Part XI, Chapter 53

Stokes equations: Basic ideas

The Stokes equations constitute the basic linear model for incompressible fluid mechanics. We first derive a weak formulation of the Stokes equations and establish its well-posedness. The approximation is then realized by means of mixed finite elements, that is, we consider a pair of finite elements, where the first component of the pair is used to approximate the velocity and the second component is used to approximate the pressure. Following the ideas of Chapter 50, the finite element pair is said to be stable whenever the discrete velocity and the discrete pressure spaces satisfy an inf-sup condition. In this chapter, we list some classical unstable pairs. Examples of stable pairs are reviewed in the following two chapters.

53.1 Incompressible fluid mechanics

Let D be a Lipschitz domain in $\mathbb{R}^d$. We are interested in modeling the behavior of incompressible fluid flows in D in the time-independent Stokes regime, i.e., the inertial forces are assumed to be negligible. Given a vector-valued field $\boldsymbol{f} : D \to \mathbb{R}^d$ (the body force acting on the fluid) and a scalar-valued field $g : D \to \mathbb{R}$ (the mass production rate), the Stokes problem consists of seeking the velocity field $\boldsymbol{u} : D \to \mathbb{R}^d$ and the pressure field $p : D \to \mathbb{R}$ such that the following balance equations hold true:

$$-\nabla{\cdot}\mathbb{s}(\boldsymbol{u}) + \nabla p = \boldsymbol{f} \quad \text{in } D, \tag{53.1a}$$

$$\nabla{\cdot}\boldsymbol{u} = g \quad \text{in } D, \tag{53.1b}$$

$$\boldsymbol{u}_{|\partial D_{\mathrm{d}}} = \boldsymbol{a}_{\mathrm{d}}, \quad \mathbb{s}(\boldsymbol{u})_{|\partial D_{\mathrm{n}}}\boldsymbol{n} - p_{|\partial D_{\mathrm{n}}}\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}} \quad \text{on } \partial D. \tag{53.1c}$$

The equations (53.1a)-(53.1b) express, respectively, the balance of momentum and mass. The second-order tensor $\mathbb{s}(\boldsymbol{u})$ in (53.1a) is the *viscous stress tensor*. Notice that we abuse the notation in (53.1a) since we should write $\nabla{\cdot}(\mathbb{s}(\boldsymbol{u}))$ instead of $\nabla{\cdot}\mathbb{s}(\boldsymbol{u})$. As for linear elasticity (see §42.1), the principle of conservation of angular momentum implies that $\mathbb{s}(\boldsymbol{u})$ is symmetric and, assuming the fluid to be Newtonian, Galilean invariance implies that

$$\mathbb{s}(\boldsymbol{u}) = 2\mu\mathbb{e}(\boldsymbol{u}) + \lambda(\nabla\cdot\boldsymbol{u})\mathbb{I}, \qquad \mathbb{e}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}), \tag{53.2}$$

where $\mathbb{I}$ is the $d\times d$ identity tensor. The quantity $\mathbb{e}(\boldsymbol{u})$ is called (linearized) *strain rate tensor*, and the constants $\mu > 0$, $\lambda \geq 0$ are the dynamic and bulk viscosities, respectively. In (53.1c), the subsets $\partial D_{\mathrm{d}}, \partial D_{\mathrm{n}}$ form a partition of the boundary ∂D, and we assume for simplicity that $|\partial D_{\mathrm{d}}| > 0$. The boundary data are the prescribed velocity $\boldsymbol{a}_{\mathrm{d}}$ on ∂D_{d} (Dirichlet condition) and the prescribed normal force $\boldsymbol{a}_{\mathrm{n}}$ on ∂D_{n} (Neumann condition).

Remark 53.1 (Total stress tensor). After introducing the total stress tensor $\mathbb{r}(\boldsymbol{u},p) := \mathbb{s}(\boldsymbol{u}) - p\mathbb{I}$, one can rewrite the momentum balance equation (53.1a) in the form $-\nabla\cdot\mathbb{r}(\boldsymbol{u},p) = \boldsymbol{f}$, and the Neumann condition on ∂D_{n} as $\mathbb{r}(\boldsymbol{u},p)_{|\partial D_{\mathrm{n}}}\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}}$. □

Remark 53.2 (Incompressibility). The field $\boldsymbol{u}$ is said to be incompressible, or divergence-free, if $\nabla\cdot\boldsymbol{u} = g = 0$. In the incompressible regime, (53.2) simplifies to $\mathbb{s}(\boldsymbol{u}) = 2\mu\mathbb{e}(\boldsymbol{u})$. □

Remark 53.3 (Laplacian/Cauchy–Navier form). When $g = 0$ and the dynamic viscosity is constant, the momentum equation can be simplified by observing that $\nabla\cdot((\nabla\boldsymbol{u})^{\mathsf{T}}) = \nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{0}$. The momentum equation can then be rewritten in the Laplacian (or *Cauchy–Navier*) form $-\mu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}$, and the Neumann boundary condition becomes $\mu\partial_n\boldsymbol{u}_{|\partial D_{\mathrm{n}}} - p_{|\partial D_{\mathrm{n}}}\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}}$. □

Remark 53.4 (Pressure constant). When $\partial D = \partial D_{\mathrm{d}}$, the data fields g and $\boldsymbol{a}_{\mathrm{d}}$ must satisfy the compatibility condition $\int_D g\,\mathrm{d}x = \int_{\partial D} \boldsymbol{a}_{\mathrm{d}}\cdot\boldsymbol{n}\,\mathrm{d}s$, and the pressure is determined up to an additive constant. This indetermination is usually removed by assuming that $\int_D p\,\mathrm{d}x = 0$. □

Remark 53.5 ($\lambda = 0$). Since $\nabla\cdot(\lambda(\nabla\cdot\boldsymbol{u})\mathbb{I}) = \nabla(\lambda\nabla\cdot\boldsymbol{u})$, we can redefine the pressure and the viscous stress tensor by setting $p' := p - \lambda\nabla\cdot\boldsymbol{u}$ and $\mathbb{s}'(\boldsymbol{u}) := 2\mu\mathbb{e}(\boldsymbol{u})$. Then the momentum balance equation (53.1a) becomes $-\nabla\cdot\mathbb{s}'(\boldsymbol{u}) + \nabla p' = \boldsymbol{f}$. We adopt this change of variable in what follows, i.e., we assume that $\mathbb{s}(\boldsymbol{u}) := 2\mu\mathbb{e}(\boldsymbol{u})$ from now on. □

Remark 53.6 (Homogeneous Dirichlet condition). Let us assume that there is a function $\boldsymbol{u}_{\mathrm{d}}$ (smooth enough) s.t. $(\boldsymbol{u}_{\mathrm{d}})_{|\partial D_{\mathrm{d}}} = \boldsymbol{a}_{\mathrm{d}}$. Then we can make the change of variable $\boldsymbol{u}' := \boldsymbol{u} - \boldsymbol{u}_{\mathrm{d}}$ so that $\boldsymbol{u}'$ satisfies the homogeneous boundary condition $\boldsymbol{u}'_{|\partial D_{\mathrm{d}}} = \boldsymbol{0}$. Upon denoting $\boldsymbol{f}' := \boldsymbol{f} + \nabla\cdot(\mathbb{s}(\boldsymbol{u}_{\mathrm{d}}))$, $g' := g - \nabla\cdot\boldsymbol{u}_{\mathrm{d}}$, and inserting the definition $\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{u}_{\mathrm{d}}$ into (53.1), one observes that the pair $(\boldsymbol{u}', p)$ solves a Stokes problem with homogeneous Dirichlet data and with source terms $\boldsymbol{f}'$ and g'. From now on, we abuse the notation and use the symbols $\boldsymbol{u}$, $\boldsymbol{f}$, g instead of $\boldsymbol{u}'$, $\boldsymbol{f}'$, g'. This is equivalent to assuming that $\boldsymbol{a}_{\mathrm{d}} = \boldsymbol{0}$. □

53.2 Weak formulation and well-posedness

In this section, we present a weak formulation of the Stokes equations and we establish its well-posedness using the Babuška–Brezzi theorem (Theorem 49.13).

53.2.1 Weak formulation

Let $\boldsymbol{w}$ be a sufficiently smooth $\mathbb{R}^d$-valued test function. Since the velocity $\boldsymbol{u}$ vanishes on ∂D_{d}, we only consider test functions $\boldsymbol{w}$ that vanish on ∂D_{d}. Multiplying (53.1a) by $\boldsymbol{w}$ and integrating over D gives

$$-\int_D (\nabla{\cdot}\mathbb{s}(\boldsymbol{u})){\cdot}\boldsymbol{w}\,\mathrm{d}x + \int_D \nabla p{\cdot}\boldsymbol{w}\,\mathrm{d}x = \int_D \boldsymbol{f}{\cdot}\boldsymbol{w}\,\mathrm{d}x.$$

Integrating by parts the term involving the viscous stress tensor, we obtain

$$-\int_D (\nabla{\cdot}\mathbb{s}(\boldsymbol{u})){\cdot}\boldsymbol{w}\,\mathrm{d}x = \int_D \mathbb{s}(\boldsymbol{u}){:}\nabla\boldsymbol{w}\,\mathrm{d}x - \int_{\partial D_{\mathrm{n}}} (\mathbb{s}(\boldsymbol{u})\boldsymbol{n}){\cdot}\boldsymbol{w}\,\mathrm{d}s,$$

where $\boldsymbol{n} := (n_1,\ldots,n_d)^{\mathsf{T}}$ is the outward unit normal to D. The boundary integral over ∂D_{d} is zero since $\boldsymbol{w}$ vanishes on ∂D_{d}. The symmetry of $\mathbb{s}(\boldsymbol{u})$ implies that $\mathbb{s}(\boldsymbol{u}){:}\nabla\boldsymbol{w} = \mathbb{s}(\boldsymbol{u}){:}\mathbb{e}(\boldsymbol{w})$. Similarly, the term $\int_D \nabla p{\cdot}\boldsymbol{w}\,\mathrm{d}x$ is equal to $-\int_D p\nabla{\cdot}\boldsymbol{w}\,\mathrm{d}x + \int_{\partial D_{\mathrm{n}}} p\boldsymbol{n}{\cdot}\boldsymbol{w}\,\mathrm{d}s$. Combining the above equations and using the Neumann boundary condition $\mathbb{s}(\boldsymbol{u})_{|\partial D_{\mathrm{n}}}\boldsymbol{n} - p_{|\partial D_{\mathrm{n}}}\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}}$, the weak form of the momentum equation is

$$\int_D (\mathbb{s}(\boldsymbol{u}){:}\mathbb{e}(\boldsymbol{w}) - p\nabla{\cdot}\boldsymbol{w})\,\mathrm{d}x = \int_D \boldsymbol{f}{\cdot}\boldsymbol{w}\,\mathrm{d}x + \int_{\partial D_{\mathrm{n}}} \boldsymbol{a}_{\mathrm{n}}{\cdot}\boldsymbol{w}\,\mathrm{d}s.$$

The three integrals are well defined if $p \in L^2(D)$, $\boldsymbol{f} \in \boldsymbol{L}^2(D)$, $\boldsymbol{a}_{\mathrm{n}} \in L^2(\partial D_{\mathrm{n}})$, and if $\boldsymbol{u}$, $\boldsymbol{w}$ are in the space

$$\boldsymbol{V}_{\mathrm{d}}(D) := \{\boldsymbol{v} \in \boldsymbol{H}^1(D) \mid \gamma^{\mathrm{g}}(\boldsymbol{v})_{|\partial D_{\mathrm{d}}} = \boldsymbol{0}\}, \tag{53.3}$$

with the $\mathbb{R}^d$-valued trace operator $\gamma^{\mathrm{g}} : \boldsymbol{H}^1(D) \to \boldsymbol{H}^{\frac{1}{2}}(\partial D)$ acting componentwise as the scalar-valued trace operator $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$. We equip the space $\boldsymbol{V}_{\mathrm{d}}$ with the norm $\|\boldsymbol{v}\|_{\boldsymbol{V}_{\mathrm{d}}} := |\boldsymbol{v}|_{\boldsymbol{H}^1(D)} = \|\nabla\boldsymbol{v}\|_{\mathbb{L}^2(D)}$. Since $|\partial D_{\mathrm{d}}| > 0$, we infer from the Poincaré–Steklov inequality (42.9) that there is a constant $\tilde{C}_{\mathrm{PS}} > 0$ s.t. $\tilde{C}_{\mathrm{PS}}\|\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} \le \ell_D\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2(D)}$ for all $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}$ (recall that ℓ_D is a length scale associated with D, e.g., $\ell_D := \mathrm{diam}(D)$). This argument shows that $\|{\cdot}\|_{\boldsymbol{V}_{\mathrm{d}}}$ is a norm on $\boldsymbol{V}_{\mathrm{d}}$, equivalent to the $\|{\cdot}\|_{\boldsymbol{H}^1(D)}$-norm.

A weak formulation of the mass conservation (53.1b) is obtained as above by testing the equation against a sufficiently smooth scalar-valued function q. No integration by parts needs to be performed, and we simply write

$$\int_D q\nabla\cdot\boldsymbol{u}\,\mathrm{d}x = \int_D gq\,\mathrm{d}x.$$

The left-hand side is well defined provided $q \in L^2(D)$ and $\boldsymbol{u}$ is in $\boldsymbol{V}_\mathrm{d}$. Note that if $\partial D = \partial D_\mathrm{d}$, the compatibility condition $\int_D g\,\mathrm{d}x = 0$ implies the equality $\int_D \nabla\cdot\boldsymbol{u}\,\mathrm{d}x = \int_D g\,\mathrm{d}x$, meaning that the mass conservation equation need not be tested against constant functions. In this particular case, the test functions q must be restricted to be of zero mean over D. This motivates the following definition:

$$Q := \begin{cases} L^2(D) & \text{if } \partial D \neq \partial D_\mathrm{d}, \\ L^2_*(D) := \{q \in L^2(D) \mid \int_D q\,\mathrm{d}x = 0\} & \text{if } \partial D = \partial D_\mathrm{d}. \end{cases} \tag{53.4}$$

We equip the space Q with the L^2-norm. Let us define the bilinear forms

$$a(\boldsymbol{v},\boldsymbol{w}) := \int_D \mathbb{s}(\boldsymbol{v}){:}\mathbb{e}(\boldsymbol{w})\,\mathrm{d}x, \qquad b(\boldsymbol{w},q) := -\int_D q\nabla\cdot\boldsymbol{w}\,\mathrm{d}x, \tag{53.5}$$

on $\boldsymbol{V}_\mathrm{d}\times\boldsymbol{V}_\mathrm{d}$ and $\boldsymbol{V}_\mathrm{d}\times Q$, respectively. We also define the linear forms $F(\boldsymbol{w}) := \int_D \boldsymbol{f}\cdot\boldsymbol{w}\,\mathrm{d}x + \int_{\partial D_\mathrm{n}} \boldsymbol{a}_\mathrm{n}\cdot\boldsymbol{w}\,\mathrm{d}s$, $G(q) := -\int_D gq\,\mathrm{d}x$ on $\boldsymbol{V}_\mathrm{d}$ and Q, respectively. Assuming enough smoothness on $\boldsymbol{f}$, $\boldsymbol{a}_\mathrm{n}$, and g, it is reasonable to expect that $F \in \mathcal{L}(\boldsymbol{V}_\mathrm{d};\mathbb{R})$ and $G \in \mathcal{L}(Q;\mathbb{R})$. We obtain the following weak formulation:

$$\begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V}_\mathrm{d} \text{ and } p \in Q \text{ such that} \\ a(\boldsymbol{u},\boldsymbol{w}) + b(\boldsymbol{w},p) = F(\boldsymbol{w}), & \forall \boldsymbol{w} \in \boldsymbol{V}_\mathrm{d}, \\ b(\boldsymbol{u},q) = G(q), & \forall q \in Q. \end{cases} \tag{53.6}$$

Proposition 53.7 (Weak solution). *Assume $\boldsymbol{f} \in \boldsymbol{L}^2(D)$, $g \in Q$, and $\boldsymbol{a}_\mathrm{n} \in \boldsymbol{L}^2(\partial D_\mathrm{n})$. Any weak solution $(\boldsymbol{u},p)$ to* (53.6) *satisfies* (53.1a)-(53.1b) *a.e. in D and satisfies the boundary condition* (53.1c) *a.e. on ∂D.*

Proof. Let us set $\mathbb{r}(\boldsymbol{u},p) := \mathbb{s}(\boldsymbol{u}) - p\mathbb{I} \in \mathbb{L}^2(D)$. Testing the momentum equation in (53.6) against an arbitrary function $\boldsymbol{w} \in \boldsymbol{C}_0^\infty(D)$, we infer that $\mathbb{r}(\boldsymbol{u},p)$ has a weak divergence in $\boldsymbol{L}^2(D)$ equal to $-\boldsymbol{f}$. Since $\nabla\cdot\mathbb{r}(\boldsymbol{u},p) = \nabla\cdot\mathbb{s}(\boldsymbol{u}) - \nabla p$, we infer that (53.1a) is satisfied a.e. in D. Testing the mass equation in (53.6) against an arbitrary function $q \in C_0^\infty(D)$, we infer that (53.1b) is satisfied a.e. in D (if $\partial D = \partial D_\mathrm{d}$, the compatibility condition $\int_D g\,\mathrm{d}x = 0$ implies that $b(\boldsymbol{u},q) = G(q)$ for all $q \in L^2(D)$). The Dirichlet boundary condition $\boldsymbol{u}_{|\partial D_\mathrm{d}} = \boldsymbol{0}$ is a natural consequence of the trace theorem (Theorem 3.10) and $\boldsymbol{u}$ being in $\boldsymbol{V}_\mathrm{d}$. To derive the Neumann condition, we proceed as in §31.3.3. Since $\nabla\cdot\mathbb{r}(\boldsymbol{u},p) = -\boldsymbol{f} \in \boldsymbol{L}^2(D)$, we have $\mathbb{r}(\boldsymbol{u},p) \in \mathbb{H}(\mathrm{div};D)$ (i.e., each row of $\mathbb{r}(\boldsymbol{u},p)$ is in $\boldsymbol{H}(\mathrm{div};D)$). Owing to Theorem 4.15, we infer that $\mathbb{r}(\boldsymbol{u},p)\boldsymbol{n} \in \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$. As a result, we have

$$\langle \mathbb{r}(\boldsymbol{u},p)\boldsymbol{n}, \gamma^{\mathrm{g}}(\boldsymbol{w})\rangle_{\partial D} = \int_D (\mathbb{r}(\boldsymbol{u},p){:}\nabla\boldsymbol{w} + (\nabla{\cdot}\mathbb{r}(\boldsymbol{u},p)){\cdot}\boldsymbol{w})\,\mathrm{d}x$$

$$= \int_D (\mathbb{s}(\boldsymbol{u}){:}\mathbb{e}(\boldsymbol{w}) - p\nabla{\cdot}\boldsymbol{w} - \boldsymbol{f}{\cdot}\boldsymbol{w})\,\mathrm{d}x = \int_{\partial D_{\mathrm{n}}} \boldsymbol{a}_{\mathrm{n}}{\cdot}\gamma^{\mathrm{g}}(\boldsymbol{w})\,\mathrm{d}s, \qquad \forall \boldsymbol{w} \in \boldsymbol{V}_{\mathrm{d}},$$

which implies that the Neumann condition $\mathbb{r}(\boldsymbol{u},p)\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}}$ is satisfied in $\widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\partial D_{\mathrm{n}})'$, with $\widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\partial D_{\mathrm{n}}) := \{\boldsymbol{v} \in \boldsymbol{H}^{\frac{1}{2}}(\partial D_{\mathrm{n}}) \mid \widetilde{\boldsymbol{v}} \in \boldsymbol{H}^{\frac{1}{2}}(\partial D)\}$ (recall that $\widetilde{\boldsymbol{v}}$ is the zero extension of $\boldsymbol{v}$ to ∂D). Actually, the Neumann condition is satisfied a.e. on ∂D_{n} since we assumed $\boldsymbol{a}_{\mathrm{n}} \in \boldsymbol{L}^2(\partial D_{\mathrm{n}})$. □

Remark 53.8 (Neumann data). The above proof shows that it is possible to take more generally $\boldsymbol{a}_{\mathrm{n}} \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\partial D_{\mathrm{n}})'$. □

53.2.2 Well-posedness

One readily sees that the bilinear form $a(\boldsymbol{v},\boldsymbol{w}) := (\mathbb{s}(\boldsymbol{v}), \mathbb{e}(\boldsymbol{w}))_{\mathbb{L}^2(D)}$ defined in (53.5) is coercive and bounded on $\boldsymbol{V}_{\mathrm{d}}{\times}\boldsymbol{V}_{\mathrm{d}}$. The coercivity of a has been established in Theorem 42.11 as a consequence of Korn's inequalities. In particular, there is $C_{\mathrm{K}} > 0$ s.t. $\|\mathbb{e}(\boldsymbol{v})\|_{\mathbb{L}^2(D)} \ge C_{\mathrm{K}}|\boldsymbol{v}|_{\boldsymbol{H}^1(D)}$ for all $\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}$, and this implies that (see (42.15) with $\rho_{\min} := 2\mu$ in the present setting)

$$a(\boldsymbol{v},\boldsymbol{v}) \ge 2\mu C_{\mathrm{K}}^2 |\boldsymbol{v}|^2_{\boldsymbol{H}^1(D)}, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}}. \tag{53.7}$$

Moreover, the Cauchy–Schwarz inequality and the bound $\|\mathbb{e}(\boldsymbol{v})\|_{\mathbb{L}^2(D)} \le |\boldsymbol{v}|_{\boldsymbol{H}^1(D)}$ show that the boundedness constant of the bilinear form a satisfies $\|a\| \le 2\mu$. Hence, the key argument for the well-posedness of the Stokes problem is the surjectivity of the divergence operator $\nabla{\cdot} : \boldsymbol{V}_{\mathrm{d}} \to Q$. This result is a bit more subtle than Lemma 51.2 since $\boldsymbol{V}_{\mathrm{d}}$ is a smaller space than $\boldsymbol{H}(\mathrm{div}; D)$.

Lemma 53.9 ($\nabla\cdot$ is surjective). *Let D be a Lipschitz domain in $\mathbb{R}^d$.* (i) *Case $\partial D = \partial D_{\mathrm{d}}$. $\nabla{\cdot} : \boldsymbol{H}^1_0(D) \to L^2_*(D)$ is surjective.* (ii) *Case $\partial D \ne \partial D_{\mathrm{d}}$. Consider the partition $\partial D = \partial D_{\mathrm{d}} \cup \partial D_{\mathrm{n}}$ with $|\partial D_{\mathrm{d}}| > 0$. Assume that $|\partial D_{\mathrm{n}}| > 0$ and that there exists a subset $\mathcal{O}$ of ∂D_{n} with $|\mathcal{O}| > 0$ and $\boldsymbol{n}_{|\mathcal{O}} \in \boldsymbol{H}^{\frac{1}{2}}(\mathcal{O})$. Then the operator $\nabla{\cdot} : \boldsymbol{X} := \{\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}} \mid \gamma^{\mathrm{g}}(\boldsymbol{v})_{|\partial D_{\mathrm{n}}}{\times}\boldsymbol{n} = \boldsymbol{0}\} \to L^2(D)$ is surjective.* (iii) *In all the cases, identifying Q' with Q we have*

$$\inf_{q\in Q} \sup_{\boldsymbol{v}\in\boldsymbol{V}_{\mathrm{d}}} \frac{|\int_D q\nabla{\cdot}\boldsymbol{v}\,\mathrm{d}x|}{\|q\|_{L^2(D)}|\boldsymbol{v}|_{\boldsymbol{H}^1(D)}} := \beta_D > 0. \tag{53.8}$$

Proof. (i) We refer the reader to Girault and Raviart [217, pp. 18-26] for a proof of the surjectivity of $\nabla{\cdot} : \boldsymbol{H}^1_0(D) \to L^2_*(D)$, (see also Exercise 53.1 if D is a smooth domain). (ii) Let us now consider the second case. Let q be in $L^2(D)$. Let ρ be a smooth nonnegative function compactly supported in $\mathcal{O}$ such that $\int_{\mathcal{O}} \rho\,\mathrm{d}s > 0$ (this is possible since $|\mathcal{O}| > 0$). Let $\boldsymbol{g} := c\rho\boldsymbol{n}$ be a vector field in $\mathcal{O}$, where the constant c is chosen s.t. $\int_{\mathcal{O}} \boldsymbol{g}{\cdot}\boldsymbol{n}\,\mathrm{d}s = \int_D q\,\mathrm{d}x$. Let $\widetilde{\boldsymbol{g}}$ be the zero extension of $\boldsymbol{g}$ to ∂D. Since $\boldsymbol{n}_{|\mathcal{O}} \in \boldsymbol{H}^{\frac{1}{2}}(\mathcal{O})$, we have $\rho\boldsymbol{n} \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\mathcal{O})$. Hence, $\widetilde{\boldsymbol{g}}$ is in $\boldsymbol{H}^{\frac{1}{2}}(\partial D)$ so that it is possible to find a function $\boldsymbol{w}$ in $\boldsymbol{H}^1(D)$ s.t. $\gamma^{\mathrm{g}}(\boldsymbol{w}) = \widetilde{\boldsymbol{g}}$ on ∂D. We have $\gamma^{\mathrm{g}}(\boldsymbol{w})_{|\partial D_{\mathrm{d}}} = \boldsymbol{0}$ and $\gamma^{\mathrm{g}}(\boldsymbol{w})_{|\partial D_{\mathrm{n}}} \times \boldsymbol{n} = \boldsymbol{0}$, i.e., $\boldsymbol{w} \in \boldsymbol{X}$. Now let $q_0 := \nabla\cdot\boldsymbol{w} - q$. The above definitions and the divergence formula imply that $q_0 \in L^2(D)$ and $\int_D q_0 \,\mathrm{d}x = 0$. Hence, q_0 is in $L^2_*(D)$. Since $\nabla\cdot : \boldsymbol{H}^1_0(D) \to L^2_*(D)$ is surjective, there is $\boldsymbol{w}_0 \in \boldsymbol{H}^1_0(D)$ such that $\nabla\cdot\boldsymbol{w}_0 = -q_0$. Thus, for all q in $L^2(D)$ the function $\boldsymbol{w} + \boldsymbol{w}_0$ is in $\boldsymbol{X}$ with $\nabla\cdot(\boldsymbol{w} + \boldsymbol{w}_0) = q$, that is, $\nabla\cdot\boldsymbol{X} \to L^2(D)$ is surjective. This also implies that $\nabla\cdot : \boldsymbol{V}_{\mathrm{d}} \to Q$ is surjective. (iii) The inf-sup condition (53.8) follows from the surjectivity of $\nabla\cdot : \boldsymbol{V}_{\mathrm{d}} \to Q$ and Lemma C.40. □

Remark 53.10 (Inf-sup condition in $\boldsymbol{W}^{1,p}$-$L^{p'}$). Let $p \in (1,\infty)$ and let $p' \in (1,\infty)$ be s.t. $\frac{1}{p} + \frac{1}{p'} = 1$. Then the operator $\nabla\cdot : \boldsymbol{W}^{1,p}_0(D) \to L^p_*(D) := \{q \in L^p(D) \mid \int_D q \,\mathrm{d}x = 0\}$ is surjective (see Auscher et al. [30, Lem. 10]), that is, identifying $(L^p_*(D))'$ with $L^{p'}_*(D)$, we have

$$\inf_{q\in L^{p'}_*(D)} \sup_{\boldsymbol{v}\in \boldsymbol{W}^{1,p}_0(D)} \frac{|\int_D q\nabla\cdot\boldsymbol{v}\,\mathrm{d}x|}{\|q\|_{L^{p'}(D)}|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}} := \beta_{D,p} > 0. \tag{53.9}$$

The assumption that D is Lipschitz can be weakened. For instance, the inf-sup condition (53.9) holds true also if D is a bounded open set in $\mathbb{R}^d$ and if D is star-shaped with respect to an open ball $B \subset D$, i.e., for all $\boldsymbol{x} \in D$ and $\boldsymbol{z} \in B$, the segment joining $\boldsymbol{x}$ and $\boldsymbol{z}$ is contained in D; see Bogovskiĭ [66], Galdi [210, Lem. 3.1, Chap. III], Durán and Muschietti [181], Durán et al. [180], Solonnikov [349, Prop. 2.1], Costabel and McIntosh [146]. □

Let $B : \boldsymbol{V}_{\mathrm{d}} \to Q'$ be s.t. $\langle B(\boldsymbol{v}), q\rangle_{Q',Q} := b(\boldsymbol{v}, q) = -\int_D q(\nabla\cdot\boldsymbol{v})\,\mathrm{d}x$. Identifying Q and Q', we have $B(\boldsymbol{v}) = -\nabla\cdot\boldsymbol{v}$, and $\ker(B) := \{\boldsymbol{v} \in \boldsymbol{V}_{\mathrm{d}} \mid \nabla\cdot\boldsymbol{v} = 0\}$.

Theorem 53.11 (Well-posedness). (i) *The weak formulation* (53.6) *of the Stokes problem is well-posed.* (ii) *There is c such that for all $\boldsymbol{f} \in \boldsymbol{L}^2(D)$, all $g \in Q$, and all $\boldsymbol{a}_{\mathrm{n}} \in \boldsymbol{L}^2(\partial D_{\mathrm{n}})$,*

$$2\mu|\boldsymbol{u}|_{\boldsymbol{H}^1(D)} + \|p\|_{L^2(D)} \le c\left(\ell_D\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)} + \mu\|g\|_{L^2(D)} + \ell_D^{\frac{1}{2}}\|\boldsymbol{a}_{\mathrm{n}}\|_{\boldsymbol{L}^2(\partial D_{\mathrm{n}})}\right).$$

Proof. We apply the Babuška–Brezzi theorem (Theorem 49.13). The inf-sup condition (49.37) on the bilinear form b follows from Lemma 53.9. The two conditions in (49.36) are satisfied owing to the coercivity of the bilinear form a on $\boldsymbol{V}_{\mathrm{d}}$ (see (53.7)). Finally, the stability estimate follows from (49.38). □

One can formulate a more precise stability result on the product space $Y := \boldsymbol{V}_{\mathrm{d}}\times Q$ equipped with the norm $\|(\boldsymbol{v}, q)\|^2_Y := \mu|\boldsymbol{v}|^2_{\boldsymbol{H}^1(D)} + \mu^{-1}\|p\|^2_{L^2(D)}$, and the bilinear form $t((\boldsymbol{v}, q), (\boldsymbol{w}, r)) := a(\boldsymbol{v}, \boldsymbol{w}) + b(\boldsymbol{w}, q) - b(\boldsymbol{v}, r)$ on $Y\times Y$.

Lemma 53.12 (Inf-sup condition). *The following holds true:*

$$\inf_{(\boldsymbol{v},q)\in Y} \sup_{(\boldsymbol{w},r)\in Y} \frac{|t((\boldsymbol{v},q),(\boldsymbol{w},r))|}{\|(\boldsymbol{v},q)\|_Y \|(\boldsymbol{w},r)\|_Y} =: \gamma > 0, \tag{53.10}$$

where γ is uniform w.r.t. $\mu > 0$.

Proof. Let $(\boldsymbol{v},q) \in Y$ and let us set $\mathbb{S} := \sup_{(\boldsymbol{w},r)\in Y} \frac{|t((\boldsymbol{v},q),(\boldsymbol{w},r))|}{\|(\boldsymbol{w},r)\|_Y}$. Owing to (53.7), we have

$$2\mu C_{\mathrm{K}}^2 |\boldsymbol{v}|^2_{\boldsymbol{H}^1(D)} \le a(\boldsymbol{v},\boldsymbol{v}) = t((\boldsymbol{v},q),(\boldsymbol{v},q)) \le \mathbb{S}\|(\boldsymbol{v},q)\|_Y. \tag{53.11}$$

Moreover, owing to Lemma 53.9, there is $\boldsymbol{w}_q \in \boldsymbol{V}_{\mathrm{d}}$ s.t. $\nabla\cdot\boldsymbol{w}_q = -\mu^{-1}q$ and $|\boldsymbol{w}_q|_{\boldsymbol{H}^1(D)} \le (\beta_D\mu)^{-1}\|q\|_{L^2(D)}$. We obtain

$$\begin{aligned}
\mu^{-1}\|q\|^2_{L^2(D)} &= -(q,\nabla\cdot\boldsymbol{w}_q) = -t((\boldsymbol{v},q),(\boldsymbol{w}_q,0)) + a(\boldsymbol{v},\boldsymbol{w}_q)\\
&\le \mathbb{S}\mu^{\frac12}|\boldsymbol{w}_q|_{\boldsymbol{H}^1(D)} + 2\mu^{\frac12}|\boldsymbol{v}|_{\boldsymbol{H}^1(D)}\mu^{\frac12}|\boldsymbol{w}_q|_{\boldsymbol{H}^1(D)}\\
&\le c'\,(\mathbb{S} + \mathbb{S}^{\frac12}\|(\boldsymbol{v},q)\|_Y^{\frac12})\mu^{\frac12}|\boldsymbol{w}_q|_{\boldsymbol{H}^1(D)},
\end{aligned}$$

where we used that $|a(\boldsymbol{v},\boldsymbol{w})| \le 2\mu|\boldsymbol{v}|_{\boldsymbol{H}^1(D)}|\boldsymbol{w}|_{\boldsymbol{H}^1(D)}$ and then (53.11). Using the bound on $|\boldsymbol{w}_q|_{\boldsymbol{H}^1(D)}$ and Young's inequality leads to

$$\mu^{-1}\|q\|^2_{L^2(D)} \le c\,(\mathbb{S}^2 + \mathbb{S}\|(\boldsymbol{v},q)\|_Y).$$

We can now combine this bound with (53.11) to infer that

$$\|(\boldsymbol{v},q)\|^2_Y \le c\,(\mathbb{S}^2 + \mathbb{S}\|(\boldsymbol{v},q)\|_Y).$$

Applying one more time Young's inequality yields $\|(\boldsymbol{v},q)\|_Y \le c\mathbb{S}$. □

Remark 53.13 (Helmholtz decomposition). Letting $H^1_*(D) := H^1(D)\cap L^2_*(D)$ and $\boldsymbol{\mathcal{H}} := \{\boldsymbol{v} \in \boldsymbol{L}^2(D) \mid \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{n} = 0\}$, the following $\boldsymbol{L}^2$-orthogonal Helmholtz decomposition holds true: $\boldsymbol{L}^2(D) = \boldsymbol{\mathcal{H}}\oplus\nabla(H^1_*(D))$ (see Lemma 74.1). The $\boldsymbol{L}^2$-orthogonal projection $\boldsymbol{P_{\mathcal{H}}} : \boldsymbol{L}^2(D) \to \boldsymbol{\mathcal{H}}$ resulting from this decomposition is often called *Leray projection*. Let $(\boldsymbol{u},p)$ solve (53.6). Assume for simplicity that the homogeneous Dirichlet condition $\boldsymbol{u}_{|\partial D} = \boldsymbol{0}$ is enforced over the whole boundary and assume that $g = 0$. Since $\boldsymbol{u}$ is divergence-free and vanishes at the boundary, we have $(\boldsymbol{f},\boldsymbol{u})_{\boldsymbol{L}^2(D)} = (\boldsymbol{P_{\mathcal{H}}}(\boldsymbol{f}),\boldsymbol{u})_{\boldsymbol{L}^2(D)}$. Then taking $\boldsymbol{w} := \boldsymbol{u}$ in (53.6) and invoking the coercivity of a shows that $2\mu C_{\mathrm{K}}^2|\boldsymbol{u}|^2_{\boldsymbol{H}^1(D)} \le a(\boldsymbol{u},\boldsymbol{u}) = (\boldsymbol{P_{\mathcal{H}}}(\boldsymbol{f}),\boldsymbol{u})_{\boldsymbol{L}^2(D)}$. Owing to the Cauchy–Schwarz inequality and the Poincaré–Steklov inequality, we get

$$2\mu|\boldsymbol{u}|_{\boldsymbol{H}^1(D)} \le C_{\mathrm{PS}}^{-1}C_{\mathrm{K}}^{-2}\ell_D\|\boldsymbol{P_{\mathcal{H}}}(\boldsymbol{f})\|_{\boldsymbol{L}^2(D)}.$$

This a priori estimate on the velocity is sharper than the one from Theorem 53.11 since $\|\boldsymbol{P_{\mathcal{H}}}(\boldsymbol{f})\|_{\boldsymbol{L}^2(D)}$ appears on the right-hand side instead of $\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}$. One should bear in mind that, even if $p \in H^1_*(D)$, the fields

$-\nabla{\cdot}\mathbb{s}(\boldsymbol{u})$ and $\boldsymbol{P}_{\mathcal{H}}(\boldsymbol{f})$ are generally different since the normal component of $\nabla{\cdot}\mathbb{s}(\boldsymbol{u})$ at ∂D is generally nonzero. □

53.2.3 Regularity pickup

Regularity properties for the Stokes problems can be established when μ and λ are both constant (or smooth) and $|\partial D_{\mathrm{n}}| = 0$. For instance, if ∂D is of class C^{∞}, for all $s > 0$ there is c, depending on D and s, such that

$$\mu\ell_D^{-1}\|\boldsymbol{u}\|_{\boldsymbol{H}^{1+s}(D)} + \|p\|_{H^s(D)} \le c\,(\ell_D\|\boldsymbol{f}\|_{\boldsymbol{H}^{s-1}(D)} + \mu\|g\|_{H^s(D)}). \tag{53.12}$$

There is an upper limit on s when D is not smooth. For instance, let D be a two-dimensional convex polygon. Let $\rho : D \to \mathbb{R}$ be the distance to the closest vertex of D. It is shown in Kellogg and Osborn [266, Thm. 2] that there is a constant c that depends only on D such that

$$\mu|\boldsymbol{u}|_{\boldsymbol{H}^2(D)}+|p|_{H^1(D)} \le c\left(\|\boldsymbol{f}\|_{\boldsymbol{L}^2(D)}+\mu\ell_D^{-1}(|g|_{H^1(D)}+\|\rho^{-1}g\|_{L^2(D)})\right). \tag{53.13}$$

The situation is a bit more complicated in dimension three. We refer to Dauge [153] for an overview of the problem. Assuming that $g = 0$, it is shown in [153, p. 75] that (53.12) holds true in the following situations: (i) For all $s \le 1$ if D is a convex polyhedron; (ii) For all $s < \frac{3}{2}$ if D is any convex domain with wedge angles $\le \frac{2}{3}\pi$; (iii) For all $s < \frac{1}{2}$ if D has a piecewise smooth boundary, and its faces meet two by two or three by three with independent normal vectors at the meeting points.

53.3 Conforming approximation

In the rest of this chapter, we assume that D is a polyhedron in $\mathbb{R}^d$ and $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is a shape-regular sequence of matching meshes so that each mesh covers D exactly. We also assume that ∂D_{d} is a union of mesh faces. Let $(\boldsymbol{V}_{h\mathrm{d}} \subset \boldsymbol{V}_{\mathrm{d}})_{h\in\mathcal{H}}$ and $(Q_h \subset Q)_{h\in\mathcal{H}}$ be sequences of finite-dimensional spaces built using $(\mathcal{T}_h)_{h\in\mathcal{H}}$. Notice that the inclusion $\boldsymbol{V}_{h\mathrm{d}} \subset \boldsymbol{V}_{\mathrm{d}}$ means that the homogeneous Dirichlet condition on the velocity is strongly enforced on ∂D_{d}. The discrete counterpart of the problem (53.6) is as follows:

$$\begin{cases} \text{Find } \boldsymbol{u}_h \in \boldsymbol{V}_{h\mathrm{d}} \text{ and } p_h \in Q_h \text{ such that} \\ a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = F(\boldsymbol{v}_h), & \forall \boldsymbol{v}_h \in \boldsymbol{V}_{h\mathrm{d}}, \\ b(\boldsymbol{u}_h, q_h) = G(q_h), & \forall q_h \in Q_h. \end{cases} \tag{53.14}$$

Since $\boldsymbol{V}_{h\mathrm{d}}$ is $\boldsymbol{V}_{\mathrm{d}}$-conforming, the discrete formulation inherits the coercivity of a. Unfortunately, there is no reason a priori for the discrete formulation to inherit the surjectivity of the divergence operator established in Lemma 53.9.

Verifying this condition is the crucial step in devising stable mixed finite elements for the Stokes problem.

Proposition 53.14 (Well-posedness). *The discrete problem* (53.14) *is well-posed if and only if the following inf-sup condition holds true:*

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h\mathrm{d}}} \frac{|\int_D q_h \nabla\cdot \boldsymbol{v}_h \,\mathrm{d}x|}{\|q_h\|_{L^2(D)} |\boldsymbol{v}_h|_{\boldsymbol{H}^1(D)}} =: \beta_h > 0. \tag{53.15}$$

Proof. Apply Proposition 50.1. □

We henceforth say that the inf-sup condition (53.15) holds uniformly w.r.t. $h \in \mathcal{H}$ if $\inf_{h\in\mathcal{H}} \beta_h =: \beta_0 > 0$.

Definition 53.15 (Stable/unstable pair). *We say that a pair of finite elements used to approximate the velocity and the pressure is* stable *if the inf-sup condition* (53.15) *holds true uniformly w.r.t.* $h \in \mathcal{H}$*, and we say that it is* unstable *otherwise.*

Remark 53.16 (Inf-sup condition in $\boldsymbol{W}^{1,p}$-$L^{p'}$). Let $p \in (1,\infty)$ and let $p' \in (1,\infty)$ be s.t. $\frac{1}{p} + \frac{1}{p'} = 1$. As in Remark 53.10, a more general variant of the inf-sup condition (53.15) is

$$\inf_{q_h \in Q_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h\mathrm{d}}} \frac{|\int_D q_h \nabla\cdot \boldsymbol{v}_h \,\mathrm{d}x|}{\|q_h\|_{L^{p'}(D)} |\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}} =: \beta_h > 0. \tag{53.16}$$

We will see in the next chapters that many stable finite element pairs for the Stokes equations satisfy this more general inf-sup condition. □

Let us define the discrete operator $B_h : \boldsymbol{V}_{h\mathrm{d}} \to Q_h'$ s.t. $\langle B_h(\boldsymbol{v}_h), q_h\rangle_{Q_h',Q_h} := b(\boldsymbol{v}_h, q_h) = -\int_D q_h \nabla\cdot \boldsymbol{v}_h \,\mathrm{d}x$ for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{h\mathrm{d}} \times Q_h$. We have

$$(53.15) \iff B_h \text{ is surjective}, \tag{53.17a}$$

$$\ker(B_h) = \{\boldsymbol{v}_h \in \boldsymbol{V}_{h\mathrm{d}} \mid (q_h, \nabla\cdot \boldsymbol{v}_h)_{L^2(D)} = 0,\ \forall q_h \in Q_h\}. \tag{53.17b}$$

The operator $B_h : \boldsymbol{V}_{h\mathrm{d}} \to Q_h'$ is the discrete counterpart of the divergence operator $B : \boldsymbol{V}_{\mathrm{d}} \to Q'$ introduced just above Theorem 53.11. We observe that the inf-sup condition (53.15) is equivalent to asserting the surjectivity of B_h. Moreover, assuming for simplicity that $g = 0$ in the mass conservation equation, the discrete Stokes problem (53.14) produces a velocity field $\boldsymbol{u}_h \in \ker(B_h)$. One then says that the discrete velocity field is weakly divergence-free. However, $\ker(B_h)$ may not be a subspace of $\ker(B)$, i.e., the discrete velocity field $\boldsymbol{u}_h$ is not necessarily strongly (or pointwise) divergence-free.

Several techniques are available to prove the inf-sup condition (53.15), and we refer the reader to the next two chapters for various examples. Recall in particular that (53.15) is equivalent to the existence of a Fortin operator $\boldsymbol{\Pi}_h \in \mathcal{L}(\boldsymbol{V}_{\mathrm{d}}; \boldsymbol{V}_{h\mathrm{d}})$ s.t. $b(\boldsymbol{\Pi}_h(\boldsymbol{v}) - \boldsymbol{v}, q_h) = 0$ for all $q_h \in Q_h$ (see Lemma 26.9).

Theorem 53.17 (Error estimate). *Let* $(\boldsymbol{u}, p)$ *solve* (53.6). *Assume* (53.15) *and let* $(\boldsymbol{u}_h, p_h)$ *solve* (53.14). *Then we have*

$$|\boldsymbol{u}-\boldsymbol{u}_h|_{\boldsymbol{H}^1(D)} \le c_{1h} \inf_{\boldsymbol{v}_h\in \boldsymbol{V}_{h\mathrm{d}}} |\boldsymbol{u}-\boldsymbol{v}_h|_{\boldsymbol{H}^1(D)} + c_{2h} \inf_{q_h\in Q_h} \|p-q_h\|_{L^2(D)},$$
$$\|p-p_h\|_{L^2(D)} \le c_{3h} \inf_{\boldsymbol{v}_h\in \boldsymbol{V}_{h\mathrm{d}}} |\boldsymbol{u}-\boldsymbol{v}_h|_{\boldsymbol{H}^1(D)} + c_{4h} \inf_{q_h\in Q_h} \|p-q_h\|_{L^2(D)},$$

where $c_{1h} := (1+\frac{\|a\|}{\alpha})(1+\|\boldsymbol{\Pi}_h\|_{\mathcal{L}(\boldsymbol{V}_\mathrm{d};\boldsymbol{V}_{h\mathrm{d}})})$ *for any Fortin operator* $\boldsymbol{\Pi}_h \in \mathcal{L}(\boldsymbol{V}_\mathrm{d};\boldsymbol{V}_{h\mathrm{d}})$, $c_{2h} := 0$ *if* $\ker(B_h)\subset\ker(B)$ *and* $c_{2h} := \frac{\|b\|}{\alpha}$ *otherwise,* $c_{3h} := c_{1h}\frac{\|a\|}{\beta_h}$, *and* $c_{4h} := 1+\frac{\|b\|}{\beta_h}+c_{2h}\frac{\|a\|}{\beta_h}$. *Here,* $\alpha \ge 2\mu C_\mathrm{K}^2$ *is the coercivity constant of the bilinear form* a *on* $\boldsymbol{V}_\mathrm{d}\times\boldsymbol{V}_\mathrm{d}$, $\|a\|\le 2\mu$ *its norm, and* $\|b\|\le 1$ *the norm of the bilinear form* b *on* $\boldsymbol{V}_\mathrm{d}\times Q$.

Proof. This is a direct application of Corollary 50.5. □

Remark 53.18 (β_h vs. β_0). The estimates from Theorem 53.17 show that it is important that the inf-sup condition (53.15) be satisfied uniformly w.r.t. $h\in\mathcal{H}$. Indeed, the factor $\frac{1}{\beta_h}$ appears in the coefficients c_{3h} and c_{4h} in the pressure error bound, and a factor $\frac{1}{\beta_h}$ may appear in the constant c_{1h} affecting both error bounds if $\|\boldsymbol{\Pi}_h\|_{\mathcal{L}(\boldsymbol{V}_\mathrm{d};\boldsymbol{V}_{h\mathrm{d}})} \sim \frac{\|b\|}{\beta_h}$ for every Fortin operator. □

We say that the pair $(\boldsymbol{\xi}(\boldsymbol{r}), \phi(\boldsymbol{r})) \in \boldsymbol{V}_\mathrm{d}\times Q$ is the solution to the adjoint problem of (53.6) with source term $\boldsymbol{r}\in \boldsymbol{L}^2(D)$ if $a(\boldsymbol{v},\boldsymbol{\xi}(\boldsymbol{r})) + b(\boldsymbol{v},\phi(\boldsymbol{r})) = \int_D \boldsymbol{r}{\cdot}\boldsymbol{v}\,\mathrm{d}x$ for all $\boldsymbol{v}\in\boldsymbol{V}_\mathrm{d}$ and $b(\boldsymbol{\xi}(\boldsymbol{r}),q)=0$ for all $q\in Q$.

Theorem 53.19 ($\boldsymbol{L}^2$-velocity error estimate). *Let* $(\boldsymbol{u}, p)$ *solve* (53.6). *Assume* (53.15) *and let* $(\boldsymbol{u}_h, p_h)$ *solve* (53.14). *Assume that there exist real numbers* c_smo *and* $s\in(0,1]$ *s.t.* $\mu\ell_D^{-1}\|\boldsymbol{\xi}(\boldsymbol{r})\|_{\boldsymbol{H}^{1+s}(D)} + \|\phi(\boldsymbol{r})\|_{H^s(D)} \le c_\mathrm{smo}\ell_D\|\boldsymbol{r}\|_{\boldsymbol{L}^2(D)}$ *for all* $\boldsymbol{r}\in\boldsymbol{L}^2(D)$, *and that there is* c *such that for all* $h\in\mathcal{H}$, $\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_{h\mathrm{d}}}|\boldsymbol{v}-\boldsymbol{v}_h|_{\boldsymbol{H}^1(D)} \le ch^s|\boldsymbol{v}|_{\boldsymbol{H}^{1+s}(D)}$ *for all* $\boldsymbol{v}\in\boldsymbol{V}_\mathrm{d}\cap\boldsymbol{H}^{1+s}(D)$ *and* $\inf_{q_h\in Q_h}\|q-q_h\|_{L^2(D)} \le ch^s|q|_{H^s(D)}$ *for all* $q\in Q\cap H^s(D)$. *Then there is* c *s.t. for all* $h\in\mathcal{H}$,

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(D)} \le c\,h^s\ell_D^{1-s}\bigg(\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_{h\mathrm{d}}}|\boldsymbol{u}-\boldsymbol{v}_h|_{\boldsymbol{H}^1(D)} + \frac{\|b\|}{\|a\|}\inf_{q_h\in Q_h}\|p-q_h\|_{L^2(D)}\bigg).$$

Proof. Apply Lemma 50.11 or see Exercise 53.3. □

Let us give some further insight into the velocity error estimate from Theorem 53.17. For simplicity, we assume that $g=0$. Let us define the projection operator $\boldsymbol{P}_h^\mathrm{s} : \boldsymbol{V}_\mathrm{d}\to\ker(B_h)$ such that

$$a(\boldsymbol{P}_h^\mathrm{s}(\boldsymbol{v}),\boldsymbol{w}_h) = a(\boldsymbol{v},\boldsymbol{w}_h), \qquad \forall(\boldsymbol{v},\boldsymbol{w}_h)\in\boldsymbol{V}_\mathrm{d}\times\ker(B_h). \tag{53.18}$$

Lemma 53.20 (Quasi-optimality). *Assume* (53.15). *The following holds true for all* $\boldsymbol{v}\in\boldsymbol{V}_\mathrm{d}$ *and any Fortin operator* $\boldsymbol{\Pi}_h\in\mathcal{L}(\boldsymbol{V}_\mathrm{d};\boldsymbol{V}_{h\mathrm{d}})$:

$$|\boldsymbol{v} - \boldsymbol{P}_h^{\mathrm{S}}(\boldsymbol{v})|_{\boldsymbol{H}^1(D)} \le \tilde{c}_{1h} \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_{h\mathrm{d}}} |\boldsymbol{v} - \boldsymbol{v}_h|_{\boldsymbol{H}^1(D)}, \tag{53.19}$$

with $\tilde{c}_{1h} := \frac{\|a\|}{\alpha}(1 + \|\boldsymbol{\Pi}_h\|_{\mathcal{L}(\boldsymbol{V}_{\mathrm{d}};\boldsymbol{V}_{h\mathrm{d}})})$.

Proof. Since the bilinear form a is bounded and coercive, we have

$$|\boldsymbol{v} - \boldsymbol{P}_h^{\mathrm{S}}(\boldsymbol{v})|_{\boldsymbol{H}^1(D)} \le \frac{\|a\|}{\alpha} \inf_{\boldsymbol{v}_h \in \ker(B_h)} |\boldsymbol{v} - \boldsymbol{v}_h|_{\boldsymbol{H}^1(D)}.$$

The assertion then follows from Lemma 50.3 (notice that $\boldsymbol{\Pi}_h(\boldsymbol{u}) \in \ker(B_h)$ since $\nabla{\cdot}\boldsymbol{u} = g = 0$ by assumption, see Remark 50.4). □

Lemma 53.21 (Discrete velocity estimate). *Let* $(\boldsymbol{u}, p)$ *solve* (53.6). *Assume* (53.15) *and let* $(\boldsymbol{u}_h, p_h)$ *solve* (53.14). *As in Theorem* 53.17, *set* $c_{2h} := 0$ *if* $\ker(B_h) \subset \ker(B)$ *and* $c_{2h} := \frac{\|b\|}{\alpha}$ *otherwise. The following holds true:*

$$|\boldsymbol{u}_h - \boldsymbol{P}_h^{\mathrm{S}}(\boldsymbol{u})|_{\boldsymbol{H}^1(D)} \le c_{2h} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2(D)}. \tag{53.20}$$

Proof. The proof follows a similar, yet simpler, path to that of Lemma 50.2. Since $a(\boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h) = F(\boldsymbol{w}_h) = a(\boldsymbol{u}, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p) = a(\boldsymbol{P}_h^{\mathrm{S}}(\boldsymbol{u}), \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p)$ for all $\boldsymbol{w}_h \in \ker(B_h) \subset \boldsymbol{V}_{h\mathrm{d}} \subset \boldsymbol{V}_{\mathrm{d}}$, setting $\boldsymbol{e}_h := \boldsymbol{u}_h - \boldsymbol{P}_h^{\mathrm{S}}(\boldsymbol{u}) \in \ker(B_h)$, we infer that $a(\boldsymbol{e}_h, \boldsymbol{w}_h) = b(\boldsymbol{w}_h, p - p_h)$ for all $\boldsymbol{w}_h \in \ker(B_h)$. Since $\boldsymbol{e}_h \in \ker(B_h)$, invoking the coercivity of a then yields

$$\alpha\, |\boldsymbol{e}_h|^2_{\boldsymbol{H}^1(D)} \le b(\boldsymbol{e}_h, p - p_h).$$

If $\ker(B_h) \subset \ker(B)$, then $|\boldsymbol{e}_h|_{\boldsymbol{H}^1(D)} = 0$ which proves (53.20). Otherwise, we use that $\boldsymbol{e}_h \in \ker(B_h)$ to write $\alpha |\boldsymbol{e}_h|^2_{\boldsymbol{H}^1(D)} \le b(\boldsymbol{e}_h, p - q_h)$ for all $q_h \in Q_h$, and invoke the boundedness of b to prove (53.20). □

The bound (53.20) implies that $\boldsymbol{u}_h = \boldsymbol{P}_h^{\mathrm{S}}(\boldsymbol{u})$ whenever $\ker(B_h) \subset \ker(B)$. Moreover, in the general case, combining (53.20) with (53.19) and using the triangle inequality we obtain again the velocity error estimate from Theorem 53.17 with the slightly sharper constant $\tilde{c}_{1h}$ instead of c_{1h}.

Remark 53.22 (Well-balanced scheme). In the particular case where $\boldsymbol{f} = \nabla\phi$ for some $\phi \in H^1(D) \cap L^2_*(D)$, the solution to the Stokes problem (53.6) is $(\boldsymbol{u}, p) = (\boldsymbol{0}, \phi)$. This situation is encountered with hydrostatic (or curl-free) body forces. One says that the discrete problem (53.14) is *well-balanced* w.r.t. hydrostatic body forces if $\boldsymbol{u}_h = \boldsymbol{0}$ as well. (One also sometimes says that the discretization is *pressure robust.*) A well-balanced discretization of the Stokes equations can be desirable even if $\boldsymbol{f}$ is not curl-free, but has a relatively large curl-free component. In this case, a discretization that is not well-balanced can lead to a rather poor velocity approximation, even on meshes that seem rather fine. Lemma 53.21 shows that (53.14) is well-balanced whenever $\ker(B_h) \subset \ker(B)$. The scheme can be made well-balanced when $\ker(B_h) \not\subset \ker(B)$ by slightly modifying the discrete momentum equation. Considering Dirichlet conditions over the whole boundary for simplicity, one introduces a lifting operator $L : \boldsymbol{V}_{h\mathrm{d}} \to \boldsymbol{V}_{\mathrm{d}}$ such that $L(\ker(B_h)) \subset \ker(B)$ and then replaces the first equation in (53.14) by $a(\boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h) = (\boldsymbol{f}, L(\boldsymbol{w}_h))_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{w}_h \in \boldsymbol{V}_{h\mathrm{d}}$. The lifting operator L must satisfy some consistency conditions to preserve the optimal decay rates of the error estimate. This idea has been introduced by Linke [283] and explored more thoroughly by Lederer et al. [279] in the context of mixed finite elements with continuous pressures; see also John et al. [261] for an overview. Examples of curl-free body forces in fluid mechanics are the Coriolis force if $d = 2$, the gravity, and the centrifugal force. Obviously, if $\boldsymbol{f} \approx \nabla\phi$ and ϕ is explicitly known, one can always make the change of variable $p \to p - \phi$ to alleviate the above difficulty if the scheme is not well-balanced. □

53.4 Classical examples of unstable pairs

We study in this section three pairs of finite elements that look appealing at first sight, but that unfortunately do not satisfy the inf-sup condition (53.15). For simplicity, we consider a homogeneous Dirichlet condition on the velocity over the whole boundary, so that $\boldsymbol{V}_{\mathrm{d}} := \boldsymbol{H}_0^1(D)$ and we write $\boldsymbol{V}_{h0}$ instead of $\boldsymbol{V}_{h\mathrm{d}}$ for the discrete velocity space. Since the approximation setting is conforming, we have $\boldsymbol{V}_{h0} \subset \boldsymbol{H}_0^1(D)$ in all cases.

Recall that the inf-sup condition (53.15) is not satisfied if and only if $B_h^* : Q_h \to \boldsymbol{V}_{h0}'$ is not injective (or, once global shape functions have been chosen, the associated matrix does not have full column rank). In this case, a nonzero pressure field in $\ker(B_h^*)$ is called *spurious pressure mode*. Equivalently, the inf-sup condition is not satisfied if and only if $B_h : \boldsymbol{V}_{h0} \to Q_h'$ is not surjective.

53.4.1 The $(\mathbb{Q}_1, \mathbb{P}_0)$ pair: Checkerboard instability

A well-known pair of incompatible finite elements is the $(\mathbb{Q}_1, \mathbb{P}_0)$ pair obtained when approximating the velocity with continuous piecewise bilinear polynomials and the pressure with piecewise constants. This pair produces an instability often called *checkerboard instability*.

Let us restrict ourselves to the two-dimensional setting and assume that $D := (0,1)^2$. We define a uniform Cartesian mesh on D as follows: Let N be an integer larger than 2. Set $h := \frac{1}{I}$, and for all $i, j \in \{0{:}I{-}1\}$, denote by $\boldsymbol{a}_{ij}$ the point with Cartesian coordinates (ih, jh). Let K_{ij} be the square cell whose bottom left node is $\boldsymbol{a}_{ij}$; see Figure 53.1. The resulting mesh is denoted by $\mathcal{T}_h := \bigcup_{i,j} K_{ij}$. Consider the following finite element spaces:

$$\boldsymbol{V}_{h0} := \{\boldsymbol{v}_h \in \boldsymbol{C}^0(\overline{D}) \mid \forall K_{ij} \in \mathcal{T}_h,\, \boldsymbol{v}_h \circ \boldsymbol{T}_{K_{ij}} \in \mathbb{Q}_{1,d},\ \boldsymbol{v}_{h|\partial D} = \boldsymbol{0}\}, \quad (53.21\text{a})$$

$$Q_h := \{q_h \in L^2_*(D) \mid \forall K_{ij} \in \mathcal{T}_h,\, q_h \circ \boldsymbol{T}_{K_{ij}} \in \mathbb{P}_{0,d}\}. \quad (53.21\text{b})$$

Recall that for all $K \in \mathcal{T}_h$, $\boldsymbol{T}_K : \widehat{K} \to K$ denotes the geometric mapping; see §8.1. For all $p_h \in Q_h$, set $p_{i+\frac{1}{2},j+\frac{1}{2}} := p_{h|K_{ij}}$, and for all $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}$, denote by (u_{ij}, v_{ij}) the values of the two Cartesian components of $\boldsymbol{v}_h$ at the node $\boldsymbol{a}_{ij}$.

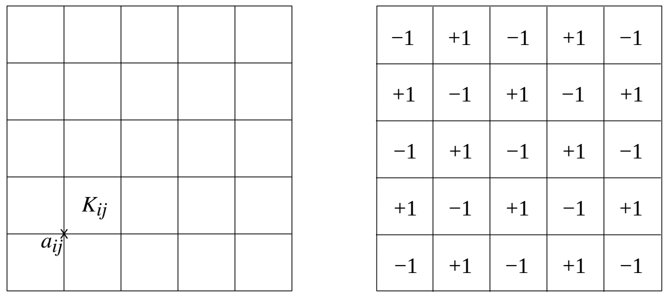

Fig. 53.1 $(\mathbb{Q}_1, \mathbb{P}_0)$ pair: mesh (left) and spurious pressure mode (right).

To prove that the inf-sup constant is zero, it is sufficient to prove the existence of a nonzero pressure field $p_h \in \ker(B_h^*)$, i.e., $\int_D p_h \nabla{\cdot}\boldsymbol{v}_h \,\mathrm{d}x = 0$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}$. Since p_h is constant on each cell, we have

$$\begin{aligned}\int_{K_{ij}} p_h \nabla{\cdot}\boldsymbol{v}_h \,\mathrm{d}x &= p_{i+\frac{1}{2},j+\frac{1}{2}} \int_{\partial K_{ij}} \boldsymbol{v}_h{\cdot}\boldsymbol{n} \,\mathrm{d}s \\ &= \tfrac{1}{2} h p_{i+\frac{1}{2},j+\frac{1}{2}} \big(u_{i+1,j} + u_{i+1,j+1} + v_{i+1,j+1} + v_{i,j+1} \\ &\qquad - u_{i,j} - u_{i,j+1} - v_{i,j} - v_{i+1,j}\big).\end{aligned}$$

Summing over all the cells and rearranging the sum yields $\int_D p_h \nabla{\cdot}\boldsymbol{v}_h \,\mathrm{d}x = -h^2 \sum_{i,j\in\{0:N-1\}} (u_{i,j} G_{1,ij}(p_h) + v_{i,j} G_{2,ij}(p_h))$, where

$$\begin{aligned}G_{1,ij}(p_h) &:= \tfrac{1}{2h}\big(p_{i+\frac{1}{2},j+\frac{1}{2}} + p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}\big), \\ G_{2,ij}(p_h) &:= \tfrac{1}{2h}\big(p_{i+\frac{1}{2},j+\frac{1}{2}} + p_{i-\frac{1}{2},j+\frac{1}{2}} - p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}\big).\end{aligned}$$

We infer that $\int_D p_h \nabla{\cdot}\boldsymbol{v}_h \,\mathrm{d}x = 0$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}$ if and only if

$$p_{i+\frac{1}{2},j+\frac{1}{2}} = p_{i-\frac{1}{2},j-\frac{1}{2}} \qquad \text{and} \qquad p_{i-\frac{1}{2},j+\frac{1}{2}} = p_{i+\frac{1}{2},j-\frac{1}{2}}.$$

The solution set of this linear system is a two-dimensional vector space. One dimension is spanned by the constant field $p_h = 1$, but span$\{1\}$ must be excluded from the solution set since the elements in Q_h must have a zero mean. The other dimension is spanned by the field whose value is alternatively $+1$ and -1 on adjacent cells in a checkerboard pattern, as shown on the right panel of Figure 53.1. This is a spurious pressure mode, and if N is even, this spurious mode is in Q_h (i.e., it satisfies the zero-mean condition). In this case, the inf-sup condition is not satisfied, i.e., the $(\mathbb{Q}_1, \mathbb{P}_0)$ pair is incompatible for the Stokes problem.

Remark 53.23 (Filtering). Since the $(\mathbb{Q}_1, \mathbb{P}_0)$ pair is very simple to program, one may be tempted to cure its deficiencies by restricting the size of Q_h. For instance, one could enforce the pressure to be orthogonal (in the L^2-sense) to the space spanned by the spurious pressure mode. Unfortunately, this remedy is not strong enough to produce a healthy finite element pair, since it can be shown that in this case there are positive constants c, c' s.t. $ch \le \beta_h \le c'h$ uniformly w.r.t. $h \in \mathcal{H}$; see Boland and Nicolaides [68] or Girault and Raviart [217, p. 164]. This shows that the method may not converge since the factor $\frac{1}{\beta_h}$ appears in the error bound on the velocity and the factor $\frac{1}{\beta_h^2}$ appears in the error bound on the pressure (see Theorem 53.17). $\square$

53.4.2 The $(\mathbb{P}_1, \mathbb{P}_1)$ pair: Checkerboard-like instability

Because it is very simple to program, the continuous $\mathbb{P}_1$ finite element for both the velocity and the pressure is a natural choice for approximating the Stokes problem. Unfortunately, the $(\boldsymbol{\mathbb{P}}_1, \mathbb{P}_1)$ pair does not satisfy the inf-sup condition (53.15). To understand the origin of the problem, let us construct a two-dimensional counterexample in $D := (0,1)^2$. Consider a uniform Cartesian mesh composed of squares of side h and split each square along one diagonal as shown in the left panel of Figure 53.2. Let $\mathcal{T}_h$ be the resulting triangulation and let the velocity and the pressure finite element spaces be

$$\boldsymbol{V}_{h0} := \boldsymbol{P}^{\mathrm{g}}_{1,0}(\mathcal{T}_h), \qquad Q_h := P^{\mathrm{g}}_1(\mathcal{T}_h) \cap L^2_*(D), \tag{53.22}$$

where $\boldsymbol{P}^{\mathrm{g}}_{1,0}(\mathcal{T}_h)$ is the vector-valued version of the space $P^{\mathrm{g}}_{1,0}(\mathcal{T}_h)$ defined in §19.4. Let $\{\boldsymbol{z}_{n,K}\}_{n\in\{0:2\}}$ be the three vertices of the mesh cell $K \in \mathcal{T}_h$.

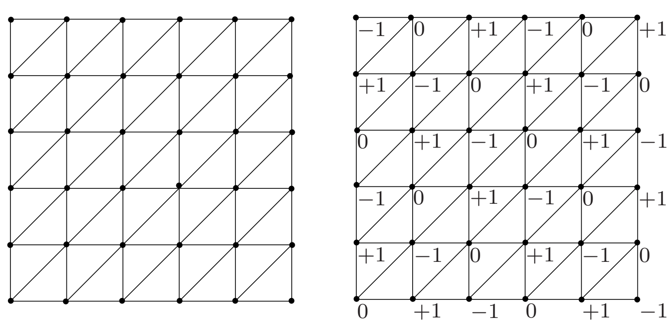

Fig. 53.2 $(\boldsymbol{\mathbb{P}}_1, \mathbb{P}_1)$ pair: the mesh (left) and one spurious pressure mode (right).

Now consider a pressure field p_h such that $\sum_{n\in\{0:2\}} p_h(\boldsymbol{z}_{n,K})$ is zero on each triangle K. An example of such a *spurious pressure mode* is shown in the right panel of Figure 53.2. Then we have for all $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}$,

$$\int_D p_h \nabla\cdot \boldsymbol{v}_h \,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} (\nabla\cdot\boldsymbol{v}_h)_{|K} \int_K p_h \,\mathrm{d}x,$$
$$= \sum_{K\in\mathcal{T}_h} (\nabla\cdot\boldsymbol{v}_h)_{|K} \frac{|K|}{3} \sum_{n\in\{0:2\}} p_h(\boldsymbol{z}_{n,K}) = 0.$$

Hence, p_h satisfies $\int_D p_h\nabla\cdot\boldsymbol{v}_h \,\mathrm{d}x = 0$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}$. In other words, the field p_h is a spurious pressure mode, and the inf-sup constant is zero.

53.4.3 The $(\mathbb{P}_1, \mathbb{P}_0)$ pair: Locking effect

A simple alternative to the $(\mathbb{Q}_1, \mathbb{P}_0)$ pair consists of using the $(\mathbb{P}_1, \mathbb{P}_0)$ pair, i.e., assuming that $\mathcal{T}_h$ is composed of simplices, the velocity is approximated with continuous, piecewise linear polynomials and the pressure with piecewise constants. We observe that $\ker(B_h) \subset \ker(B)$ in this case since the divergence of the velocity is piecewise constant. Unfortunately, the $(\mathbb{P}_1, \mathbb{P}_0)$ pair does not satisfy the inf-sup condition (53.15). Let us produce a two-dimensional counterexample. Assume that D is a simply connected polygon. Let N_{c}, $N_{\mathrm{v}}^{\mathrm{i}}$, and $N_{\mathrm{e}}^{\partial}$ denote the number of elements, internal vertices, and boundary edges in $\mathcal{T}_h$, respectively. The Euler relations give $N_{\mathrm{c}} = 2N_{\mathrm{v}}^{\mathrm{i}} + N_{\mathrm{e}}^{\partial} - 2$ (see Remark 8.13 and Exercise 8.2). Since $\dim(Q_h) = N_{\mathrm{c}} - 1$ and $\dim(\boldsymbol{V}_{h0}) = 2N_{\mathrm{v}}^{\mathrm{i}}$, the rank nullity theorem implies that

$$\dim(\ker(B_h^*)) = \dim(Q_h) - \dim(\mathrm{im}(B_h^*)) \geq \dim(Q_h) - \dim(\boldsymbol{V}_{h0})$$
$$= N_{\mathrm{c}} - 1 - 2N_{\mathrm{v}}^{\mathrm{i}} = N_{\mathrm{e}}^{\partial} - 3.$$

Hence, there are at least $N_{\mathrm{e}}^{\partial} - 3$ spurious pressure modes. This means that the space Q_h is far too rich for B_h to be surjective. Actually, in some cases, it can be shown that B_h is injective, i.e., the only member of $\ker(B_h)$ is zero. This situation is referred to as *locking* in the literature.

Remark 53.24 (Comparison with $(\mathbb{P}_1, \mathbb{P}_1)$). Note that the dimension of the pressure finite element space is smaller for the $(\mathbb{P}_1, \mathbb{P}_1)$ pair (where $\dim(Q_h) = N_{\mathrm{v}} - 1$) than for the $(\mathbb{P}_1, \mathbb{P}_0)$ pair (where $\dim(Q_h) = N_{\mathrm{c}} - 1$). Indeed, we have $N_{\mathrm{c}} \sim 2N_{\mathrm{v}}$ on fine meshes (see Exercise 8.2). □

Exercises

Exercise 53.1 ($\nabla\cdot$ is surjective). Let $D \subset \mathbb{R}^2$ be a domain of class C^2. Prove that $\nabla\cdot : \boldsymbol{H}_0^1(D) \to L_*^2(D)$ is continuous and surjective. (*Hint*: construct $\boldsymbol{v} \in \boldsymbol{H}_0^1(D)$ such that $\boldsymbol{v} = \nabla q + \nabla\times\psi$, where q solves a Poisson problem, ψ solves a biharmonic problem, and $\nabla\times\psi := (\partial_2\psi, -\partial_1\psi)^{\mathsf{T}}$.)

Exercise 53.2 (de Rham). Let D be a bounded open set in $\mathbb{R}^d$ and assume that D is star-shaped with respect to an open ball $B \subset D$. Prove that the continuous linear forms on $\boldsymbol{W}_0^{1,p}(D)$ that are zero on $\ker(\nabla\cdot)$ are gradients of functions in $L_*^{p'}(D)$. (*Hint*: use Remark 53.10 and the closed range theorem.)

Exercise 53.3 ($\boldsymbol{L}^2$-estimate). Prove Theorem 53.19 directly, i.e., without invoking Lemma 50.11.

Exercise 53.4 (Projection). Let $(\boldsymbol{V}_{h0}, Q_h)_{h\in\mathcal{H}}$ be a sequence of pairs of finite element spaces. Let $p \in [1,\infty]$ and let $p' \in [1,\infty]$ be s.t. $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\Pi_h^Z : Q_h \to Z_h$ be an operator, where Z_h is a finite-dimensional subspace of $L^p(D)$. Assume that there are $\beta_1, \beta_2 > 0$ such that for all $h \in \mathcal{H}$, $\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|\int_D q_h \nabla\cdot\boldsymbol{v}_h \,\mathrm{d}x|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}} \geq \beta_1 \|q_h - \Pi_h^Z(q_h)\|_{L^{p'}(D)}$ for all $q_h \in Q_h$ and $\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|\int_D q_h \nabla\cdot\boldsymbol{v}_h \,\mathrm{d}x|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}} \geq \beta_2 \|q_h\|_{L^{p'}(D)}$ for all $q_h \in Z_h$. (i) Show that Π_h^Z is bounded uniformly w.r.t. $h \in \mathcal{H}$. (ii) Show that the $(\boldsymbol{V}_{h0}, Q_h)$ pair satisfies an inf-sup condition uniformly w.r.t. $h \in \mathcal{H}$.

Exercise 53.5 (Spurious mode for the $(\mathbb{Q}_1, \mathbb{Q}_1)$ pair). (i) Let $\widehat{K} := [0,1]^2$ be the unit square. Let $\widehat{\boldsymbol{a}}_{ij} := (\frac{i}{2}, \frac{j}{2})$, for all $i, j \in \{0{:}2\}$. Show that the quadrature $\int_{\widehat{K}} f(\widehat{\boldsymbol{x}}) \,\mathrm{d}\widehat{x} \approx \sum_{i,j} w_{ij} f(\widehat{\boldsymbol{a}}_{ij})$, where $w_{ij} := \frac{1}{36}(3i(2-i)+1)(3j(j-2)+1)$ ($w_{ij} := \frac{1}{36}$ for the four vertices of $\widehat{K}$, $w_{ij} := \frac{1}{9}$ for the four edge midpoints, and $w_{ij} := \frac{4}{9}$ at the barycenter of $\widehat{K}$) is exact for all $f \in \mathbb{Q}_2$. (*Hint*: write the $\mathbb{Q}_2$ Lagrange shape functions in tensor-product form and use Simpson's rule in each direction.) (ii) Consider $D := (0,1)^2$ and a mesh composed of $I \times I$ squares, $I \geq 2$. Consider the points $\boldsymbol{a}_{lm} := (\frac{l}{2I}, \frac{m}{2I})$ for all $l, m \in \{0{:}2I\}$. Let p_h be the continuous, piecewise bilinear function such that $p_h(\boldsymbol{a}_{2k,2n}) := (-1)^{k+n}$ for all $k, n \in \{0{:}I\}$. Show that p_h is a spurious pressure mode for the $(\mathbb{Q}_1, \mathbb{Q}_1)$ pair (continuous velocity and pressure).