbol{V}_{h0}$ are compatible in the sense that $\nabla M_{h0} \subset \boldsymbol{V}_{h0}$. Using this property and proceeding as in Lemma 44.1 proves the following discrete Helmholtz decomposition:

$$\boldsymbol{V}_{h0} = \boldsymbol{X}_{h0\nu} \oplus \nabla M_{h0}. \tag{44.15}$$

Lemma 44.5 (Discrete solution). *Let $\boldsymbol{A}_h \in \boldsymbol{V}_{h0}$ be the unique solution to* (44.13)*. Then $\boldsymbol{A}_h \in \boldsymbol{X}_{h0\nu}$.*

Proof. We must show that $(\nu \boldsymbol{A}_h, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0$ for all $m_h \in M_{h0}$. Since $\nabla m_h \in \nabla M_{h0} \subset \boldsymbol{V}_{h0}$, ∇m_h is an admissible test function in (44.13). Recalling that $\nabla\cdot \boldsymbol{f} = 0$, we infer that

$$0 = \ell(\nabla m_h) = a_{\nu,\kappa}(\boldsymbol{A}_h, \nabla m_h) = (\nu \boldsymbol{A}_h, \nabla m_h)_{\boldsymbol{L}^2(D)},$$

since $\nabla\times(\nabla m_h) = \boldsymbol{0}$. This completes the proof. □

We now establish a discrete counterpart to the Poincaré–Steklov inequality (44.9). This result is not straightforward since $\boldsymbol{X}_{h0\nu}$ is not a subspace of $\boldsymbol{X}_{0\nu}$. The key tool that we are going to invoke is the stable commuting quasi-interpolation projections from §23.3.3.

Theorem 44.6 (Discrete Poincaré–Steklov). *Under the assumptions of* Lemma 44.4*, there is a constant $\hat{C}'_{\mathrm{PS}} > 0$ (depending on $\hat{C}_{\mathrm{PS}}$, the polynomial degree k, the regularity of the mesh sequence, and the contrast factor $\nu_{\sharp/\flat}$, but not on $\nu_\flat$ alone) s.t. for all $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}$ and all $h \in \mathcal{H}$,*

$$\hat{C}'_{\mathrm{PS}} \ell_D^{-1} \|\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} \le \|\nabla\times \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}. \tag{44.16}$$

Proof. Let $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}$ be a nonzero discrete field. Let $\phi(\boldsymbol{x}_h) \in M_0 := H^1_0(D)$ be the solution to the following well-posed Poisson problem:

$$(\nu\nabla\phi(\boldsymbol{x}_h), \nabla m)_{\boldsymbol{L}^2(D)} = (\nu \boldsymbol{x}_h, \nabla m)_{\boldsymbol{L}^2(D)}, \qquad \forall m \in M_0.$$

Let us define the *curl-preserving lifting* of $\boldsymbol{x}_h$ s.t. $\boldsymbol{\xi}(\boldsymbol{x}_h) := \boldsymbol{x}_h - \nabla\phi(\boldsymbol{x}_h)$, and let us notice that $\boldsymbol{\xi}(\boldsymbol{x}_h) \in \boldsymbol{X}_{0\nu}$. Upon invoking the quasi-interpolation operators $\mathcal{J}^{\mathrm{c}}_{h0}$ and $\mathcal{J}^{\mathrm{d}}_{h0}$ introduced in §23.3.3, we observe that

$$\boldsymbol{x}_h - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)) = \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{x}_h - \boldsymbol{\xi}(\boldsymbol{x}_h)) = \mathcal{J}^{\mathrm{c}}_{h0}(\nabla(\phi(\boldsymbol{x}_h))) = \nabla(\mathcal{J}^{\mathrm{g}}_{h0}(\phi(\boldsymbol{x}_h))),$$

where we used that $\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{x}_h)=\boldsymbol{x}_h$ and the commuting properties of $\mathcal{J}_{h0}^{\mathrm{g}}$ and $\mathcal{J}_{h0}^{\mathrm{c}}$. Since $\boldsymbol{x}_h\in\boldsymbol{X}_{h0\nu}$, we infer that $(\nu\boldsymbol{x}_h,\nabla(\mathcal{J}_{h0}^{\mathrm{g}}(\phi(\boldsymbol{x}_h))))_{\boldsymbol{L}^2(D)}=0$, so that

$$\begin{aligned}(\nu\boldsymbol{x}_h,\boldsymbol{x}_h)_{\boldsymbol{L}^2(D)}&=(\nu\boldsymbol{x}_h,\boldsymbol{x}_h-\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)}+(\nu\boldsymbol{x}_h,\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)}\\&=(\nu\boldsymbol{x}_h,\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)}.\end{aligned}$$

Multiplying by $e^{\mathrm{i}\theta}$, taking the real part, and using the Cauchy–Schwarz inequality, we infer that

$$\nu_\flat\|\boldsymbol{x}_h\|^2_{\boldsymbol{L}^2(D)}\le\nu_\sharp\|\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}\|\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h))\|_{\boldsymbol{L}^2(D)}.$$

The uniform boundedness of $\mathcal{J}_{h0}^{\mathrm{c}}$ on $\boldsymbol{L}^2(D)$, together with the Poincaré–Steklov inequality (44.9) and the identity $\nabla\times\boldsymbol{\xi}(\boldsymbol{x}_h)=\nabla\times\boldsymbol{x}_h$, implies that

$$\begin{aligned}\|\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h))\|_{\boldsymbol{L}^2(D)}&\le\|\mathcal{J}_{h0}^{\mathrm{c}}\|_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}\\&\le\|\mathcal{J}_{h0}^{\mathrm{c}}\|_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\hat{C}_{\mathrm{PS}}^{-1}\ell_D\|\nabla\times\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)},\end{aligned}$$

so that (44.16) holds true with $\hat{C}'_{\mathrm{PS}}:=\nu_{\sharp/\flat}^{-1}\|\mathcal{J}_{h0}^{\mathrm{c}}\|^{-1}_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\hat{C}_{\mathrm{PS}}$. □

Remark 44.7 (Literature). There are many ways to prove the discrete Poincaré–Steklov inequality (44.16). One route described in Hiptmair [244, §4.2] consists of invoking subtle regularity estimates from Amrouche et al. [10, Lem. 4.7]. Another one, which avoids invoking regularity estimates, is based on an argument by Kikuchi [267] which is often called *discrete compactness*; see also Monk and Demkowicz [304], Caorsi et al. [106]. The proof is not constructive and is based on an argument by contradiction. The technique used in the proof of Theorem 44.6, inspired from Arnold et al. [23, Thm. 5.11] and Arnold et al. [26, Thm. 3.6], is more recent, and uses the stable commuting quasi-interpolation projections $\mathcal{J}_h^{\mathrm{c}}$ and $\mathcal{J}_{h0}^{\mathrm{c}}$. It was already observed in Boffi [61] that the existence of stable commuting quasi-interpolation operators would imply the discrete compactness property. □

44.2.2 $\boldsymbol{H}(\mathrm{curl})$-error analysis

We are now in a position to revisit the error analysis of §43.3. Let us first show that $\boldsymbol{X}_{h0\nu}$ has the same approximation properties as $\boldsymbol{V}_{h0}$ in $\boldsymbol{X}_{0\nu}$.

Lemma 44.8 (Approximation in $\boldsymbol{X}_{h0\nu}$). *There is c, uniform w.r.t. the model parameters, s.t. for all $\boldsymbol{A}\in\boldsymbol{X}_{0\nu}$ and all $h\in\mathcal{H}$,*

$$\inf_{\boldsymbol{x}_h\in\boldsymbol{X}_{h0\nu}}\|\boldsymbol{A}-\boldsymbol{x}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}\le c\,\nu_{\sharp/\flat}\inf_{\boldsymbol{b}_h\in\boldsymbol{V}_{h0}}\|\boldsymbol{A}-\boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}.\qquad(44.17)$$

Proof. Let $\boldsymbol{A}\in\boldsymbol{X}_{0\nu}$. We start by computing the Helmholtz decomposition of $\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A})$ in $\boldsymbol{V}_{h0}$ as stated in (44.15). Let $p_h\in M_{h0}$ be the unique solution to the discrete Poisson problem $(\nu\nabla p_h,\nabla q_h)_{\boldsymbol{L}^2(D)}=(\nu\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A}),\nabla q_h)_{\boldsymbol{L}^2(D)}$ for all $q_h\in M_{h0}$. Let us define $\boldsymbol{y}_h:=\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A})-\nabla p_h$. By construction, $\boldsymbol{y}_h\in\boldsymbol{X}_{h0\nu}$ and

$\nabla\times\boldsymbol{y}_h = \nabla\times\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})$. Hence, $\|\nabla\times(\boldsymbol{A}-\boldsymbol{y}_h)\|_{\boldsymbol{L}^2(D)} = \|\nabla\times(\boldsymbol{A}-\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A}))\|_{\boldsymbol{L}^2(D)}$. Since $\nabla\cdot(\nu\boldsymbol{A}) = 0$, we also infer that

$$(\nu\nabla p_h, \nabla p_h)_{\boldsymbol{L}^2(D)} = (\nu\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A}), \nabla p_h)_{\boldsymbol{L}^2(D)} = (\nu(\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A}) - \boldsymbol{A}), \nabla p_h)_{\boldsymbol{L}^2(D)},$$

which in turn implies that $\|\nabla p_h\|_{\boldsymbol{L}^2(D)} \le \nu_{\sharp/\flat}\|\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A}) - \boldsymbol{A}\|_{\boldsymbol{L}^2(D)}$. The above argument shows that

$$\begin{aligned}\|\boldsymbol{A} - \boldsymbol{y}_h\|_{\boldsymbol{L}^2(D)} &\le \|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)} + \|\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A}) - \boldsymbol{y}_h\|_{\boldsymbol{L}^2(D)} \\ &\le \|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)} + \|\nabla p_h\|_{\boldsymbol{L}^2(D)} \\ &\le (1 + \nu_{\sharp/\flat})\|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)}.\end{aligned}$$

In conclusion, we have proved that

$$\begin{aligned}\inf_{\boldsymbol{x}_h\in\boldsymbol{X}_{h0\nu}} \|\boldsymbol{A} - \boldsymbol{x}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} &\le \|\boldsymbol{A} - \boldsymbol{y}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \\ &\le (1 + \nu_{\sharp/\flat})\|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|_{\boldsymbol{H}(\mathrm{curl};D)}.\end{aligned}$$

Invoking the commutation and approximation properties of the quasi-interpolation operators, we infer that

$$\begin{aligned}\|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{H}(\mathrm{curl};D)} &= \|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\times(\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A}))\|^2_{\boldsymbol{L}^2(D)} \\ &= \|\boldsymbol{A} - \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{A} - \mathcal{J}^{\mathrm{d}}_{h0}(\nabla\times\boldsymbol{A})\|^2_{\boldsymbol{L}^2(D)} \\ &\le c \inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{\mathrm{c}}_0(\mathcal{T}_h)} \|\boldsymbol{A} - \boldsymbol{b}_h\|^2_{\boldsymbol{L}^2(D)} + c'\ell_D^2 \inf_{\boldsymbol{d}_h\in\boldsymbol{P}^{\mathrm{d}}_0(\mathcal{T}_h)} \|\nabla\times\boldsymbol{A} - \boldsymbol{d}_h\|^2_{\boldsymbol{L}^2(D)} \\ &\le c \inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{\mathrm{c}}_0(\mathcal{T}_h)} \|\boldsymbol{A} - \boldsymbol{b}_h\|^2_{\boldsymbol{L}^2(D)} + c'\ell_D^2 \inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{\mathrm{c}}_0(\mathcal{T}_h)} \|\nabla\times(\boldsymbol{A} - \boldsymbol{b}_h)\|^2_{\boldsymbol{L}^2(D)},\end{aligned}$$

where the last bound follows by restricting the minimization set to $\nabla\times\boldsymbol{P}^{\mathrm{c}}_0(\mathcal{T}_h)$ since $\nabla\times\boldsymbol{P}^{\mathrm{c}}_0(\mathcal{T}_h) \subset \boldsymbol{P}^{\mathrm{d}}_0(\mathcal{T}_h)$. The conclusion follows readily. □

Theorem 44.9 ($\boldsymbol{H}$(curl)-error estimate). *Let $\boldsymbol{A}$ solve (44.1) and let $\boldsymbol{A}_h$ solve (44.13). Assume that ∂D is connected and that ν is piecewise smooth. There is c, which depends on the discrete Poincaré–Steklov constant $\hat{C}'_{\mathrm{PS}}$ and the contrast factors $\nu_{\sharp/\flat}$ and $\kappa_{\sharp/\flat}$, s.t. for all $h \in \mathcal{H}$,*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \le c\,\hat{\gamma}_{\nu,\kappa} \inf_{\boldsymbol{b}_h\in\boldsymbol{V}_{h0}} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}, \tag{44.18}$$

with $\hat{\gamma}_{\nu,\kappa} := \max(1, \gamma_{\nu,\kappa})$ and the magnetic Reynolds number $\gamma_{\nu,\kappa} := \nu_\sharp\ell_D^2\kappa_\sharp^{-1}$.

Proof. Owing to Lemma 44.5, $\boldsymbol{A}_h$ also solves the following problem: Find $\boldsymbol{A}_h \in \boldsymbol{X}_{h0\nu}$ s.t.

$$a_{\nu,\kappa}(\boldsymbol{A}_h, \boldsymbol{x}_h) = \ell(\boldsymbol{x}_h), \forall \boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}.$$

Using the discrete Poincaré–Steklov inequality (44.16) and proceeding as in (44.10), we infer that

$$\Re\big(e^{\mathrm{i}\theta}a_{\nu,\kappa}(\boldsymbol{x}_h,\boldsymbol{x}_h)\big) \geq \frac{1}{2}\kappa_\flat \ell_D^{-2}\min(1,(\hat{C}'_{\rm PS})^2)\|\boldsymbol{x}_h\|^2_{\boldsymbol{H}(\mathrm{curl};D)},$$

for all $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}$. Hence, the above problem is well-posed. Recalling the boundedness property (43.13b) of the sesquilinear form $a_{\nu,\kappa}$ and invoking the abstract error estimate (26.18) leads to

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \leq \frac{2\max(\nu_\sharp,\ell_D^{-2}\kappa_\sharp)}{\kappa_\flat\ell_D^{-2}\min(1,(\hat{C}'_{\rm PS})^2)} \inf_{\boldsymbol{x}_h\in\boldsymbol{X}_{h0\nu}} \|\boldsymbol{A}-\boldsymbol{x}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}.$$

We conclude the proof by invoking Lemma 44.8. □

Remark 44.10 (Neumann boundary condition). The above analysis can be adapted to handle the Neumann condition $(\kappa\nabla\times\boldsymbol{A})_{|\partial D}\times\boldsymbol{n} = \boldsymbol{0}$; see Exercise 44.3. This condition implies that $(\nabla\times(\kappa\nabla\times\boldsymbol{A}))_{|\partial D}\cdot\boldsymbol{n} = 0$. Moreover, assuming $\boldsymbol{f}_{|\partial D}\cdot\boldsymbol{n} = 0$ and taking the normal component of the equation $\nu\boldsymbol{A} + \nabla\times(\kappa\nabla\times\boldsymbol{A}) = \boldsymbol{f}$ at the boundary gives $\boldsymbol{A}_{|\partial D}\cdot\boldsymbol{n} = 0$. Since $\nabla\cdot\boldsymbol{f} = 0$, we also have $\nabla\cdot(\nu\boldsymbol{A}) = 0$. In other words, we have

$$\boldsymbol{A} \in \boldsymbol{X}_{*\nu} := \{\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl};D) \mid (\nu\boldsymbol{b},\nabla m)_{\boldsymbol{L}^2(D)} = 0,\ \forall m \in M_*\}.$$

The discrete spaces are now $\boldsymbol{V}_h := \boldsymbol{P}^{\rm c}_k(\mathcal{T}_h)$ and $M_{h*} := P^{\rm g}_{k+1}(\mathcal{T}_h;\mathbb{C}) \cap M_*$. Using $\boldsymbol{V}_h$ for the discrete trial and test spaces, we infer that

$$\boldsymbol{A}_h \in \boldsymbol{X}_{h*\nu} := \{\boldsymbol{b}_h \in \boldsymbol{V}_h \mid (\nu\boldsymbol{b}_h,\nabla m_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall m_h \in M_{h*}\}.$$

The Poincaré–Steklov inequality (44.16) still holds true provided the assumption that ∂D is connected is replaced by the assumption that D is simply connected. The error analysis from Theorem 44.9 can be readily adapted. □

44.3 The duality argument for edge elements

Our goal is to derive an improved error estimate in the $\boldsymbol{L}^2$-norm using a duality argument that invokes a weak control on the divergence. The subtlety is that, as already mentioned, the setting is nonconforming since $\boldsymbol{X}_{h0\nu}$ is not a subspace of $\boldsymbol{X}_{0\nu}$. We assume in the section that the boundary ∂D is connected and that the domain D is simply connected. Recalling the smoothness indices $s, s' > 0$ from Lemma 44.2 together with the index $\tau > 0$ from the multiplier property (44.11) and letting $s'' := \min(s',\tau)$, we have $\boldsymbol{A} \in \boldsymbol{H}^s(D)$ and $\nabla\times\boldsymbol{A} \in \boldsymbol{H}^{s''}(D)$ with $s, s'' > 0$. In what follows, we set

$$\sigma := \min(s, s''). \tag{44.19}$$

Let us first start with an approximation result on the curl-preserving lifting operator $\boldsymbol{\xi} : \boldsymbol{X}_{h0\nu} \to \boldsymbol{X}_{0\nu}$ defined in the proof of Theorem 44.6. Recall that for all $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}$, the field $\boldsymbol{\xi}(\boldsymbol{x}_h) \in \boldsymbol{X}_{0\nu}$ is s.t. $\boldsymbol{\xi}(\boldsymbol{x}_h) := \boldsymbol{x}_h - \nabla\phi(\boldsymbol{x}_h)$ with $\phi(\boldsymbol{x}_h) \in H^1_0(D)$, implying that $\nabla\times\boldsymbol{\xi}(\boldsymbol{x}_h) = \nabla\times\boldsymbol{x}_h$.

Lemma 44.11 (Curl-preserving lifting). *Let $s > 0$ be the smoothness index introduced in (44.7). There is c, depending on the constant $\check{C}_D$ from (44.7) and the contrast factor $\nu_{\sharp/\flat}$, s.t. for all $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}$ and all $h \in \mathcal{H}$,*

$$\|\boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} \le c\, h^s \ell_D^{1-s} \|\nabla\times\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}. \tag{44.20}$$

Proof. Let us set $\boldsymbol{e}_h := \boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h$. We have seen in the proof of Theorem 44.6 that $\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)) - \boldsymbol{x}_h \in \nabla M_{h0}$, so that $(\nu\boldsymbol{e}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)) - \boldsymbol{x}_h)_{\boldsymbol{L}^2(D)} = 0$ since $\boldsymbol{\xi}(\boldsymbol{x}_h) \in \boldsymbol{X}_{0\nu}$, $M_{h0} \subset M_0$, and $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}$. Since $\boldsymbol{e}_h = (I - \mathcal{J}^{\mathrm{c}}_{h0})(\boldsymbol{\xi}(\boldsymbol{x}_h)) + (\mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)) - \boldsymbol{x}_h)$, we infer that

$$(\nu\boldsymbol{e}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} = (\nu\boldsymbol{e}_h, (I - \mathcal{J}^{\mathrm{c}}_{h0})(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)},$$

thereby implying that $\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} \le \nu_{\sharp/\flat}\|(I - \mathcal{J}^{\mathrm{c}}_{h0})(\boldsymbol{\xi}(\boldsymbol{x}_h))\|_{\boldsymbol{L}^2(D)}$. Using the approximation properties of $\mathcal{J}^{\mathrm{c}}_{h0}$ yields

$$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} \le c\,\nu_{\sharp/\flat} h^s |\boldsymbol{\xi}(\boldsymbol{x}_h)|_{\boldsymbol{H}^s(D)},$$

and we conclude using the bound $|\boldsymbol{\xi}(\boldsymbol{x}_h)|_{\boldsymbol{H}^s(D)} \le \check{C}_D \ell_D^{1-s} \|\nabla\times\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}$ which follows from (44.7) since $\boldsymbol{\xi}(\boldsymbol{x}_h) \in \boldsymbol{X}_{0,\nu}$ and $\nabla\times\boldsymbol{\xi}(\boldsymbol{x}_h) = \nabla\times\boldsymbol{x}_h$. □

Lemma 44.12 (Adjoint solution). *Let $\boldsymbol{y} \in \boldsymbol{X}_{0\nu}$ and let $\boldsymbol{\zeta} \in \boldsymbol{X}_{0\nu}$ solve the (adjoint) problem $\nu\boldsymbol{\zeta} + \nabla\times(\kappa\nabla\times\boldsymbol{\zeta}) := \nu_\flat^{-1}\nu\boldsymbol{y}$. There is c, depending on the constants $\hat{C}_{\mathrm{PS}}$ from (44.9), $\check{C}$, $\check{C}'$ from (44.7)-(44.8), and the contrast factors $\nu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, and $\kappa_\sharp C_{\kappa^{-1}}$, s.t. for all $h \in \mathcal{H}$,*

$$|\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \le c\,\nu_\sharp^{-1}\gamma_{\nu,\kappa}\ell_D^{-\sigma}\|\boldsymbol{y}\|_{L^2(D)}, \tag{44.21a}$$

$$|\nabla\times\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \le c\,\nu_\sharp^{-1}\gamma_{\nu,\kappa}\hat{\gamma}_{\nu,\kappa}\ell_D^{-1-\sigma}\|\boldsymbol{y}\|_{L^2(D)}. \tag{44.21b}$$

Proof. Proof of (44.21a). Testing the adjoint problem with $e^{-\mathrm{i}\theta}\boldsymbol{\zeta}$ leads to $\kappa_\flat\|\nabla\times\boldsymbol{\zeta}\|^2_{\boldsymbol{L}^2(D)} \le \nu_{\sharp/\flat}\|\boldsymbol{y}\|_{L^2(D)}\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$. Using the Poincaré–Steklov inequality (44.9), we can bound $\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$ by $\|\nabla\times\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$, and altogether this gives

$$\|\nabla\times\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)} \le \kappa_\flat^{-1}\nu_{\sharp/\flat}\hat{C}_{\mathrm{PS}}^{-1}\ell_D\|\boldsymbol{y}\|_{\boldsymbol{L}^2(D)}. \tag{44.22}$$

Invoking (44.7) with $\sigma \le s$ yields

$$|\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \le \check{C}_D^{-1}\ell_D^{1-\sigma}\|\nabla\times\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)} \le \kappa_\flat^{-1}\nu_{\sharp/\flat}\check{C}_D^{-1}\hat{C}_{\mathrm{PS}}^{-1}\ell_D^{2-\sigma}\|\boldsymbol{y}\|_{\boldsymbol{L}^2(D)},$$

which proves (44.21a) since $\kappa_\flat^{-1}\ell_D^2 = \kappa_{\sharp/\flat}\nu_\sharp^{-1}\gamma_{\nu,\kappa}$.
Proof of (44.21b). Invoking (44.8) with $\sigma \le s'$ for $\boldsymbol{b} := \kappa\nabla\times\boldsymbol{\zeta}$, which is a member of $\boldsymbol{X}_{*\kappa^{-1}}$, we infer that

$$\check{C}'_D \ell_D^{-1+\sigma}|\boldsymbol{b}|_{\boldsymbol{H}^\sigma(D)} \le \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} = \|\nabla\times(\kappa\nabla\times\boldsymbol{\zeta})\|_{\boldsymbol{L}^2(D)} \\ \le \nu_{\sharp/\flat}\|\boldsymbol{y}\|_{\boldsymbol{L}^2(D)} + \nu_\sharp\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)},$$

by definition of the adjoint solution $\boldsymbol{\zeta}$ and the triangle inequality. Invoking again the Poincaré–Steklov inequality (44.9) to bound $\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$ by $\|\nabla\times\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$ and using (44.22) yields $\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)} \le \kappa_\flat^{-1}\nu_{\sharp/\flat}\hat{C}_{\text{PS}}^{-2}\ell_D^2\|\boldsymbol{y}\|_{L^2(D)}$. As a result, we obtain

$$\check{C}'_D \ell_D^{-1+\sigma}|\boldsymbol{b}|_{\boldsymbol{H}^\sigma(D)} \le \nu_{\sharp/\flat}(1+\nu_\sharp\kappa_\flat^{-1}\hat{C}_{\text{PS}}^{-2}\ell_D^2)\|\boldsymbol{y}\|_{\boldsymbol{L}^2(D)},$$

and this concludes the proof of (44.21b) since $|\nabla\times\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \le C_{\kappa^{-1}}|\boldsymbol{b}|_{\boldsymbol{H}^\sigma(D)}$ owing to the multiplier property (44.11) and $\sigma \le \tau$. □

We can now state the main result of this section.

Theorem 44.13 (Improved L^2-error estimate). *Let $\boldsymbol{A}$ solve* (44.1) *and let $\boldsymbol{A}_h$ solve* (44.13). *There is c, depending on the constants $\hat{C}_{\text{PS}}$ from* (44.9), *$\check{C}$, $\check{C}'$ from* (44.7)-(44.8), *and the contrast factors $\nu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, and $\kappa_\sharp C_{\kappa^{-1}}$, s.t. for all $h \in \mathcal{H}$,*

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{L}^2(D)} \le c \inf_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}}(\|\boldsymbol{A}-\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} + \hat{\gamma}_{\nu,\kappa}^3 h^\sigma \ell_D^{-\sigma}\|\boldsymbol{A}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\text{curl})}).$$

Proof. In this proof, we use the symbol c to denote a generic positive constant that can have the same parametric dependencies as in the above statement. Let $\boldsymbol{v}_h \in \boldsymbol{X}_{h0\nu}$ and let us set $\boldsymbol{x}_h := \boldsymbol{A}_h - \boldsymbol{v}_h$. We observe that $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\nu}$. Let $\boldsymbol{\xi}(\boldsymbol{x}_h)$ be the image of $\boldsymbol{x}_h$ by the curl-preserving lifting operator and let $\boldsymbol{\zeta} \in \boldsymbol{X}_{0\nu}$ be the solution to the following adjoint problem:

$$\nu\boldsymbol{\zeta} + \nabla\times(\kappa\nabla\times\boldsymbol{\zeta}) := \nu_\flat^{-1}\nu\boldsymbol{\xi}(\boldsymbol{x}_h).$$

(1) Let us first bound $\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}$ from above. Recalling that $\boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h = -\nabla\phi(\boldsymbol{x}_h)$ and that $(\nu\boldsymbol{\xi}(\boldsymbol{x}_h), \boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h)_{\boldsymbol{L}^2(D)} = -(\nu\boldsymbol{\xi}(\boldsymbol{x}_h), \nabla\phi(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} = 0$, we infer that

$$\begin{aligned}(\boldsymbol{\xi}(\boldsymbol{x}_h), \nu\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} &= (\boldsymbol{x}_h, \nu\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} \\ &= (\boldsymbol{A}-\boldsymbol{v}_h, \nu\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} + (\boldsymbol{A}_h-\boldsymbol{A}, \nu\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} \\ &= (\boldsymbol{A}-\boldsymbol{v}_h, \nu\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} + \nu_\flat a_{\nu,\kappa}(\boldsymbol{A}_h-\boldsymbol{A}, \boldsymbol{\zeta}) \\ &= (\boldsymbol{A}-\boldsymbol{v}_h, \nu\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} + \nu_\flat a_{\nu,\kappa}(\boldsymbol{A}_h-\boldsymbol{A}, \boldsymbol{\zeta}-\mathcal{J}_{h0}^{\text{c}}(\boldsymbol{\zeta})),\end{aligned}$$

where we used the Galerkin orthogonality property on the fourth line. Since we have $|a_{\nu,\kappa}(\boldsymbol{a},\boldsymbol{b})| \le \kappa_\sharp\ell_D^{-2}\hat{\gamma}_{\nu,\kappa}\|\boldsymbol{a}\|_{\boldsymbol{H}(\text{curl};D)}\|\boldsymbol{b}\|_{\boldsymbol{H}(\text{curl};D)}$ by (43.13b), we infer from the commutation and approximation properties of the quasi-interpolation operators that

$$\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|^2_{\boldsymbol{L}^2(D)} \le \nu_{\sharp/\flat}\|\boldsymbol{A}-\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)}\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}$$
$$+ c\,\kappa_\sharp \ell_D^{-2}\hat{\gamma}_{\nu,\kappa}h^\sigma\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)}(|\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)}+\ell_D|\nabla\times\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)}).$$

Owing to the bounds from Lemma 44.12 on the adjoint solution with $\boldsymbol{y} := \boldsymbol{\xi}(\boldsymbol{x}_h)$, we conclude that

$$\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)} \le \nu_{\sharp/\flat}(\|\boldsymbol{A}-\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} + c\,\hat{\gamma}^2_{\nu,\kappa}h^\sigma\ell_D^{-\sigma}\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)}).$$

(2) The triangle inequality and the identity $\boldsymbol{A}-\boldsymbol{A}_h = \boldsymbol{A}-\boldsymbol{v}_h-\boldsymbol{x}_h$ imply that

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{L}^2(D)} \le \|\boldsymbol{A}-\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{\xi}(\boldsymbol{x}_h)-\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}.$$

We use Lemma 44.11 to bound the second term on the right-hand side as

$$\|\boldsymbol{\xi}(\boldsymbol{x}_h)-\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} \le c\,h^\sigma\ell_D^{1-\sigma}\|\nabla\times\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}$$
$$\le c\,h^\sigma\ell_D^{1-\sigma}(\|\nabla\times(\boldsymbol{A}-\boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} + \|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\boldsymbol{L}^2(D)}),$$

and we use (44.18) to infer that $\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)} \le c\hat{\gamma}_{\nu,\kappa}\|\boldsymbol{A}-\boldsymbol{v}_h\|_{\boldsymbol{H}(\text{curl};D)}$. For the third term on the right-hand side, we use the bound on $\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}$ from Step (1). We conclude by taking the infimum over $\boldsymbol{v}_h \in \boldsymbol{X}_{h0\nu}$, and we use Lemma 44.8 to extend the infimum over $\boldsymbol{V}_{h0}$. □

Remark 44.14 (Literature). The construction of the curl-preserving lifting operator invoked in the proof of Theorem 44.6 and Theorem 44.13 is done in Monk [302, pp. 249-250]. The statement in Lemma 44.11 is similar to that in Monk [303, Lem. 7.6], but the present proof is simplified by the use of the commuting quasi-interpolation operators. The curl-preserving lifting of $\boldsymbol{A}-\boldsymbol{A}_h$ is invoked in Arnold et al. [23, Eq. (9.9)] and denoted therein by $\boldsymbol{\psi}$. The estimate of $\|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(D)}$ given one line above [23, Eq. (9.11)] is similar to (44.3) and is obtained by invoking the commuting quasi-interpolation operators constructed in [23, §5.4] for natural boundary conditions. Note that contrary to the above reference, we invoke the curl-preserving lifting of $\boldsymbol{A}_h-\boldsymbol{v}_h$ instead of $\boldsymbol{A}-\boldsymbol{A}_h$ and make use of Lemma 44.11, which simplifies the argument. Furthermore, the statement of Theorem 44.13 is similar to that of Zhong et al. [405, Thm. 4.1], but the present proof is simpler and does not require the smoothness index σ to be larger than $\frac{1}{2}$. □

Exercises

Exercise 44.1 (Gradient). Let $\phi \in H^1_0(D)$. Prove that $\nabla\phi \in \boldsymbol{H}_0(\text{curl}; D)$

Exercise 44.2 (Vector potential). Let $\boldsymbol{v} \in \boldsymbol{L}^2(D)$ with $(\nu\boldsymbol{v}, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0$ for all $m_h \in M_{h0}$. Prove that $(\nu\boldsymbol{v}, \boldsymbol{w}_h)_{\boldsymbol{L}^2(D)} = (\nabla\times\boldsymbol{z}_h, \nabla\times\boldsymbol{w}_h)_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{w}_h \in \boldsymbol{V}_{h0}$, where $\boldsymbol{z}_h$ solves a curl-curl problem on $\boldsymbol{X}_{h0\nu}$.

Exercise 44.3 (Neumann condition). Recall Remark 44.10. Assume that D is simply connected so that there is $\hat{C}_{\text{PS}} > 0$ such that $\hat{C}_{\text{PS}}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} \leq \|\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{b} \in \boldsymbol{X}_{*\nu}$. Prove that there is $\hat{C}'_{\text{PS}} > 0$ such that $\hat{C}'_{\text{PS}}\ell_D^{-1}|\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} \leq \|\nabla\times\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{b}_h \in \boldsymbol{X}_{h*\nu}$. (*Hint*: adapt the proof of Theorem 44.6 using $\mathcal{J}_h^{\text{c}}$.)

Exercise 44.4 (Discrete Poincaré–Steklov for $\nabla\cdot$). Let ν be as in §44.1.1. Let $\boldsymbol{Y}_{0\nu} := \{\boldsymbol{v} \in \boldsymbol{H}_0(\text{div}; D) \mid (\nu\boldsymbol{v}, \nabla\times\boldsymbol{\phi})_{\boldsymbol{L}^2(D)} = 0,\ \forall\boldsymbol{\phi} \in \boldsymbol{H}_0(\text{curl}; D)\}$ and accept as a fact that there is $\hat{C}_{\text{PS}} > 0$ such that $\hat{C}_{\text{PS}}\ell_D^{-1}\|\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} \leq \|\nabla\cdot\boldsymbol{v}\|_{L^2(D)}$ for all $\boldsymbol{v} \in \boldsymbol{Y}_{0\nu}$. Let $k \geq 0$ and consider the discrete space $\boldsymbol{Y}_{h0\nu} := \{\boldsymbol{v}_h \in \boldsymbol{P}_{k,0}^{\text{d}}(\mathcal{T}_h) \mid (\nu\boldsymbol{v}_h, \nabla\times\boldsymbol{\phi}_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall\boldsymbol{\phi}_h \in \boldsymbol{P}_{k,0}^{\text{c}}(\mathcal{T}_h; \mathbb{C})\}$. Prove that there is $\hat{C}'_{\text{PS}} > 0$ such that $\hat{C}'_{\text{PS}}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \leq \ell_D\|\nabla\cdot\boldsymbol{v}_h\|_{L^2(D)}$ for all $\boldsymbol{v}_h \in \boldsymbol{Y}_{h0\nu}$. (*Hint*: adapt the proof of Theorem 44.6 using $\mathcal{J}_{h0}^{\text{d}}$.)