Part XI, Chapter 54

Stokes equations: Stable pairs (I)

This chapter reviews various stable finite element pairs that are suitable to approximate the Stokes equations, i.e., the discrete velocity space and the discrete pressure space satisfy the inf-sup condition (53.15) (or its $\boldsymbol{W}^{1,p}$-$L^{p'}$ version (53.16)) uniformly with respect to $h \in \mathcal{H}$. We first review two standard techniques to prove the inf-sup condition, one based on the Fortin operator and one hinging on a weak control of the pressure gradient. Then we show how these techniques can be applied to finite element pairs where the discrete pressure space is H^1-conforming. The two main examples are the mini element based on the $(\mathbb{P}_1\text{-bubble}, \mathbb{P}_1)$ pair and the Taylor–Hood element based on the $(\mathbb{P}_2, \mathbb{P}_1)$ pair. In the next chapter, we introduce another technique based on macroelements to prove the inf-sup condition and we review stable finite element pairs where the discrete pressures are discontinuous. We assume in the entire chapter that Dirichlet conditions are enforced on the velocity over the whole boundary, that D is a polyhedron in $\mathbb{R}^d$, and that $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is a shape-regular sequence of affine meshes so that each mesh covers D exactly.

54.1 Proving the inf-sup condition

We briefly review two standard techniques to prove the inf-sup condition (53.15): one uses a Fortin operator and the other uses a weak control on the pressure gradient. Since this section is only meant to be a short introduction, the reader is referred to Boffi et al. [65, Chap. 8], Girault and Raviart [217, §II.1.4] for thorough reviews of the topic.

54.1.1 Fortin operator

One way to prove the inf-sup condition (53.15) consists of using the notion of Fortin operator. The theory behind the Fortin operator theory is investigated in detail §26.2.3. We now briefly summarize the main features of this theory and adapt the notation to the setting of the Stokes equations.

Let $\boldsymbol{V}, Q$ be two complex Banach spaces and let b be a bounded sesquilinear form on $\boldsymbol{V}\times Q$. Let β and $\|b\|$ be the inf-sup and the boundedness constants of b. Let $\boldsymbol{V}_{h0} \subset \boldsymbol{V}$ and let $Q_h \subset Q$ be finite-dimensional subspaces equipped, respectively, with the norms of $\boldsymbol{V}$ and Q. A map $\boldsymbol{\Pi}_h : \boldsymbol{V} \to \boldsymbol{V}_{h0}$, is called a *Fortin operator* if $b(\boldsymbol{\Pi}_h(\boldsymbol{v}) - \boldsymbol{v}, q_h) = 0$ for all $(\boldsymbol{v}, q_h) \in \boldsymbol{V}\times Q_h$, and there is real number $\gamma_h > 0$ such that $\gamma_h \|\boldsymbol{\Pi}_h(\boldsymbol{v})\|_{\boldsymbol{V}} \le \|\boldsymbol{v}\|_{\boldsymbol{V}}$ for all $\boldsymbol{v} \in \boldsymbol{V}$. The key result we are going to use is the following statement (see Lemma 26.9, Boffi et al. [65, Prop. 8.4.1], and the work by the authors [187, Thm. 1]).

Lemma 54.1 (Fortin operator). *If there exists a Fortin operator, then the inf-sup condition*

$$\inf_{q_h\in Q_h} \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_{h0}} \frac{|b(\boldsymbol{v}_h, q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}} \|q_h\|_Q} =: \beta_h > 0, \tag{54.1}$$

holds true with $\beta_h \ge \gamma_h\beta$. Conversely, if the inf-sup condition (54.1) *holds true, then there exists a Fortin operator with $\gamma_h \ge \frac{\beta_h}{\|b\|}$.*

Hence, proving the inf-sup condition (54.1) can be done by constructing a Fortin operator. A practical way to do this is given by the following result.

Lemma 54.2 (Decomposition). *Let $\boldsymbol{\Pi}_{1h}, \boldsymbol{\Pi}_{2h} : \boldsymbol{V} \to \boldsymbol{V}_{h0}$ be two operators. Assume the following:* (i) *$\boldsymbol{\Pi}_{2h}$ is linear;* (ii) *$b(\boldsymbol{v} - \boldsymbol{\Pi}_{2h}(\boldsymbol{v}), q_h) = 0$ for all $(\boldsymbol{v}, q_h) \in \boldsymbol{V}\times Q_h$;* (iii) *The real numbers*

$$c_{1h} := \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{\|\boldsymbol{\Pi}_{1h}(\boldsymbol{v})\|_{\boldsymbol{V}}}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \quad \text{and} \quad c_{2h} := \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{\|\boldsymbol{\Pi}_{2h}(\boldsymbol{v} - \boldsymbol{\Pi}_{1h}(\boldsymbol{v}))\|_{\boldsymbol{V}}}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \tag{54.2}$$

are finite. Then (recalling that $I_{\boldsymbol{V}} : \boldsymbol{V} \to \boldsymbol{V}$ is the identity)

$$\boldsymbol{\Pi}_h := \boldsymbol{\Pi}_{1h} + \boldsymbol{\Pi}_{2h}(I_{\boldsymbol{V}} - \boldsymbol{\Pi}_{1h}) \tag{54.3}$$

is a Fortin operator with $\gamma_h \ge (c_{1h} + c_{2h})^{-1}$.

Proof. Since the operator $\boldsymbol{\Pi}_{2h}$ is linear owing to the assumption (i), we have

$$b(\boldsymbol{v} - \boldsymbol{\Pi}_h(\boldsymbol{v}), q_h) = b(\boldsymbol{v} - \boldsymbol{\Pi}_{2h}(\boldsymbol{v}), q_h) - b(\boldsymbol{\Pi}_{1h}(\boldsymbol{v}) - \boldsymbol{\Pi}_{2h}(\boldsymbol{\Pi}_{1h}(\boldsymbol{v})), q_h),$$

for all $(\boldsymbol{v}, q_h) \in \boldsymbol{V}\times Q_h$, and both terms on the right-hand side are zero owing to the assumption (ii). Furthermore, we have $\sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{\|\boldsymbol{\Pi}_h(\boldsymbol{v})\|_{\boldsymbol{V}}}{\|\boldsymbol{v}\|_{\boldsymbol{V}}} \le c_{1h} + c_{2h}$, i.e., $\gamma_h \|\boldsymbol{\Pi}_h(\boldsymbol{v})\|_{\boldsymbol{V}} \le \|\boldsymbol{v}\|_{\boldsymbol{V}}$ for all $\boldsymbol{v} \in \boldsymbol{V}$ with $\gamma_h \ge (c_{1h} + c_{2h})^{-1} > 0$ owing to the assumption (iii). □

54.1.2 Weak control on the pressure gradient

A second possibility to prove the inf-sup condition (54.1) consists of establishing a weak control on the gradient of the pressure. This technique can be used when the discrete pressure space is H^1-conforming. Let us focus more specifically on the bilinear form $b(\boldsymbol{v}, q) := -(\nabla\cdot\boldsymbol{v}, q)_{L^2(D)}$. Let $p \in (1,\infty)$, $\boldsymbol{V} := \boldsymbol{W}_0^{1,p}(D)$ equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{V}} := |\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}$, and $Q := L_*^{p'}(D) := \{q \in L^{p'}(D) \mid \int_D q\,\mathrm{d}x = 0\}$ equipped with the norm $\|q\|_Q := \|q\|_{L^{p'}(D)}$ with $p' \in (1,\infty)$ s.t. $\frac{1}{p} + \frac{1}{p'} = 1$. The discrete velocity space is $\boldsymbol{V}_{h0} \subset \boldsymbol{V}$, and the discrete pressure space is $Q_h \subset Q$.

Lemma 54.3 (Pressure gradient control). *Assume that the discrete pressure space Q_h is H^1-conforming, and that there is c such that the following holds true for all $p \in (1,\infty)$ and all $h \in \mathcal{H}$:*

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|b(\boldsymbol{v}_h, q_h)|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}} \geq c\left(\sum_{K\in\mathcal{T}_h} h_K^{p'}\|\nabla q_h\|_{\boldsymbol{L}^{p'}(K)}^{p'}\right)^{\frac{1}{p'}}. \tag{54.4}$$

Then the inf-sup condition (54.1) *holds true uniformly w.r.t. $h \in \mathcal{H}$.*

Proof. Let $q_h \in Q_h$. Since $Q_h \subset Q$, the continuous inf-sup condition (53.9) implies that

$$\beta_D\|q_h\|_Q \leq \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{|b(\boldsymbol{v}, q_h)|}{|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}} \leq \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{|b(\boldsymbol{\mathcal{I}}_h^{\mathrm{av}}(\boldsymbol{v}), q_h)|}{|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}} + \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{|b(\boldsymbol{v} - \boldsymbol{\mathcal{I}}_h^{\mathrm{av}}(\boldsymbol{v}), q_h)|}{|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}},$$

where $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}$ is the $\mathbb{R}^d$-valued version of the $W_0^{1,p}$-conforming quasi-interpolation operator introduced in §22.4.2. This means that $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v}) := \sum_{i\in\{1:d\}} \mathcal{I}_{h0}^{\mathrm{av}}(v_i)\boldsymbol{e}_i$, where $\boldsymbol{v} := \sum_{i\in\{1:d\}} v_i\boldsymbol{e}_i$ and $\{\boldsymbol{e}_i\}_{i\in\{1:d\}}$ is the canonical Cartesian basis of $\mathbb{R}^d$. Let $\mathfrak{T}_1, \mathfrak{T}_2$ denote the two terms on the right-hand side. Owing to the $\boldsymbol{W}_0^{1,p}$-stability of $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}$, we have $|\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v})|_{\boldsymbol{W}^{1,p}(D)} \leq c_{\mathcal{I}}|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}$. Since $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v}) \in \boldsymbol{V}_{h0}$, we infer that

$$|\mathfrak{T}_1| \leq c_{\mathcal{I}} \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{|b(\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v}), q_h)|}{\|\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v})\|_{\boldsymbol{W}^{1,p}(D)}} \leq c_{\mathcal{I}} \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|b(\boldsymbol{v}_h, q_h)|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}}.$$

Moreover, using that Q_h is H^1-conforming to integrate by parts, and then invoking Hölder's inequality and the approximation properties of $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}$, we infer that

$$\begin{aligned}|b(\boldsymbol{v} - \boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v}), q_h)| &= \left|(\nabla q_h, \boldsymbol{v} - \boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v}))_{\boldsymbol{L}^2(D)}\right| \\ &\leq c\sum_{K\in\mathcal{T}_h} \|\nabla q_h\|_{\boldsymbol{L}^{p'}(K)} h_K \|\nabla\boldsymbol{v}\|_{\mathbb{L}^p(D_K)},\end{aligned}$$

where D_K is the set of the points composing the mesh cells touching K. Since $\sum_{K\in\mathcal{T}_h}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^p(D_K)}^p \leq c\|\nabla\boldsymbol{v}\|_{\mathbb{L}^p(D)}^p = c|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}^p$ owing to the regularity of the mesh sequence, Hölder's inequality combined with the assumption (54.4) implies that

$$|\mathfrak{T}_2| \le c' \left(\sum_{K\in\mathcal{T}_h} h_K^{p'} \|\nabla q_h\|^{p'}_{\boldsymbol{L}^{p'}(K)} \right)^{\frac{1}{p'}} \le c'' \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|b(\boldsymbol{v}_h,q_h)|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}}.$$

This completes the proof of the inf-sup condition. □

Remark 54.4 (Literature). The technique presented above is based on Bercovier and Pironneau [54, Prop. 1], Verfürth [376] (for $p := 2$). □

54.2 Mini element: the $(\mathbb{P}_1$-bubble$, \mathbb{P}_1)$ pair

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine simplicial meshes. Recall from §53.4.2 that the reason for which the $(\boldsymbol{\mathbb{P}}_1, \mathbb{P}_1)$ pair does not satisfy the inf-sup condition (53.15) is that the velocity space is not rich enough (or equivalently the pressure space is too rich). To circumvent this difficulty, we are going to enlarge the velocity space by adding one more degree of freedom per simplex for each Cartesian component of the velocity.

Let $\widehat{K}$ be the reference simplex and $\widehat{\boldsymbol{x}}_{\widehat{K}}$ be its barycenter, and let $\widehat{b}$ be a function such that

$$\widehat{b} \in W_0^{1,\infty}(\widehat{K}), \qquad 0 \le \widehat{b} \le 1, \qquad \widehat{b}(\widehat{\boldsymbol{x}}_{\widehat{K}}) = 1. \tag{54.5}$$

One can use $\widehat{b}(\widehat{\boldsymbol{x}}) := (d+1)^{d+1}\prod_{i\in\{0:d\}} \widehat{\lambda}_i(\widehat{\boldsymbol{x}})$, where $\{\widehat{\lambda}_i\}_{i\in\{0:d\}}$ are the barycentric coordinates on $\widehat{K}$. This function is usually called *bubble function* in reference to the shape of its graph as shown in Figure 54.1. Another possibility consists of dividing the simplex $\widehat{K}$ into $(d+1)$ subsimplices by connecting the $(d+1)$ vertices of $\widehat{K}$ to $\widehat{\boldsymbol{x}}_{\widehat{K}}$. Then $\widehat{b}$ is defined to be the continuous piecewise affine function on $\widehat{K}$ that is equal to one at $\widehat{\boldsymbol{x}}_{\widehat{K}}$ and zero at the vertices of $\widehat{K}$. We introduce the finite-dimensional space $\widehat{\boldsymbol{P}} := \boldsymbol{\mathbb{P}}_{1,d} \oplus (\mathrm{span}\{\widehat{b}\})^d$ and define $\widehat{\Sigma}$ to be the Lagrange degrees of freedom associated with the vertices of $\widehat{K}$ plus $\widehat{\boldsymbol{x}}_{\widehat{K}}$ for each Cartesian component of the velocity.

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine simplicial meshes so that each mesh covers D exactly. Recalling that we are enforcing homogeneous Dirichlet conditions on the velocity, the approximation spaces are defined by

$$\boldsymbol{V}_{h0} := \boldsymbol{\mathbb{P}}^{\mathrm{g}}_{1,0}(\mathcal{T}_h) \oplus \boldsymbol{B}_h, \qquad Q_h := P_1^{\mathrm{g}}(\mathcal{T}_h) \cap L^2_*(D), \tag{54.6}$$

where $\boldsymbol{B}_h := \bigoplus_{K\in\mathcal{T}_h}(\mathrm{span}\{b_K\})^d$ and $b_K := \widehat{b}\circ\boldsymbol{T}_K$ being the bubble function associated with the mesh cell $K \in \mathcal{T}_h$. Notice that

$$\boldsymbol{V}_{h0} = \{\boldsymbol{v}_h \in \boldsymbol{C}^0(\overline{D}) \mid \forall K \in \mathcal{T}_h,\, \boldsymbol{v}_h\circ\boldsymbol{T}_K \in \widehat{\boldsymbol{P}},\ \boldsymbol{v}_{h|\partial D} = \boldsymbol{0}\}, \tag{54.7}$$

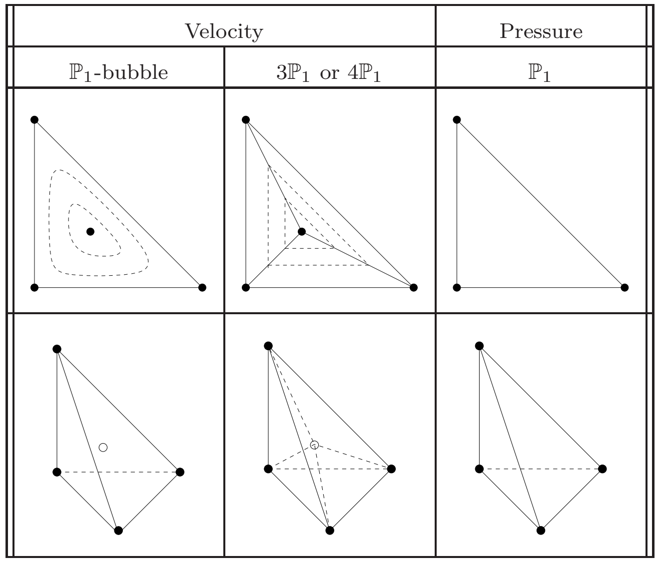

Fig. 54.1 Conventional representation of the ($\mathbb{P}_1$-bubble, $\mathbb{P}_1$) pair in dimensions two (top) and three (bottom). The degrees of freedom for the velocity are shown in the first column ($\mathbb{P}_1$-bubble) and in the second column ($3\mathbb{P}_1$ in dimension two and $4\mathbb{P}_1$ in dimension three). Some isolines of the two-dimensional bubble function are drawn. The pressure degrees of freedom are shown in the third column.

and that $\boldsymbol{V}_{h0} \subset \boldsymbol{W}_0^{1,p}(D)$ for all $p \in (1,\infty)$. We now show that the ($\boldsymbol{\mathbb{P}}_1$-bubble, $\mathbb{P}_1$) pair is stable. We do so by constructing a Fortin operator as in Lemma 54.2.

Lemma 54.5 (Stability). *Let $p \in (1,\infty)$ and let $p' \in (1,\infty)$ be s.t. $\frac{1}{p}+\frac{1}{p'} = 1$. Let $\boldsymbol{V}_{h0}$ and Q_h be defined in* (54.6). *There is β_0 such that for all $h \in \mathcal{H}$,*

$$\inf_{q_h\in Q_h} \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_{h0}} \frac{|\int_D q_h \nabla\cdot \boldsymbol{v}_h \,\mathrm{d}x|}{\|q_h\|_{L^{p'}(D)}|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}} \geq \beta_0 > 0. \tag{54.8}$$

Proof. Let us build a Fortin operator by means of the construction devised in Lemma 54.2 with $\boldsymbol{V} := \boldsymbol{W}_0^{1,p}(D)$ equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{V}} := |\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}$. We define the operator $\boldsymbol{\Pi}_{2h} : \boldsymbol{V} \to \boldsymbol{V}_{h0}$ by setting

$$\boldsymbol{\Pi}_{2h}(\boldsymbol{v}) := \sum_{K\in\mathcal{T}_h} \frac{\int_K \boldsymbol{v}\,\mathrm{d}x}{\int_K b_K\,\mathrm{d}x} b_K \in \boldsymbol{B}_h \subset \boldsymbol{V}_{h0}.$$

This operator is linear in agreement with the assumption (i) of Lemma 54.2. Moreover, the definition of $\boldsymbol{\Pi}_{2h}$ implies that $\int_K \boldsymbol{\Pi}_{2h}(\boldsymbol{v})\,\mathrm{d}x = \int_K \boldsymbol{v}\,\mathrm{d}x$ for all $\boldsymbol{v} \in \boldsymbol{V}$. Then for all $(\boldsymbol{v}, q_h) \in \boldsymbol{V}\times Q_h$ we infer that

$$b(\boldsymbol{v}, q_h) = -\int_D q_h \nabla\cdot\boldsymbol{v}\,\mathrm{d}x = \int_D \boldsymbol{v}\cdot\nabla q_h\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} \nabla q_{h|K}\cdot\int_K \boldsymbol{v}\,\mathrm{d}x$$
$$= \sum_{K\in\mathcal{T}_h} \nabla q_{h|K}\cdot\int_K \boldsymbol{\Pi}_{2h}(\boldsymbol{v})\,\mathrm{d}x = \int_D \boldsymbol{\Pi}_{2h}(\boldsymbol{v})\cdot\nabla q_h\,\mathrm{d}x = b(\boldsymbol{\Pi}_{2h}(\boldsymbol{v}), q_h),$$

which proves the assumption (ii) of Lemma 54.2. We now set $\boldsymbol{\Pi}_{1h} := \boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}$, where $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}} : \boldsymbol{V} \to \boldsymbol{V}_{h0}$ is the $\mathbb{R}^d$-valued version of the $\boldsymbol{W}_0^{1,p}$-conforming quasi-interpolation operator introduced in §22.4.2. We observe that the real number $c_{1h} := \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{|\boldsymbol{\Pi}_{1h}(\boldsymbol{v})|_{\boldsymbol{W}^{1,p}(D)}}{|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}}$ is uniformly bounded w.r.t. $h \in \mathcal{H}$. Moreover, the regularity of the mesh sequence and Lemma 11.7 imply that for all $K \in \mathcal{T}_h$,

$$|b_K|_{W^{1,p}(K)} \le c\,\|\mathbb{J}_K^{-1}\|_{\ell^2}|\det(\mathbb{J}_K)|^{\frac{1}{p}}|\widehat{b}|_{W^{1,p}(\widehat{K})} \le c'\,h_K^{-1}|K|^{\frac{1}{p}}.$$

Similar arguments show that $\int_K b_K\,\mathrm{d}x \ge c|K|$ and Hölder's inequality implies that $|\int_K \boldsymbol{v}\,\mathrm{d}x| \le |K|^{\frac{1}{p'}}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)}$. Putting these estimates together shows that

$$|\boldsymbol{\Pi}_{2h}(\boldsymbol{v})|_{\boldsymbol{W}^{1,p}(K)} \le c\,h_K^{-1}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)}.$$

Then the approximation properties of $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}$ (see Theorem 22.14) yield

$$|\boldsymbol{\Pi}_{2h}(\boldsymbol{v}-\boldsymbol{\Pi}_{1h}(\boldsymbol{v}))|_{\boldsymbol{W}^{1,p}(K)} = |\boldsymbol{\Pi}_{2h}(\boldsymbol{v}-\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v}))|_{\boldsymbol{W}^{1,p}(K)}$$
$$\le c\,h_K^{-1}\|\boldsymbol{v}-\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v})\|_{\boldsymbol{L}^p(K)} \le c'\,|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D_K)},$$

where D_K is the set of the points composing the mesh cells touching K. Summing the above bound over $K \in \mathcal{T}_h$ and using the regularity of the mesh sequence, we infer that $|\boldsymbol{\Pi}_{2h}(\boldsymbol{v}-\boldsymbol{\Pi}_{1h}(\boldsymbol{v}))|_{\boldsymbol{W}^{1,p}(D)} \le c\,|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}$. This shows that the real number $c_{2h} := \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{|\boldsymbol{\Pi}_{2h}(\boldsymbol{v}-\boldsymbol{\Pi}_{1h}(\boldsymbol{v}))|_{\boldsymbol{W}^{1,p}(D)}}{|\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}}$ is uniformly bounded w.r.t. $h \in \mathcal{H}$. In conclusion, all the assumptions of Lemma 54.2 are met, showing that $\boldsymbol{\Pi}_h := \boldsymbol{\Pi}_{1h} + \boldsymbol{\Pi}_{2h}(I_{\boldsymbol{V}} - \boldsymbol{\Pi}_{1h})$ is a Fortin operator with $\gamma_h \ge (c_{1h}+c_{2h})^{-1}$. Notice that γ_h is bounded from below away from zero uniformly w.r.t. $h \in \mathcal{H}$. Invoking Lemma 54.1, we conclude that the inf-sup condition (54.8) holds true uniformly w.r.t. $h \in \mathcal{H}$. □

Remark 54.6 (Convergence rate). Assume that the solution to (53.6) is such that $\boldsymbol{u} \in \boldsymbol{H}^2(D) \cap \boldsymbol{H}_0^1(D)$ and $p \in H^1(D) \cap L_*^2(D)$. Owing to Theorem 53.17, the discrete solution $(\boldsymbol{u}_h, p_h)$ to (53.14) with $(\boldsymbol{V}_{h0}, Q_h)$ defined in (54.6) satisfies $\mu|\boldsymbol{u}-\boldsymbol{u}_h|_{\boldsymbol{H}^1(D)} + \|p-p_h\|_{L^2(D)} \le ch(\mu|\boldsymbol{u}|_{\boldsymbol{H}^2(D)} + |p|_{H^1(D)})$. If the assumptions of Theorem 53.19 additionally hold true with $s := 1$, then $\mu\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(D)} \le ch^2(\mu|\boldsymbol{u}|_{\boldsymbol{H}^2(D)} + |p|_{H^1(D)})$. Notice that the convergence rate of the error on the velocity is that associated with the finite element space $\mathbb{P}_{1,0}^{\mathrm{g}}(\mathcal{T}_h)$, i.e., the bubble functions introduced to approximate the velocity do not contribute to the approximation error, they contribute only to the stability of the discretization (see also Exercise 54.2). □

Remark 54.7 (Literature). The idea of using bubble functions has been introduced by Crouzeix and Raviart [151]. The analysis of the mini element is due to Arnold et al. [20]. □

54.3 Taylor–Hood element: the $(\mathbb{P}_2, \mathbb{P}_1)$ pair

This section is dedicated to the analysis of the Taylor–Hood element based on the $(\mathbf{P}_2, \mathbb{P}_1)$ pair. Compared to the mini element which is based on the $(\mathbb{P}_1\text{-bubble}, \mathbb{P}_1)$ pair, the idea is to further enrich the discrete velocity space so as to improve by one order the convergence rate of the error. Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular family of affine simplicial meshes. Recalling that we are enforcing homogeneous Dirichlet conditions on the velocity, the approximation spaces are defined by

$$\boldsymbol{V}_{h0} := \boldsymbol{P}^{\mathrm{g}}_{2,0}(\mathcal{T}_h), \qquad Q_h := P^{\mathrm{g}}_1(\mathcal{T}_h) \cap L^2_*(D), \tag{54.9}$$

i.e., the velocity is approximated using continuous $\mathbf{P}_2$ elements and the pressure is approximated using continuous $\mathbb{P}_1$ elements. The conventional representation of this element is shown in Figure 54.2. We are going to prove the inf-sup condition (54.1) by using the technique described in §54.1.2, i.e., we first establish a weak control on the pressure gradient, then we invoke Lemma 54.3. As above, we set $\boldsymbol{V} := \boldsymbol{W}^{1,p}_0(D)$ and $Q := L^{p'}_*(D)$ with $p, p' \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Notice that $\boldsymbol{V}_{h0} \subset \boldsymbol{V}$ and $Q_h \subset Q$.

Lemma 54.8 (Bound on pressure gradient). *Let $\boldsymbol{V}_{h0}$, Q_h be defined in (54.9). Assume that $d \in \{2, 3\}$ and that every mesh cell has at least d internal edges (i.e., at most one face in ∂D). There is c such that the following holds true for all $p \in (1, \infty)$ and all $h \in \mathcal{H}$:*

$$\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|\int_D q_h \nabla\cdot\boldsymbol{v}_h \,\mathrm{d}x|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(D)}} \geq c \left(\sum_{K\in\mathcal{T}_h} h_K^{p'} \|\nabla q_h\|^{p'}_{\boldsymbol{L}^{p'}(K)}\right)^{\frac{1}{p'}}. \tag{54.10}$$

Proof. We only give the proof for $d = 3$ since the proof for $d = 2$ is similar. Let us number all the internal mesh edges from 1 to $N^{\mathrm{i}}_{\mathrm{e}}$. Consider an oriented edge E_i with $i \in \{1{:}N^{\mathrm{i}}_{\mathrm{e}}\}$, and denote its two endpoints by $\boldsymbol{z}_i^{\pm}$ and its midpoint by $\boldsymbol{m}_i$. Set $l_i := \|\boldsymbol{z}_i^+ - \boldsymbol{z}_i^-\|_{\ell^2}$ and $\boldsymbol{\tau}_i := l_i^{-1}(\boldsymbol{z}_i^+ - \boldsymbol{z}_i^-)$, so that l_i is the length of E_i and $\boldsymbol{\tau}_i$ is the unit tangent vector orienting E_i. Let q_h be a function in Q_h and let sgn be the sign function. Let $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}$ be (uniquely) defined by prescribing its global degrees of freedom in $\boldsymbol{V}_{h0}$ as follows:

$$\begin{cases} \boldsymbol{v}_h(\boldsymbol{a}_j) := \boldsymbol{0} & \text{if } \boldsymbol{a}_j \text{ is a mesh vertex,} \\ \boldsymbol{v}_h(\boldsymbol{m}_i) := -l_i^{p'} \operatorname{sgn}(\partial_{\tau_i} q_h) |\partial_{\tau_i} q_h|^{p'-1} \boldsymbol{\tau}_i & \text{if } E_i \not\subset \partial D, \\ \boldsymbol{v}_h(\boldsymbol{m}_i) := \boldsymbol{0} & \text{if } E_i \subset \partial D, \end{cases}$$

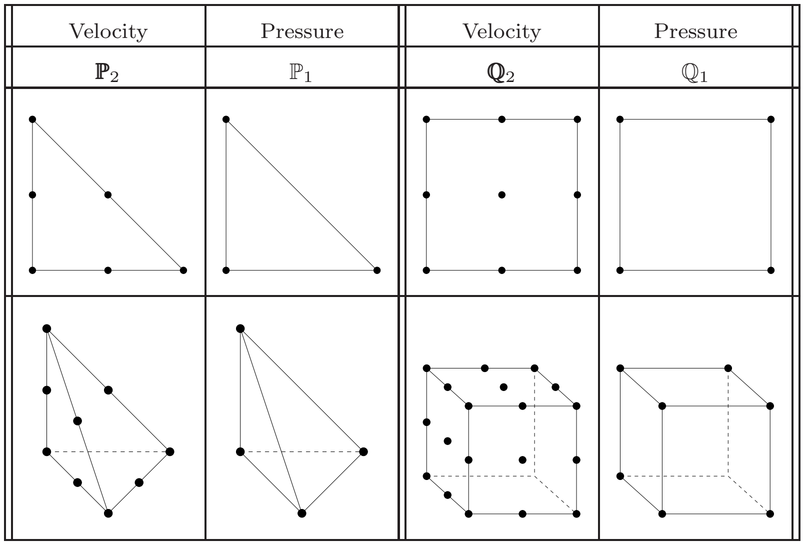

Fig. 54.2 Conventional representation of the $(\mathbb{P}_2, \mathbb{P}_1)$ pair (left) and of the $(\mathbb{Q}_2, \mathbb{Q}_1)$ pair (right) in dimensions two (top) and three (bottom, only visible degrees of freedom are shown).

where $\partial_{\tau_i} q_h := \boldsymbol{\tau}_i \cdot \nabla q_h$ denotes the tangential derivative of q_h along the oriented edge E_i. Note that $\boldsymbol{v}_h(\boldsymbol{m}_i)$ depends only on the values of q_h on E_i. Let $K \in \mathcal{T}_h$. Using the quadrature formula

$$\int_K \phi \,\mathrm{d}x = |K| \left(\sum_{\boldsymbol{m} \in \mathcal{M}_K} \frac{\phi(\boldsymbol{m})}{5} - \sum_{\boldsymbol{a} \in \mathcal{V}_K} \frac{\phi(\boldsymbol{a})}{20} \right), \qquad \forall \phi \in \mathbb{P}_2,$$

where $\mathcal{M}_K$ is the set of the midpoints of the edges of K and $\mathcal{V}_K$ is the set of the vertices of K and since Q_h is H^1-conforming, we infer that

$$\begin{aligned} \int_D q_h \nabla\cdot \boldsymbol{v}_h \,\mathrm{d}x &= -\int_D \boldsymbol{v}_h \cdot \nabla q_h \,\mathrm{d}x = -\sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{v}_h \cdot \nabla q_h \,\mathrm{d}x \\ &= -\sum_{K \in \mathcal{T}_h} |K| \sum_{\boldsymbol{m}_i \in K} \frac{1}{5} \boldsymbol{v}_h(\boldsymbol{m}_i) \cdot \nabla q_h(\boldsymbol{m}_i) \\ &= \sum_{K \in \mathcal{T}_h} |K| \sum_{\boldsymbol{m}_i \in K} \frac{1}{5} |\partial_{\tau_i} q_h(\boldsymbol{m}_i)|^{p'} l_i^{p'} \geq c \sum_{K \in \mathcal{T}_h} h_K^{p'} \|\nabla q_h\|_{\boldsymbol{L}^{p'}(K)}^{p'}. \end{aligned}$$

The last inequality results from the fact that $l_i \geq c h_K$ owing to the regularity of the mesh sequence, and that every tetrahedron $K \in \mathcal{T}_h$ has at least three edges in D, i.e., the quantities $|\partial_{\tau_i} q_h(\boldsymbol{m}_i)|$, where $\boldsymbol{m}_i$ spans the midpoints of the edges of K that are not in ∂D, control $\|\nabla q_h\|_{\ell^2}$. Finally, the inverse inequality from Lemma 12.1 (with $r := p$, $l := 1$, $m := 0$) together with Proposition 12.5 implies that for all $K \in \mathcal{T}_h$,

$$|\boldsymbol{v}_h|^p_{\boldsymbol{W}^{1,p}(K)} \leq c\, h_K^{-p}|K| \sum_{\boldsymbol{m}\in\mathcal{M}_K} \|\boldsymbol{v}_h(\boldsymbol{m})\|^p_{\ell^2},$$

and since $l_i \leq ch_K$, we have $\|\boldsymbol{v}_h(\boldsymbol{m})\|_{\ell^2} \leq ch_K^{p'}\|\nabla q_h\|^{p'-1}_{\ell^2}$. Since $p(p'-1) = p'$, combining these bounds shows that $|\boldsymbol{v}_h|^p_{\boldsymbol{W}^{1,p}(K)} \leq ch_K^{p'}\|\nabla q_h\|^{p'}_{\boldsymbol{L}^{p'}(K)}$ for all $K \in \mathcal{T}_h$. This proves (54.10). □

Lemma 54.9 (Stability). *For all* $p \in (1,\infty)$ *and under the hypotheses of Lemma* 54.8*, the* $(\mathbb{P}_2, \mathbb{P}_1)$ *pair satisfies the inf-sup condition* (54.8) *uniformly w.r.t.* $h \in \mathcal{H}$*.*

Proof. Apply Lemma 54.3. □

Remark 54.10 (Convergence rate). Owing to Theorem 53.17 and assuming that the solution to (53.6) is smooth enough, the solution to (53.14) with $(\boldsymbol{V}_{h0}, Q_h)$ defined in (54.9) satisfies $\mu|\boldsymbol{u} - \boldsymbol{u}_h|_{\boldsymbol{H}^1(D)} + \|p - p_h\|_{L^2(D)} \leq ch^2(\mu|\boldsymbol{u}|_{\boldsymbol{H}^3(D)} + |p|_{H^2(D)})$. Moreover, if the assumptions of Theorem 53.19 are met for some $s \in (0,1]$, then $\mu\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(D)} \leq ch^{2+s}\ell_D^{1-s}(\mu|\boldsymbol{u}|_{\boldsymbol{H}^3(D)} + |p|_{H^2(D)})$. □

Remark 54.11 (Literature). Further insight and alternative proofs can be found in Bercovier and Pironneau [54, Prop. 1], Girault and Raviart [217, p. 176], Stenberg [354]. We refer the reader to Mardal et al. [294] for the construction of a Fortin operator associated with the Taylor–Hood element in dimension two. Well-balanced schemes (see Remark 53.22) using Taylor–Hood mixed finite elements are analyzed in Lederer et al. [279]. □

54.4 Generalizations of the Taylor–Hood element

In this section, we briefly review some generalizations of the Taylor–Hood element: extension to quadrangles, higher-order extensions, and the use of a submesh to build the discrete velocity space.

54.4.1 The $(\mathbb{P}_k, \mathbb{P}_{k-1})$ and $(\mathbb{Q}_k, \mathbb{Q}_{k-1})$ pairs

It is possible to generalize the Taylor–Hood element to quadrangles and hexahedra. For instance, the $(\mathbb{Q}_2, \mathbb{Q}_1)$ pair has the same properties as the Taylor–Hood element; see Figure 54.2.

It is also possible to use higher-degree polynomials. For $k \geq 2$, the $(\mathbb{P}_k, \mathbb{P}_{k-1})$ pair and the $(\mathbb{Q}_k, \mathbb{Q}_{k-1})$ pair are stable in dimensions two and three. Provided the solution to (53.6) is smooth enough, these elements yield the error estimates $\mu|\boldsymbol{u}-\boldsymbol{u}_h|_{\boldsymbol{H}^1(D)} + \|p-p_h\|_{L^2(D)} \le c\,h^k(\mu|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(D)} + |p|_{H^k(D)})$ and $\mu\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(D)} \le ch^{k+s}\ell_D^{1-s}(\mu|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(D)} + |p|_{H^k(D)})$ if the assumptions of Theorem 53.19 are met for some $s \in (0,1]$. Proofs and further insight can be found in Stenberg [352, p. 18], Brezzi and Falk [92], Boffi et al. [65, p. 494], Boffi [60].

54.4.2 The $(\mathbb{P}_1\text{-iso-}\mathbb{P}_2, \mathbb{P}_1)$ and $(\mathbb{Q}_1\text{-iso-}\mathbb{Q}_2, \mathbb{Q}_1)$ pairs

An alternative to the Taylor–Hood element consists of replacing the $\mathbb{P}_2$ approximation of the velocity by a $\mathbb{P}_1$ approximation on a finer simplicial mesh. This finer mesh, say $\mathcal{T}_{\frac{h}{2}}$, is constructed as follows. In two dimensions, each triangle in $\mathcal{T}_h$ is divided into four new triangles by connecting the midpoints of the three edges. In three dimensions, each tetrahedron in $\mathcal{T}_h$ is divided into eight new tetrahedra (all having the same volume) by dividing each face into four new triangles and by connecting the midpoints of one pair of nonintersecting edges (there are three pairs of nonintersecting edges). This construction is illustrated in the top and bottom left panels of Figure 54.3. The discrete spaces are

$$\boldsymbol{V}_{h0} := \boldsymbol{P}^{\mathrm{g}}_{1,0}(\mathcal{T}_{\frac{h}{2}}), \qquad Q_h := P^{\mathrm{g}}_1(\mathcal{T}_h) \cap L^2_*(D). \tag{54.11}$$

These finite element pairs are often called $(\mathbb{P}_1\text{-iso-}\mathbb{P}_2, \mathbb{P}_1)$, or $(4\,\mathbb{P}_1, \mathbb{P}_1)$ in two dimensions and $(8\,\mathbb{P}_1, \mathbb{P}_1)$ in three dimensions.

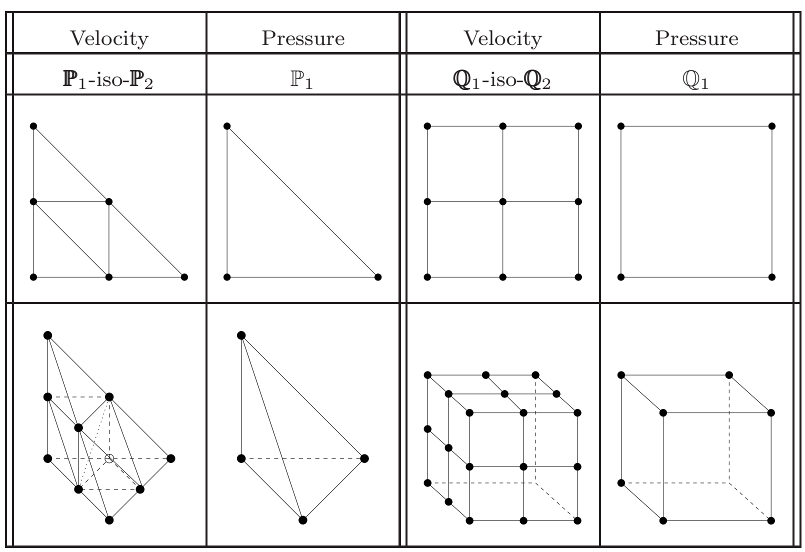

Fig. 54.3 $(\mathbb{P}_1\text{-iso-}\mathbb{P}_2, \mathbb{P}_1)$ (left) and $(\mathbb{Q}_1\text{-iso-}\mathbb{Q}_2, \mathbb{Q}_1)$ (right) pairs in dimensions two (top) and three (bottom, only visible degrees of freedom are shown for the $(\mathbb{Q}_1\text{-iso-}\mathbb{Q}_2, \mathbb{Q}_1)$ pair).

The ($\mathbb{P}_1$-iso-$\mathbb{P}_2, \mathbb{P}_1$) pair can be generalized to quadrangles in two dimensions and hexahedra in three dimensions. Assume that $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is a shape-regular sequence of meshes composed of quadrangles or hexahedra. A new mesh $\mathcal{T}_{\frac{h}{2}}$ is defined in two dimensions by dividing each quadrangle in $\mathcal{T}_h$ into four new quadrangles and by connecting the midpoints of all the pairs of nonintersecting edges. In three dimensions, we divide each hexahedron in $\mathcal{T}_h$ into eight new hexahedra by dividing each face into four quadrangles and by connecting the barycenters of all the pairs of nonintersecting faces. This construction is illustrated in the top and bottom right panels of Figure 54.3. The discrete spaces are

$$\boldsymbol{V}_{h0} := \{\boldsymbol{v}_h \in \boldsymbol{C}^0(\overline{D}) \mid \forall K \in \mathcal{T}_{\frac{h}{2}},\ \boldsymbol{v}_h \circ \boldsymbol{T}_K \in \mathbb{Q}_1,\ \boldsymbol{v}_{h|\partial D} = \boldsymbol{0}\}, \tag{54.12a}$$

$$Q_h := \{q_h \in C^0(\overline{D}) \cap L^2_*(D) \mid \forall K \in \mathcal{T}_h,\ q_h \circ \boldsymbol{T}_K \in \mathbb{Q}_1\}. \tag{54.12b}$$

These finite elements are often called ($\mathbb{Q}_1$-iso-$\mathbb{Q}_2, \mathbb{Q}_1$), or ($4\,\mathbb{Q}_1, \mathbb{Q}_1$) in dimension two and ($8\,\mathbb{Q}_1, \mathbb{Q}_1$) in dimension three.

Lemma 54.12 (Stability). *For all $p \in (1, \infty)$, and under the hypotheses of Lemma 54.8 if $\mathcal{T}_h$ is composed of simplices, the ($\mathbb{P}_1$-iso-$\mathbb{P}_2, \mathbb{P}_1$) and ($\mathbb{Q}_1$-iso-$\mathbb{Q}_2, \mathbb{Q}_1$) pairs satisfy the inf-sup condition (54.8) uniformly w.r.t. $h \in \mathcal{H}$.*

Proof. Adapt the proof of Lemma 54.8; see Bercovier and Pironneau [54] (for $d = 2$ and $p = 2$) and Exercise 54.4. □

Remark 54.13 (Convergence rate). Owing to Theorem 53.17 and assuming that the solution to (53.6) is smooth enough, the discrete solution to (53.14) with $(\boldsymbol{V}_{h0}, Q_h)$ defined in either (54.11) or (54.12) satisfies $\mu|\boldsymbol{u} - \boldsymbol{u}_h|_{\boldsymbol{H}^1(D)} + \|p - p_h\|_{L^2(D)} \leq ch(\mu|\boldsymbol{u}|_{\boldsymbol{H}^2(D)} + |p|_{H^1(D)})$, and if the assumptions of Theorem 53.19 are met for some $s \in (0, 1]$, we have $\mu\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(D)} \leq ch^{1+s}\ell_D^{1-s}(\mu|\boldsymbol{u}|_{\boldsymbol{H}^2(D)} + |p|_{H^1(D)})$. □

Exercises

Exercise 54.1 (Mini element). Show that the Fortin operator $\boldsymbol{\Pi}_h$ constructed in the proof of Lemma 54.5 is of the form $\boldsymbol{\Pi}_h(\boldsymbol{v}) := \boldsymbol{\mathcal{I}}^{\mathrm{av}}_{h0}(\boldsymbol{v}) + \sum_{K\in\mathcal{T}_h} \sum_{i\in\{1:d\}} \gamma^i_K(\boldsymbol{v}) b_K \boldsymbol{e}_i$, for some coefficients $\gamma^i_K(\boldsymbol{v})$ to be determined. Here, $\{\boldsymbol{e}_i\}_{i\in\{1:d\}}$ is the canonical Cartesian basis of $\mathbb{R}^d$.

Exercise 54.2 (Bubble⇔Stabilization). Consider the mini element defined in §54.2 and assume that the viscosity μ is constant over D. Recall that $\boldsymbol{V}_{h0} := \boldsymbol{V}^1_{h0} \oplus \boldsymbol{B}_h$ and $Q_h := P^{\mathrm{g}}_1(\mathcal{T}_h) \cap L^2_*(D)$ with $\boldsymbol{V}^1_{h0} := \mathbb{P}^{\mathrm{g}}_{1,0}(\mathcal{T}_h)$. Let $(\boldsymbol{u}_h, p_h)$ be the solution to the discrete Stokes problem (53.14). (i) Show that $a(\boldsymbol{v}_h, \boldsymbol{b}_h) = 0$ for all $\boldsymbol{v}_h \in \boldsymbol{V}^1_{h0}$ and all $\boldsymbol{b}_h \in \boldsymbol{B}_h$. (ii) Set $\boldsymbol{u}_h := \boldsymbol{u}^1_h + \boldsymbol{u}^b_h \in \boldsymbol{V}_{h0}$. Show that

$$a(\boldsymbol{u}_h^1, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = F(\boldsymbol{v}_h), \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_{h0}^1. \tag{54.13}$$

(iii) Let $b_K := \widehat{b} \circ \boldsymbol{T}_K$ be the bubble function on $K \in \mathcal{T}_h$. Let $\{\boldsymbol{e}_i\}_{i\in\{1:d\}}$ be the canonical Cartesian basis of $\mathbb{R}^d$. Let $\mathcal{S}^K \in \mathbb{R}^{d\times d}$ be defined by $\mathcal{S}_{ij}^K := \frac{1}{\int_K b_K \,\mathrm{d}x} a(b_K\boldsymbol{e}_j, b_K\boldsymbol{e}_i)$ for all $i,j \in \{1:d\}$. Let $\boldsymbol{u}_{h|K}^b := \sum_{i\in\{1:d\}} c_K^i \boldsymbol{e}_i b_K$. Show that $\boldsymbol{c}_K = (\mathcal{S}^K)^{-1}(\boldsymbol{F}_K - \nabla p_{h|K})$, where $F_K^i := \frac{1}{\int_K b_K \,\mathrm{d}x} F(b_K \boldsymbol{e}_i)$, for all $i \in \{1:d\}$. (iv) Set $c_h(p_h, q_h) := \sum_{K\in\mathcal{T}_h} \nabla q_{h|K} (\mathcal{S}^K)^{-1} \nabla p_{h|K} \int_K b_K \,\mathrm{d}x$ and $R_h(q_h) := \sum_{K\in\mathcal{T}_h} \nabla q_{h|K} (\mathcal{S}^K)^{-1} \boldsymbol{F}_K \int_K b_K \,\mathrm{d}x$. Show that the mass conservation equation becomes

$$b(\boldsymbol{u}_h^1, q_h) - c_h(p_h, q_h) = G(q_h) - R_h(q_h), \qquad \forall q_h \in Q_h. \tag{54.14}$$

Note: since $(\mathcal{S}_K)^{-1}$ scales like $\mu^{-1}h_K^2$, $c_h(p_h,q_h)$ behaves like $\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\mu} \int_K \nabla q_h \cdot \nabla p_h \,\mathrm{d}x$, and $R_h(q_h)$ scales like $\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\mu} \int_K \nabla q_{h|K} \cdot F_K \,\mathrm{d}x$. This shows that, once the bubbles are eliminated, the system (54.13)-(54.14) is equivalent to a stabilized form of the Stokes system for the $(\mathbb{P}_1, \mathbb{P}_1)$ pair; see Chapters 62 and 63.

Exercise 54.3 (Singular vertex). Let $K \subset \mathbb{R}^2$ be a quadrangle and let $\boldsymbol{z}$ be the intersection of the two diagonals of K. Let $K_1, \ldots, K_4$ be the four triangles formed by dividing K along its two diagonals (assume that $K_1 \cap K_3 = \{\boldsymbol{z}\}$ and $K_2 \cap K_4 = \{\boldsymbol{z}\}$). (i) Let ϕ be a scalar field continuous over K and of class C^1 over the triangles $K_1, \ldots, K_4$. Prove that $\sum_{i\in\{1:4\}} (-1)^i \boldsymbol{n}\cdot\nabla\phi_{|K_i}(\boldsymbol{z}) = 0$ for every unit vector $\boldsymbol{n}$. (ii) Let $\boldsymbol{v}$ be a vector field continuous over K and of class C^1 over the triangles $K_1, \ldots, K_4$. Prove that $\sum_{i\in\{1:4\}} (-1)^i \nabla\cdot\boldsymbol{v}_{|K_i}(\boldsymbol{z}) = 0$. (iii) Assume that $\boldsymbol{v}$ is linear over each triangle. Show that the four equations $\int_{K_i} \nabla\cdot\boldsymbol{v} \,\mathrm{d}x = 0$ for all $i \in \{1:4\}$ are linearly dependent.

Exercise 54.4 ($\mathbb{P}_1$-iso-$\mathbb{P}_2, \mathbb{P}_1$). Consider the setting of Lemma 54.12 with the ($\mathbb{P}_1$-iso-$\mathbb{P}_2, \mathbb{P}_1$) pair in dimension three. (i) Let $K \in \mathcal{T}_h$. Let $\mathcal{V}_K$ be the set of the vertices of K. Let $\mathcal{M}_K$ be the midpoints of the six edges of K. Let $\mathcal{M}_K^1$ be the set of the two midpoints that are connected to create the 8 new tetrahedra. Let $\mathcal{M}_K^2$ be the set of the remaining midpoints. Let $\boldsymbol{V}_{h0}$ be the $\mathbb{P}_1$ velocity space based of $\mathcal{T}_{h/2}$. Find the coefficients α, β, γ so that the following quadrature is exact for all $\boldsymbol{w}_h \in V_{h0}$: $\int_K \boldsymbol{w}_h \,\mathrm{d}x = |K|(\alpha \sum_{\boldsymbol{z}\in\mathcal{V}_K} \boldsymbol{w}_h(\boldsymbol{z}) + \beta \sum_{\boldsymbol{m}\in\mathcal{M}_K^1} \boldsymbol{w}_h(\boldsymbol{m}) + \gamma \sum_{\boldsymbol{m}\in\mathcal{M}_K^2} \boldsymbol{w}_h(\boldsymbol{m}))$. (*Hint*: on a tetrahedron K' with vertices $\{\boldsymbol{z}'\}_{\boldsymbol{z}'\in\mathcal{V}_{K'}}$, the quadrature $\int_{K'} \boldsymbol{w}_h \,\mathrm{d}x = |K'| \sum_{\boldsymbol{z}'\in\mathcal{V}_{K'}} \frac{1}{4} \boldsymbol{w}_h(\boldsymbol{z}')$ is exact on $\mathbb{P}_1$.) (ii) Prove Lemma 54.12 for the ($\mathbb{P}_1$-iso-$\mathbb{P}_2, \mathbb{P}_1$) pair in dimension three for all $p \in (1, \infty)$. (*Hint*: adapt the proof of Lemma 54.8.)