Part X, Chapter 48

Nonsymmetric problems

In this chapter, we continue our investigation of the finite element approximation of eigenvalue problems, but this time we do not assume symmetry and we explore techniques that can handle nonconforming approximation settings. The main abstract results used in the present chapter are based on a theory popularized in the landmark review article by Babuška and Osborn [38]. Some results are simplified to avoid invoking spectral projections. Our objective is to show how to apply this abstract theory to the conforming and nonconforming approximation of eigenvalue problems arising from variational formulations.

48.1 Abstract theory

In this section, we present an abstract theory for the approximation of the spectrum of compact operators in complex Banach spaces, and we show how to apply it to spectral problems arising from variational formulations.

48.1.1 Approximation of compact operators

Let L be a complex Banach space and $T \in \mathcal{L}(L)$ be a compact operator. We assume that we have at hand a sequence of compact operators $T_n : L \to L$, $n \in \mathbb{N}$, that converges in norm to T i.e., we assume that

$$\lim_{n\to\infty} \|T - T_n\|_{\mathcal{L}(L)} = 0. \tag{48.1}$$

We want to estimate how the eigenpairs of each member in the sequence $(T_n)_{n\in\mathbb{N}}$ approximate some of the eigenpairs of T.

Recall that $\sigma(T)\backslash\{0\} = \sigma_{\mathrm{p}}(T)\backslash\{0\}$ and that the nonzero eigenvalues of T are isolated since T is compact; see Items (ii)-(iii) in Theorem 46.14. Let $\mu \in \sigma_{\mathrm{p}}(T)\backslash\{0\}$ be a nonzero eigenvalue of T. Let α be the ascent of μ.

Recall that α is the smallest integer with the property that $\ker(\mu I_L - T)^\alpha = \ker(\mu I_L - T)^{\alpha+1}$. Denoting by $T^* : L' \to L'$ the adjoint of T, we set

$$G_\mu := \ker(\mu I_L - T)^\alpha, \qquad G_\mu^* := \ker(\overline{\mu} I_{L'} - T^*)^\alpha, \tag{48.2a}$$
$$m := \dim(G_\mu) = \dim(G_\mu^*). \tag{48.2b}$$

Members of G_μ and G_μ^* are called *generalized eigenvectors*. The generalized eigenvectors are all eigenvectors only if $\alpha = 1$. Recall that m is the algebraic multiplicity of μ and that $m \geq \alpha$; see (46.5). Owing to the above assumption on norm convergence, it can be shown that there are m eigenvalues of T_n, say $\{\mu_{n,j}\}_{j\in\{1:m\}}$ (counted with their algebraic multiplicities), that converge to μ as $n \to \infty$. Let $\alpha_{n,j}$ be the ascent of $\mu_{n,j}$ and let us set

$$G_{n,\mu} := \sum_{j\in\{1:m\}} \ker(\mu_{n,j} I_L - T_n)^{\alpha_{n,j}}. \tag{48.3}$$

We want to evaluate how close the subspaces G_μ and $G_{n,\mu}$ are, and for this purpose we define the notion of gap. Given two closed subspaces of L, Y, and Z, we define $\delta(Y,Z) := \sup_{y\in Y; \|y\|_L=1} \operatorname{dist}(y,Z)$, where $\operatorname{dist}(y,Z) := \inf_{z\in Z} \|y - z\|_L$. The gap between Y and Z is defined by $\widehat{\delta}(Y,Z) := \max(\delta(Y,Z), \delta(Z,Y))$.

Theorem 48.1 (Bound on eigenspace gap). *Assume* (48.1). *Let* $\mu \in \sigma_{\mathrm{p}}(T)\backslash\{0\}$. *Let* G_μ *be defined in* (48.2a) *and let* $G_{n,\mu}$ *be defined in* (48.3). *There is c, depending on* μ*, such that for all* $n \in \mathbb{N}$*,*

$$\widehat{\delta}(G_\mu, G_{n,\mu}) \leq c\,\|(T - T_n)_{|G_\mu}\|_{\mathcal{L}(G_\mu;L)}. \tag{48.4}$$

Proof. See Osborn [321, Thm. 1] or Babuška and Osborn [38, Thm. 7.1]. □

Let us now examine the convergence of the eigenvalues. When α, the ascent of μ, is larger than one, it is interesting to consider the convergence of the arithmetic mean of the eigenvalues $\mu_{n,j}$. We will see that this quantity converges faster than any of the $\mu_{n,j}$ (for instance, compare (48.5) and (48.6), and see (48.21) in Theorem 48.8).

Theorem 48.2 (Convergence of eigenvalues). *Assume* (48.1). *Let* $\mu \in \sigma_{\mathrm{p}}(T)\backslash\{0\}$ *with algebraic multiplicity m. Let* $\{\mu_{n,j}\}_{j\in\{1:m\}}$ *be the eigenvalues of* T_n *that converge to* μ *and set* $\langle\mu_n\rangle := \frac{1}{m}\sum_{j\in\{1:m\}} \mu_{n,j}$. *There is c, depending on* μ*, such that for all* $n \in \mathbb{N}$*,*

$$\begin{aligned} |\mu - \langle\mu_n\rangle| \leq \frac{1}{m} \max_{(v,w)\in G_\mu\times G_\mu^*} &\frac{|\langle w, (T-T_n)(v)\rangle_{L',L}|}{\|w\|_{L'}\|v\|_L} \\ &+ c\,\|(T-T_n)_{|G_\mu}\|_{\mathcal{L}(G_\mu;L)} \|(T-T_n)^*_{|G_\mu^*}\|_{\mathcal{L}(G_\mu^*;L')}, \end{aligned} \tag{48.5}$$

and for all $j \in \{1{:}m\}$*,*

$$|\mu - \mu_{n,j}| \le c \left(\max_{(v,w)\in G_\mu \times G_\mu^*} \frac{|\langle w, (T-T_n)(v)\rangle_{L',L}|}{\|w\|_{L'}\|v\|_L} + \|(T-T_n)_{|G_\mu}\|_{\mathcal{L}(G_\mu;L)} \|(T-T_n)^*_{|G_\mu^*}\|_{\mathcal{L}(G_\mu^*;L')} \right)^{\frac{1}{\alpha}}. \tag{48.6}$$

Proof. See [321, Thm. 3&4], [38, Thm. 7.2&7.3], and Exercise 48.3. □

Finally, we evaluate how the vectors in $G_{n,\mu}$ approximate those in G_μ.

Theorem 48.3 (Convergence of eigenvectors). *Assume* (48.1). *Let $\mu \in \sigma_{\rm p}(T)\backslash\{0\}$ with algebraic multiplicity m. Let $\{\mu_{n,j}\}_{j\in\{1:m\}}$ be the eigenvalues of T_n that converge to μ. For all integers $j \in \{1{:}m\}$ and $\ell \in \{1{:}\alpha\}$, let $w_{n,j}$ be a unit vector in $\ker(\mu_{n,j}I_L - T_n)^\ell$. There is c, depending on μ, such that for every integer $\ell' \in \{\ell{:}\alpha\}$, there is a unit vector $u_{\ell'} \in \ker(\mu I_L - T)^{\ell'} \subset G_\mu$ such that for all $n \in \mathbb{N}$,*

$$\|u_{\ell'} - w_{n,j}\|_L \le c\, \|(T-T_n)_{|G_\mu}\|^{\frac{\ell'-\ell+1}{\alpha}}_{\mathcal{L}(G_\mu;L)}. \tag{48.7}$$

Proof. See [321, Thm. 5] or [38, Thm. 7.4]. □

Remark 48.4 (Literature). The above theory has been developed by Bramble and Osborn [78], Osborn [321], Descloux et al. [161, 162]; see Vainikko [368, 369], Strang and Fix [359] for earlier references. Overviews can also be found in Boffi [62], Chatelin [116, Chap. 6]. □

Remark 48.5 (Sharper bounds). The bounds in Theorem 48.2 are simplified versions of the estimates given in [321, Thm. 3&4]. Therein, instead of $\max_{(v,w)\in G_\mu\times G_\mu^*} \frac{|\langle w,(T-T_n)(v)\rangle_{L',L}|}{\|w\|_{L'}\|v\|_L}$, one has $\sum_{j\in\{1:m\}} |\langle \phi_j^*, (T-T_n)(\phi_j)\rangle_{L',L}|$, where $\{\phi_j\}_{j\in\{1:m\}}$ is a basis of G_μ and $\{\phi_j^*\}_{j\in\{1:m\}}$ is a dual basis of G_μ^*, i.e., $\langle \phi_j^*, \phi_k\rangle_{L',L} = \delta_{jk}$ and the action of the forms ϕ_j^* outside G_μ is defined by selecting an appropriate complement of G_μ. The expressions given in Theorem 48.2 will suffice for our purpose. □

48.1.2 Application to variational formulations

Let $V \hookrightarrow L$ be a complex Banach space with compact embedding and let $a : V\times V \to \mathbb{C}$ be a bounded sesquilinear form. We assume that the sesquilinear form a satisfies the two conditions of the BNB theorem (Theorem 25.9), but we do not assume that a is Hermitian. Let $b : L\times L \to \mathbb{C}$ be another bounded sesquilinear form. We now consider the following eigenvalue problem:

$$\begin{cases} \text{Find } \psi \in V\backslash\{0\} \text{ and } \lambda \in \mathbb{C} \text{ such that} \\ a(\psi, w) = \lambda b(\psi, w), \quad \forall w \in V. \end{cases} \tag{48.8}$$

If (λ, ψ) solves (48.8), we say that (λ, ψ) is an eigenpair of the form a relative to the form b, or simply (λ, ψ) is an eigenpair of (48.8) when the context is unambiguous.

To reformulate (48.8) so as to fit the approximation theory of the spectrum of compact operators from §48.1.1, we define the solution operator $T : L \to V \hookrightarrow L$ such that

$$a(T(v), w) := b(v, w), \qquad \forall v \in L,\ \forall w \in V. \tag{48.9}$$

Note that $T(v)$ is well defined for all $v \in L$ since a satisfies the two BNB conditions. Notice also that $\mathrm{im}(T) \subset V$ and that T is injective.

Proposition 48.6 (Spectrum of T). (i) $0 \notin \sigma_{\mathrm{p}}(T)$. (ii) $(\mu, \psi) \in \mathbb{C}\times V$ *is an eigenpair of* T *iff* $(\mu^{-1}, \psi) \in \mathbb{C}\times V$ *is an eigenpair of* (48.8).

Proof. (i) If $(0, \psi)$ is an eigenpair of T (i.e., $\psi \neq 0$), then $a(\psi, v) = 0$ for all $v \in V$, and the inf-sup condition on a implies that $\psi = 0$, which is a contradiction.
(ii) Let (μ, ψ) be an eigenpair of T, i.e., $\mu^{-1}T(\psi) = \psi$ (notice that $\mu \neq 0$ since T is injective). We infer that

$$\mu^{-1}b(\psi, w) = b(\mu^{-1}\psi, w) = a(T(\mu^{-1}\psi), w) = a(\mu^{-1}T(\psi), w) = a(\psi, w),$$

for all $w \in V$. Hence, (μ^{-1}, ψ) is an eigenpair of (48.8). The proof of the converse statement is identical. □

We refer the reader to §46.2 for various examples of spectral problems that can be put into the variational form (48.8). For instance, the model problem (46.21) leads to a sesquilinear form a that is not Hermitian since we have $a(v, w) := \int_D (g'(u_{\mathrm{sw}})\nabla v{\cdot}\nabla\overline{w} + v g''(u_{\mathrm{sw}})\nabla u_{\mathrm{sw}}{\cdot}\nabla\overline{w} - f'(u_{\mathrm{sw}})v\overline{w})\,\mathrm{d}x$, $V := H^1_{\mathrm{per}}(D)$, and $b(v, w) := \int_D v\overline{w}\,\mathrm{d}x$. An example with a sesquilinear form b that is not the L^2-inner product is obtained from the vibrating string model from §46.2.1 by assuming that the string has a nonuniform bounded linear density ρ. In this case, one recovers the model problem (48.8) with $V := H^1_0(D;\mathbb{R})$, $D := (0, \ell)$, where ℓ is the length of the string, $a(v, w) := \int_D \tau\partial_x v\partial_x w\,\mathrm{d}x$, where $\tau > 0$ is the uniform tension of the string, and $b(v, w) := \int_D \rho v w\,\mathrm{d}x$.

48.2 Conforming approximation

The goal of this section is to illustrate the approximation theory from §48.1 when applied to the conforming approximation of the model problem (48.8). Let V be a closed subspace of $H^1(D)$ which, depending on the boundary conditions that are enforced, satisfies $H^1_0(D) \subseteq V \subseteq H^1(D)$. We assume that V is equipped with a norm that is equivalent to that of $H^1(D)$. We assume also that the V-norm is rescaled so the operator norm of the embedding $V \hookrightarrow L^2(D)$ is at most one, e.g., one could set $\|v\|_V := C_{\mathrm{PS}}^{-1}\ell_D\|\nabla v\|_{\boldsymbol{L}^2(D)}$ if

$V := H_0^1(D)$, where C_{PS} is the constant from the Poincaré–Steklov inequality (31.12) in $H_0^1(D)$ and ℓ_D is a characteristic length associated with D, e.g., $\ell_D := \text{diam}(D)$.

Let $T : L^2(D) \to L^2(D)$ be the compact operator defined in (48.9). We identify L and L', so that $T^* = T^{\mathsf{H}}$ (see Lemma 46.15). We want to approximate the spectrum of T assuming that we have at hand an H^1-conforming approximation setting. More precisely, assume that D is a Lipschitz polyhedron and let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine meshes so that each mesh covers D exactly. Let $k \geq 1$ be the polynomial degree of the approximation. We denote by V_h the H^1-conforming finite element space based on $\mathcal{T}_h$ such that $P_{k,0}^{\mathrm{g}}(\mathcal{T}_h) \subseteq V_h \subseteq P_k^{\mathrm{g}}(\mathcal{T}_h)$ and $V_h \subset V$ (see §19.2.1 or §19.4). To avoid being specific on the type of finite element we use, we assume the following best-approximation result:

$$\min_{v_h\in V_h} \|v - v_h\|_V \leq c\, h^r \ell_D |v|_{H^{1+r}(D)}, \tag{48.10}$$

for all $v \in H^{1+r}(D) \cap V$ and all $r \in [0, k]$. We assume that there is $\alpha_0 > 0$ such that for all $h \in \mathcal{H}$,

$$\inf_{v_h\in V_h} \sup_{w_h\in V_h} \frac{|a(v_h, w_h)|}{\|v_h\|_{H^1(D)}\|w_h\|_{H^1(D)}} \geq \alpha_0. \tag{48.11}$$

Since the sesquilinear form b may differ from the L^2-inner product, we additionally introduce the linear operator $S_* : L^2(D) \to V \hookrightarrow L^2(D)$ s.t.

$$a(v, S_*(w)) = (v, w)_{L^2(D)}, \qquad \forall v \in V,\ \forall w \in L^2(D). \tag{48.12}$$

Notice that we use the L^2-inner product on the right-hand side of (48.12) instead of the sesquilinear form b as we did for the definition of T in (48.9). We also assume that the following elliptic regularity pickup holds true for T and S_* (see §31.4.2): There are real numbers $\tau, \tau^* \in (0, 1]$ such that

$$T \in \mathcal{L}(L^2(D); H^{1+\tau}(D)), \qquad S_* \in \mathcal{L}(L^2(D); H^{1+\tau^*}(D)). \tag{48.13}$$

We have $\tau = \tau^* := 1$ when maximal elliptic regularity occurs.

The discrete counterpart of the eigenvalue problem (48.8) is formulated as follows:

$$\begin{cases} \text{Find } \psi_h \in V_h\backslash\{0\} \text{ and } \lambda_h \in \mathbb{C} \text{ such that} \\ a(\psi_h, w_h) = \lambda_h b(\psi_h, w_h), \quad \forall w_h \in V_h. \end{cases} \tag{48.14}$$

We define the discrete solution operator $T_h : L^2(D) \to V_h \subset L^2(D)$ s.t. for all $v \in L^2(D)$, $T_h(v) \in V_h$ is the unique solution to the following problem:

$$a(T_h(v), w_h) = b(v, w_h), \qquad \forall w_h \in V_h.$$

Notice that 0 cannot be an eigenvalue of (48.14) owing to the inf-sup condition (48.11) satisfied by a on $V_h \times V_h$. Moreover, (λ_h, ψ_h) is an eigenpair of (48.14) iff (λ_h^{-1}, ψ_h) is an eigenpair of T_h.

Lemma 48.7 (Bound on $(T - T_h)$). *There is c such that for all $t, t^* \in [0, k]$, all $v \in L^2(D)$ s.t. $T(v) \in H^{1+t}(D)$, all $w \in L^2(D)$ s.t. $S_*(w) \in H^{1+t^*}(D)$, and all $h \in \mathcal{H}$,*

$$\left|((T - T_h)(v), w)_{L^2(D)}\right| \le c\, h^{t+t^*} \|a\| \ell_D^2 |T(v)|_{H^{1+t}(D)} |S_*(w)|_{H^{1+t^*}(D)}. \quad (48.15)$$

Proof. Lemma 26.14 and the best-approximation property (48.10) imply that $\|(T - T_h)(v)\|_V \le ch^t \ell_D |T(v)|_{H^{1+t}(D)}$. Since $(T - T_h)(v) \in V$, the Galerkin orthogonality property and the boundedness of a imply that

$$\begin{aligned}\left|((T - T_h)(v), w)_{L^2(D)}\right| &= |a((T - T_h)(v), S_*(w))| \\ &\le \inf_{w_h \in V_h} |a((T - T_h)(v), S_*(w) - w_h)| \\ &\le \|a\| \, \|(T - T_h)(v)\|_V \inf_{w_h \in V_h} \|S_*(w) - w_h\|_V.\end{aligned}$$

Using the above bound on $(T - T_h)(v)$ and the best-approximation property (48.10) to bound $\|S_*(w) - w_h\|_V$ leads to the expected estimate. □

The estimate (48.15) with $t := \tau$ and $t^* := \tau^*$ combined with the regularity property (48.13) implies that

$$\|T - T_h\|_{\mathcal{L}(L^2;L^2)} \le c\, h^{\tau+\tau^*} \big(\|a\| \ell_D^2 \|T\|_{\mathcal{L}(L^2;H^{1+\tau})} \|S_*\|_{\mathcal{L}(L^2;H^{1+\tau^*})}\big). \quad (48.16)$$

Since $\tau + \tau^* > 0$, this means that $T_h \to T$ in operator norm as $h \to 0$, that is, the key assumption (48.1) holds true. It is then legitimate to use the approximation results for compact operators stated in Theorems 48.1 to 48.3.

Let μ be a nonzero eigenvalue of T of ascent α and algebraic multiplicity m, and let

$$G_\mu := \ker(\mu I_{L^2} - T)^\alpha, \qquad G_\mu^* := \ker(\overline{\mu} I_{L^2} - T^{\mathsf{H}})^\alpha, \quad (48.17)$$

so that $m := \dim(G_\mu) = \dim(G_\mu^*)$ (see (48.2)). Recall that Proposition 48.6 implies that $\lambda := \mu^{-1}$ is an eigenvalue for (48.8). Since the smoothness of the generalized eigenvectors may differ from one eigenvalue to the other, we now define τ_μ and τ_μ^* to be the two largest real numbers in $(0, k]$ such that

$$T_{|G_\mu} \in \mathcal{L}(G_\mu; H^{1+\tau_\mu}(D)), \qquad S_{*|G_\mu^*} \in \mathcal{L}(G_\mu^*; H^{1+\tau_\mu^*}(D)), \quad (48.18)$$

where G_μ and G_μ^* are equipped with the L^2-norm. The two real numbers τ_μ and τ_μ^* measure the smoothness of the generalized eigenvectors in G_μ and G_μ^*, respectively. Notice that $\tau_\mu \in [\tau, k]$ and $\tau_\mu^* \in [\tau^*, k]$, where τ and τ^* are defined in (48.13) and are both in $(0, 1]$. We can set $\tau_\mu = \tau_\mu^* := k$ when maximal smoothness is available. It may happen that $\tau_\mu < \tau_\mu^*$ even if a is

Hermitian. For instance, this may be the case if $b(v,w) := \int_D \rho v\overline{w}\,\mathrm{d}x$, where the function ρ is a bounded discontinuous function.

Owing to the norm convergence of T_h to T as $h \to 0$, there are m eigenvalues of T_h, say $\{\mu_{h,j}\}_{j\in\{1:m\}}$ (counted with their algebraic multiplicities), that converge to μ as $h \to 0$. Let

$$G_{h,\mu} := \sum_{j\in\{1:m\}} \ker(\mu_{h,j} I_{L^2} - T_h)^{\alpha_{h,j}}, \tag{48.19}$$

where $\alpha_{h,j}$ is the ascent of $\mu_{h,j}$. We are now in the position to state the main result of this section.

Theorem 48.8 (Convergence of eigenspace gap, eigenvalues, and eigenvectors). *Let $\mu \in \sigma_{\mathrm{p}}(T) \setminus \{0\}$ with algebraic multiplicity m and let $\{\mu_{h,j}\}_{j\in\{1:m\}}$ be the eigenvalues of T_h that converge to μ. Let G_μ be defined in* (48.17) *and let $G_{h,\mu}$ be defined in* (48.19)*. There is c, depending on μ, such that for all $h \in \mathcal{H}$,*

$$\widehat{\delta}(G_\mu, G_{h,\mu}) \le c\, h^{\tau_\mu + t^*}, \tag{48.20}$$

and letting $\langle \mu_h \rangle := \frac{1}{m}\sum_{j\in\{1:m\}} \mu_{h,j}$, we have

$$|\mu - \langle \mu_h \rangle| \le c\, h^{\tau_\mu + \tau_\mu^*}, \qquad |\mu - \mu_{h,j}| \le c\, h^{\frac{1}{\alpha}(\tau_\mu + \tau_\mu^*)},\ \forall j \in \{1{:}m\}. \tag{48.21}$$

Moreover, for all integers $j \in \{1{:}m\}$ and $\ell \in \{1{:}\alpha\}$, let $w_{h,j}$ be a unit vector in $\ker(\mu_{h,j} I_{L^2} - T_h)^\ell$. There is c, depending on μ, such that for every integer $\ell' \in \{\ell{:}\alpha\}$, there is a unit vector $u_{\ell'} \in \ker(\mu I_{L^2} - T)^{\ell'} \subset G_\mu$ such that for all $h \in \mathcal{H}$,

$$\|u_{\ell'} - w_{h,j}\|_{L^2(D)} \le c\, h^{\frac{\ell' - \ell + 1}{\alpha}(\tau_\mu + \tau^*)}. \tag{48.22}$$

In the above estimates, the constant c depends on $\|a\|\ell_D^2$ and on the operator norms resulting from (48.13) *and* (48.18)*.*

Proof. Using $t := \tau_\mu$ and $t^* := \tau^*$ in (48.15), we infer that

$$\|(T - T_h)_{|G_\mu}\|_{\mathcal{L}(G_\mu;L^2)} = \sup_{v\in G_\mu} \sup_{w\in L^2} \frac{((T-T_h)(v),w)_{L^2}}{\|v\|_{L^2}\|w\|_{L^2}} \le c\, h^{\tau_\mu + \tau^*}.$$

Similarly, using $t := \tau$ and $t^* := \tau_\mu^*$ in (48.15), and recalling that $T^* = T^{\mathsf{H}}$ in the present case, we infer that

$$\begin{aligned}\|(T - T_h)^*|_{G_\mu^*}\|_{\mathcal{L}(G_\mu^*;L^2)} &= \sup_{v\in L^2} \sup_{w\in G_\mu^*} \frac{(v,(T^{\mathsf{H}} - T_h^{\mathsf{H}})(w))_{L^2}}{\|v\|_{L^2}\|w\|_{L^2}} \\ &= \sup_{v\in L^2} \sup_{w\in G_\mu^*} \frac{((T-T_h)(v),w)_{L^2}}{\|v\|_{L^2}\|w\|_{L^2}} \le c\, h^{\tau + \tau_\mu^*}.\end{aligned}$$

Finally, using $t := \tau_\mu$ and $t^* := \tau_\mu^*$ in (48.15), we infer that

$$\sup_{v\in G_\mu}\sup_{w\in G^*_\mu}\frac{((T-T_h)(v),w)_{L^2}}{\|v\|_{L^2}\|w\|_{L^2}}\le c\,h^{\tau_\mu+\tau^*_\mu}.$$

The conclusion follows by applying Theorems 48.1-48.3. □

Remark 48.9 (Convergence rates). Notice that among the two terms that compose the right-hand side in (48.5), it is the first one that dominates when the meshsize goes to zero. The first term scales like $\mathcal{O}(h^{\tau_\mu+\tau^*_\mu})$, whereas the second one scales like $\mathcal{O}(h^{\tau_\mu+\tau^*_\mu+\tau+\tau^*})$ with $\tau+\tau^*>0$. The same observation is valid for (48.6). □

Remark 48.10 (Symmetric case). The estimate (48.21) coincides with the estimate (47.16), and the estimate (48.22) (with $\alpha=\ell=\ell':=1$) coincides with the estimate (47.21) when T is symmetric. Notice though that the estimates from Chapter 47 for the i-th eigenpair depend on the smoothness of all the unit eigenfunctions $\{\psi_n\}_{n\in\{1:i\}}$ (counting the multiplicities), whereas the estimates (48.21)-(48.22) depend only on the smoothness of the unit eigenvectors in G_{μ_i}; see Remark 47.12. □

48.3 Nonconforming approximation

We revisit the theory presented above in a nonconforming context. Typical examples we have in mind are the Crouzeix–Raviart approximation from Chapter 36, Nitsche's boundary penalty technique from Chapter 37, and the discontinuous Galerkin method from Chapter 38. The theory is also applicable to the hybrid high-order method from Chapter 39.

48.3.1 Discrete formulation

We consider again the model problem (48.8) and we want to approximate the spectrum of the operator $T:L^2(D)\to L^2(D)$ defined in (48.9) using an approximation setting that is not conforming in V.

To stay general, we assume that we have at hand a sequence of discrete spaces $(V_h)_{h\in\mathcal{H}}$ with $V_h\not\subset V$. For all $h\in\mathcal{H}$, the sesquilinear form a is approximated by a discrete sesquilinear form $a_h:V_h\times V_h\to\mathbb{C}$, and for simplicity we assume that the sesquilinear form b is meaningful on $V_h\times V_h$, i.e., we assume that $V_h\subset L^2(D)$. The discrete eigenvalue problem is formulated as follows:

$$\begin{cases}\text{Find } \psi_h\in V_h\backslash\{0\} \text{ and } \lambda_h\in\mathbb{C} \text{ such that}\\ a_h(\psi_h,w_h)=\lambda_h b(\psi_h,w_h),\quad \forall w_h\in V_h.\end{cases}\tag{48.23}$$

The discrete solution operator $T_h:L^2(D)\to V_h\subset L^2(D)$ and the adjoint discrete solution operator $S_{*h}:L^2(D)\to V_h\subset L^2(D)$ are defined as follows:

$$a_h(T_h(v), w_h) := b(v, w_h), \qquad \forall (v, w_h) \in L^2(D)\times V_h, \tag{48.24a}$$
$$a_h(v_h, S_{*h}(w)) := (v_h, w)_{L^2(D)}, \qquad \forall (v_h, w) \in V_h\times L^2(D). \tag{48.24b}$$

We assume that T_h and S_{*h} are both well defined, i.e., we assume that a_h satisfies an inf-sup condition on $V_h\times V_h$ uniformly w.r.t. $h \in \mathcal{H}$. As above, (λ_h, ψ_h) is an eigenpair of (48.23) iff (λ_h^{-1}, ψ_h) is an eigenpair of T_h.

To avoid unnecessary technicalities and to stay general, we make the following assumptions:
(i) There exists a dense subspace $V_{\rm S} \hookrightarrow V$ such that the solution operators T and S_* satisfy

$$T(v) \in V_{\rm S}, \qquad S_*(w) \in V_{\rm S}, \qquad \forall v, w \in L^2(D). \tag{48.25}$$

(ii) There is a sesquilinear form $a_\sharp$ extending a_h to $V_\sharp\times V_\sharp$, with $V_\sharp := V_{\rm S} + V_h$, i.e., $a_\sharp(v_h, w_h) = a_h(v_h, w_h)$ for all $v_h, w_h \in V_h$. The space $V_\sharp$ is equipped with a norm $\|\cdot\|_{V_\sharp}$ s.t. there is $\|a_\sharp\|$ such that

$$|a_\sharp(v, w)| \le \|a_\sharp\| \, \|v\|_{V_\sharp} \|w\|_{V_\sharp}, \quad \forall v, w \in V_\sharp,\ \forall h \in \mathcal{H}. \tag{48.26}$$

(iii) The sesquilinear forms $a_\sharp$ and a coincide on $V_{\rm S}\times V_{\rm S}$ so that

$$a_\sharp(T(v), S_*(w)) = a(T(v), S_*(w)), \qquad \forall v, w \in L^2(D). \tag{48.27}$$

(iv) Restricted Galerkin orthogonality and restricted adjoint Galerkin orthogonality, i.e., we have the following identities:

$$a_\sharp(T(v), w_h) = a_h(T_h(v), w_h), \qquad \forall (v, w_h) \in L^2(D)\times(V_h \cap V), \tag{48.28a}$$
$$a_\sharp(v_h, S_*(w)) = a_h(v_h, S_{*h}(w)), \qquad \forall (v_h, w) \in (V_h \cap V)\times L^2(D). \tag{48.28b}$$

(Notice that discrete test functions are restricted to $V_h \cap V$.)
(v) There is c such that for all $h \in \mathcal{H}$,

$$\|T(v) - T_h(v)\|_{V_\sharp} \le c \inf_{v_h \in V_h\cap V} \|T(v) - v_h\|_{V_\sharp}, \tag{48.29a}$$
$$\|S_*(w) - S_{*h}(w)\|_{V_\sharp} \le c \inf_{w_h \in V_h\cap V} \|S_*(w) - w_h\|_{V_\sharp}. \tag{48.29b}$$

Moreover, there is an integer $k \ge 1$, and there is c such that the following best-approximation property holds true for all $t \in [0, k]$, all $v \in H^{1+t}(D)\cap V$, and all $h \in \mathcal{H}$:

$$\inf_{v_h \in V_h\cap V} \|v - v_h\|_{V_\sharp} \le c\, \ell_D h^t |v|_{H^{1+t}(D)}. \tag{48.30}$$

The reader is invited to verify whether all the above conditions are satisfied, with $V_{\rm S} := V \cap H^{1+r}(D)$ and $r > \frac{1}{2}$, by the Crouzeix–Raviart approximation from Chapter 36, Nitsche's boundary penalty technique from Chapter 37, and the Discontinuous Galerkin method from Chapter 38.

48.3.2 Error analysis

We are going to use the general approximation results for compact operators stated in Theorems 48.1-48.3. Let $t_0 \geq 0$ be the smallest real number such that $H^{1+t_0}(D) \cap V \subset V_{\rm S}$. We assume that $t_0 \leq k$, i.e., the interval $[t_0, k]$ is nonempty. In the applications we have in mind, t_0 is a number close to $\frac{1}{2}$ and $k \geq 1$.

Lemma 48.11 (Bound on $(T - T_h)$). *There is c s.t. for all $t, t^* \in [t_0, k]$, all $v \in L^2(D)$ s.t. $T(v) \in H^{1+t}(D)$, all $w \in L^2(D)$ s.t. $S_*(w) \in H^{1+t^*}(D)$, and all $h \in \mathcal{H}$,*

$$\big|((T - T_h)(v), w)_{L^2(D)}\big| \leq c\, h^{t+t^*} \|a_\sharp\| \ell_D^2 |T(v)|_{H^{1+t}(D)} |S_*(w)|_{H^{1+t^*}(D)}. \tag{48.31}$$

Proof. Let $v \in L^2(D)$ be s.t. $T(v) \in H^{1+t}(D)$, and let $w \in L^2(D)$ be s.t. $S_*(w) \in H^{1+t^*}(D)$. We have $T(v) \in H^{1+t}(D) \cap V \subset V_{\rm S}$ since $t \geq t_0$, and $S_*(w) \in H^{1+t^*}(D) \cap V \subset V_{\rm S}$ since $t^* \geq t_0$. Using the definitions of S_* and S_{*h}, the assumption (48.27), i.e., that $a_\sharp$ and a coincide on $V_{\rm S} \times V_{\rm S}$ (and that $a_\sharp$ and a_h coincide over $V_h \times V_h$), and elementary manipulations, we infer that

$$\begin{aligned}
((T - T_h)(v), w)_{L^2(D)} &= a(T(v), S_*(w)) - a_h(T_h(v), S_{*h}(w)) \\
&= a_\sharp(T(v), S_*(w)) - a_\sharp(T_h(v), S_{*h}(w)) \\
&= a_\sharp(T(v) - T_h(v), S_*(w)) + a_\sharp(T_h(v), S_*(w) - S_{*h}(w)) \\
&= a_\sharp(T(v) - T_h(v), S_*(w) - S_{*h}(w)) + a_\sharp(T(v) - T_h(v), S_{*h}(w)) \\
&\quad + a_\sharp(T_h(v), S_*(w) - S_{*h}(w)) =: \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3.
\end{aligned}$$

Owing to the boundedness of $a_\sharp$ on $V_\sharp \times V_\sharp$ and the approximation properties (48.29)-(48.30), we have

$$|\mathfrak{T}_1| \leq c\, h^{t+t^*} \|a_\sharp\| \ell_D^2 |T(v)|_{H^{1+t}(D)} |S_*(w)|_{H^{1+t^*}(D)}.$$

The other two terms have a similar structure that can be dealt with by invoking the restricted Galerkin orthogonality (48.28). For instance, we have

$$\begin{aligned}
|\mathfrak{T}_2| &= \inf_{w_h \in V_h \cap V} |a_\sharp((T - T_h)(v), S_{*h}(w) - w_h)| \\
&\leq \|a_\sharp\| \|(T - T_h)(v)\|_{V_\sharp} \inf_{w_h \in V_h \cap V} \|S_{*h}(w) - w_h\|_{V_\sharp} \\
&\leq c\, \|a_\sharp\| \|(T - T_h)(v)\|_{V_\sharp} (\|S_{*h}(w) - S_*(w)\|_{V_\sharp} + \inf_{v_h \in V_h \cap V} \|S_*(w) - w_h\|_{V_\sharp}) \\
&\leq c'\, h^{t+t^*} \|a_\sharp\| \ell_D^2 |T(v)|_{H^{1+t}(D)} |S_*(w)|_{H^{1+t^*}(D)}.
\end{aligned}$$

The term $\mathfrak{T}_3$ is estimated similarly. □

We assume now that the following elliptic regularity pickup holds true for T and S_* (see §31.4.2): There are real numbers $\tau \in (0, 1]$ and $\tau^* \in (0, 1]$ such that (48.13) holds true. The estimate (48.31) with $t := \tau$ and $t^* := \tau^*$ implies that $\|T - T_h\|_{\mathcal{L}(L^2,L^2)} \leq c\,\|a_\sharp\|\ell_D^2 \|T\|_{\mathcal{L}(L^2;H^{1+\tau})}\|S_*\|_{\mathcal{L}(L^2;H^{1+\tau^*})}h^{\tau+\tau^*}$. Since $\tau+\tau^* > 0$, this means that $T_h \to T$ in operator norm as $h \to 0$, that is, the key assumption (48.1) holds true. It is then legitimate to use the approximation results for compact operators stated in Theorems 48.1-48.3.

Let μ be a nonzero eigenvalue of T of ascent α and algebraic multiplicity m, and let G_μ, G_μ be defined in (48.17). Proposition 48.6 implies that $\lambda := \mu^{-1}$ is an eigenvalue for (48.8). Let τ_μ and τ_μ^* be the two largest real numbers less than or equal to k satisfying (48.18). Recall that $\tau_\mu \in [\tau, k]$ and $\tau_\mu^* \in [\tau^*, k]$. Moreover, we can set $\tau_\mu = \tau_\mu^* := k$ when maximal smoothness is available.

Owing to the norm convergence T_h to T as $h \to 0$, there are m eigenvalues of T_h, say $\{\mu_{h,j}\}_{j\in\{1:m\}}$ (counted with their algebraic multiplicities), that converge to μ as $h \to 0$. Let $G_{h,\mu}$ be defined in (48.19). We are now in the position to state the main result of this section.

Theorem 48.12 (Convergence of eigenspace gap, eigenvalues, and eigenvectors). *Let $\mu \in \sigma_{\mathrm{p}}(T)\backslash\{0\}$ with algebraic multiplicity m and let $\{\mu_{h,j}\}_{j\in\{1:m\}}$ be the eigenvalues of T_h that converge to μ. There is c, depending on μ, s.t. for all $h \in \mathcal{H}$,*

$$\widehat{\delta}(G_\mu, G_{h,\mu}) \leq c\,h^{\tau_\mu+\tau_\mu^*}, \tag{48.32}$$

and letting $\langle\mu_h\rangle := \frac{1}{m}\sum_{j\in\{1:m\}}\mu_{h,j}$, we have

$$|\mu - \langle\mu_h\rangle| \leq c\,h^{\tau_\mu+\tau_\mu^*}, \qquad |\mu - \mu_{h,j}| \leq c\,h^{\frac{1}{\alpha}(\tau_\mu+\tau_\mu^*)},\ \forall j \in \{1:m\}. \tag{48.33}$$

Moreover, for all integers $j \in \{1:m\}$ and $\ell \in \{1:\alpha\}$, let $w_{h,j}$ be a unit vector in $\ker(\mu_{h,j}I_{L^2} - T_h)^\ell$. There is c, depending on μ, such that for every integer $\ell' \in \{\ell:\alpha\}$, there is a unit vector $u_{\ell'} \in \ker(\mu I_{L^2} - T)^{\ell'} \subset G_\mu$ such that for all $h \in \mathcal{H}$,

$$\|u_{\ell'} - w_{h,j}\|_{L^2(D)} \leq c\,h^{\frac{\ell'-\ell+1}{\alpha}(\tau_\mu+\tau^*)}. \tag{48.34}$$

In the above estimates, the constant c depends on $\|a_\sharp\|\ell_D^2$ and on the operator norms defined in (48.13) and (48.18).

Proof. See Exercise 48.4. □

Remark 48.13 (Literature). The nonconforming approximation of the elliptic eigenvalue problem has been studied in Antonietti et al. [12] for discontinuous Galerkin (dG) methods, Gopalakrishnan et al. [220] for hybridizable discontinuous Galerkin (HDG) methods, and Calo et al. [103] for hybrid high-order (HHO) methods. We refer the reader to Canuto [105], Mercier et al. [301], Durán et al. [182], Boffi et al. [63] for mixed and hybrid mixed methods and to Carstensen and Gedicke [109], Liu [287] for guaranteed eigenvalue lower bounds using Crouzeix–Raviart elements. □

Exercises

Exercise 48.1 (Linearity). Consider the setting of §48.1.2. Let $V \hookrightarrow L$ be two complex Banach spaces and $a : V \times V \to \mathbb{C}$ be a bounded sesquilinear form satisfying the two conditions of the BNB theorem. Let $b : L \times L \to \mathbb{C}$ be bounded sesquilinear form. (i) Let $T : L \to L$ be such that $a(T(v), w) := b(v, w)$ for all $v \in L$ and all $w \in V$. Show that T is well defined and linear. (ii) Let $T_* : L \to L$ be such that $a(v, T_*(w)) := b(v, w)$ for all $v \in V$ and all $w \in L$. Show that T_* is well defined and linear.

Exercise 48.2 (Invariant sets). (i) Let $S, T \in \mathcal{L}(V)$ be such that $ST = TS$. Prove that $\ker(S)$ and $\operatorname{im}(S)$ are invariant under T. (ii) Let $T \in \mathcal{L}(V)$ and let $W_1, \ldots, W_m$ be subspaces of V that are invariant under T. Prove that $W_1 + \ldots + W_m$ and $\bigcap_{i\in\{1:m\}} W_i$ are invariant under T. (iii) Let $T \in \mathcal{L}(V)$ and let $\{v_1, \ldots, v_n\}$ be a basis of V. Show that the following statements are equivalent: (a) The matrix of T with respect to $\{v_1, \ldots, v_n\}$ is upper triangular; (b) $T(v_j) \in \operatorname{span}\{v_1, \ldots, v_j\}$ for all $j \in \{1:n\}$; (c) $\operatorname{span}\{v_1, \ldots, v_j\}$ is invariant under T for all $j \in \{1:n\}$. (iv) Let $T \in \mathcal{L}(V)$. Let μ be an eigenvalue of T. Prove that $\operatorname{im}(\mu I_V - T)$ is invariant under T. Prove that $\ker(\mu I_V - T)^\alpha$ is invariant under T for every integer $\alpha \geq 1$.

Exercise 48.3 (Trace). (i) Let V be a complex Banach space. Let $G \subset V$ be a subspace of V of dimension m. Let $\{\phi_j\}_{j\in\{1:m\}}$ and $\{\psi_j\}_{j\in\{1:m\}}$ be two bases of G, and let $\{\phi'_j\}_{j\in\{1:m\}}$ and $\{\psi'_j\}_{j\in\{1:m\}}$ be corresponding dual bases, i.e., $\langle \phi'_i, \phi_j\rangle_{V',V} = \delta_{ij}$, etc. (the way the antilinear forms $\{\phi'_j\}_{j\in\{1:m\}}$ and $\{\psi'_j\}_{j\in\{1:m\}}$ are extended to V does not matter). Let $T \in \mathcal{L}(V)$ and assume that G is invariant under T. Show that $\sum_{j\in\{1:m\}} \langle \psi'_j, T(\psi_j)\rangle_{V',V} = \sum_{j\in\{1:m\}} \langle \phi'_j, T(\phi_j)\rangle_{V',V}$. (ii) Let $B \in \mathbb{C}^{m\times m}$ be s.t. $T(\phi_i) =: \sum_{j\in\{1:m\}} B_{ji}\phi_j$ (recall that G is invariant under T). Let $\mathsf{V} := (\langle \phi'_j, v\rangle_{V',V})^{\mathsf{T}}_{j\in\{1:m\}}$ for all $v \in G$. Prove that $T^\alpha(v) = \sum_{j\in\{1:m\}} (B^\alpha \mathsf{V})_j \phi_j$ for all $\alpha \in \mathbb{N}$. (*Hint*: use an induction argument.) (iii) Let $\mu \in \mathbb{C}$, $\alpha \geq 1$, and $S \in \mathcal{L}(V)$. Assume that $G := \ker(\mu I_V - S)^\alpha$ is finite-dimensional and nontrivial (i.e., $\dim(G) := m \geq 1$). Prove that $\sum_{j\in\{1:m\}} \langle \phi'_j, S(\phi_j)\rangle_{V,V'} = m\mu$. (*Hint*: consider the $m\times m$ matrix A with entries $\langle \phi'_i, (\mu I_V - S)(\phi_j)\rangle_{V',V}$ and show that $A^\alpha = 0$.)

Exercise 48.4 (Theorem 48.12). Prove the estimates in Theorem 48.12. (*Hint*: see the proof of Theorem 48.8.)

Exercise 48.5 (Nonconforming approximation). Consider the Laplace operator with homogeneous Dirichlet boundary conditions in a Lipschitz polyhedron D with $b(v, w) := \int_D \rho v w \,\mathrm{d}x$, where $\rho \in C^\infty(D; \mathbb{R})$. Verify that the assumptions (48.25) to (48.30) hold true for the Crouzeix–Raviart approximation.