



High-order time stepping for the Navier–Stokes equations with minimal computational complexity



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ABSTRACT

In this paper we present extensions of the schemes proposed in Guermond and Minev (2015) that lead to a decoupling of the velocity components in the momentum equation. The new schemes reduce the solution of the incompressible Navier–Stokes equations to a set of classical scalar parabolic problems for each Cartesian component of the velocity. The pressure is explicitly recovered after the velocity is computed.

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1. Introduction

In [1], we considered the possibility to construct high-order artificial compressibility schemes for incompressible flow by using a combination of the first-order scheme proposed by Shen [2] and the defect (or deferred) correction approach (see e.g. [3]). The resulting schemes require the solution of several problems of the type $\mathbf{u} - \tau(\mu\Delta\mathbf{u} + \omega\nabla\nabla\cdot)\mathbf{u} = \tau\mathbf{f}$ at each time step, τ being the time step. The corresponding discrete problems clearly have a condition number of order of τh^{-2} , h being the spatial step. Although these problems are relatively well conditioned, the operator $\nabla\nabla\cdot$ couples all components of the velocity and, denoting by d the space dimension, the associated linear system is d times larger than the linear system associated with a scalar PDE. The first objective of the present paper is to introduce algorithms that discretize explicitly the mixed derivatives in the operator $\nabla\nabla\cdot$ in order to decouple the Cartesian components of the velocity and thereby reduce the problem to a series of scalar-valued parabolic problems. In fact, such strategies based on the direction splitting approach, which was popular at that time, have been proposed in the literature in the 1960s and 70s. For instance, a direction splitting scheme that includes the splitting of the $\nabla\nabla\cdot$ operator has been proposed in the Russian literature by the groups of Yanenko (see [4,5], Section 8.3) and Ladyzhenskaya (see [6, chapter VI, Section 9.2], and the references therein). In the Western literature, such schemes have been proposed and analyzed by Temam [7, chapter III, Section 8.3]. In the present paper we generalize the approach to make it applicable to non-Cartesian grids without splitting, and we combine it

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with the defect correction approach discussed in [1] to increase the accuracy order. The second objective of the paper is to show that it is possible to reduce further the complexity of the proposed algorithms and still be unconditionally stable and accurate by applying direction splitting techniques.

The paper is organized as follows. In Section 2 we introduce notation, the model problem and recall the defect correction method from [1]. The implicit–explicit coordinate splitting of the operator $\nabla \nabla \cdot$ is done in Section 3 (first objective of the paper). Further simplifications of the algorithms introduced in Section 3 are proposed in Section 4; these simplifications essentially consist of using direction splitting (second objective of the paper). It is shown in Section 5 that the algorithms introduced in Section 3 and Section 4 can be made second and higher-order accurate by using the defect correction method of [1]. All the proposed algorithms are tested and illustrated numerically in Section 6. Concluding remarks are reported in Section 7.

2. Preliminaries

2.1. Formulation of the problem

We consider the time-dependent Navier–Stokes equations on a finite time interval $[0, T]$ and in a domain Ω in \mathbb{R}^d with a Lipschitz boundary. Since the nonlinear term in the Navier–Stokes equations has no significant influence on the pressure–velocity coupling and since this term is usually made explicit, we henceforth mostly consider the time-dependent Stokes equations written in terms of velocity \mathbf{u} and pressure p :

$$\begin{cases} \partial_t \mathbf{u} + \mathbb{A} \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \times [0, T], \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times [0, T], \\ \mathbf{u}|_{\partial \Omega} = 0 & \text{in } [0, T], \quad \text{and } \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where \mathbf{f} is a smooth source term and \mathbf{u}_0 is a solenoidal initial velocity field with zero normal trace at the boundary of Ω . The operator \mathbb{A} is assumed to be linear, \mathbf{H}^1 -coercive and bounded, i.e., there are two constants $\nu > 0$ and $M < \infty$ such that $\int_{\Omega} \mathbb{A} \mathbf{u} \cdot \mathbf{u} \, dx \geq \nu \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}^2$ and $|\int_{\Omega} \mathbb{A} \mathbf{u} \cdot \mathbf{v} \, dx| \leq M \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}$, for all $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$. For the sake of simplicity, we consider homogeneous Dirichlet boundary conditions on the velocity.

Remark 2.1 (Other Boundary Conditions). All the stability results presented below generalize to other types of boundary conditions provided that they guarantee the coerciveness of the bilinear form generated by \mathbb{A} . We can formalize this statement by assuming that there exists a closed subspace \mathbf{V} of $\mathbf{H}^1(\Omega)$ and a bilinear form $a : V \times V \rightarrow \mathbb{R}$ such that the boundary conditions imposed in (2.1) are such that $\int_{\Omega} (\mathbb{A} \mathbf{u} + \nabla p) \cdot \mathbf{v} \, dx = a(\mathbf{u}, \mathbf{v}) - \int_{\Omega} p \nabla \cdot \mathbf{v} \, dx$ for all $\mathbf{v} \in \mathbf{V}$ and $a(\mathbf{u}, \mathbf{u}) \geq \nu \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}^2$, $a(\mathbf{u}, \mathbf{v}) \leq M \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}$, for all $\mathbf{u}, \mathbf{v} \in V$.

We are going to be mainly concerned with time discretizations of the above problem. Let $\tau > 0$ be a time step and set $t^n = n\tau$ for $0 \leq n \leq N = \lfloor T/\tau \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Let $\phi^0, \phi^1, \dots, \phi^N$ be some sequence of functions in a Hilbert space E . We denote by ϕ_{τ} this sequence, and we define the following discrete norms: $\|\phi_{\tau}\|_{\ell^2(E)} := (\tau \sum_{n=0}^N \|\phi^n\|_E^2)^{\frac{1}{2}}$, $\|\phi_{\tau}\|_{\ell^\infty(E)} := \max_{0 \leq n \leq N} (\|\phi^n\|_E)$. In addition, we denote the first differences of the elements of the sequence by $\delta_t \phi^n = (\phi^n - \phi^{n-1})$, $n = 1, \dots, N$, and their average by $\bar{\phi}^n = (\phi^n + \phi^{n-1})/2$, $n = 1, \dots, N$. The sequences $\delta_t \phi^1, \dots, \delta_t \phi^N$ and $\bar{\phi}^1, \dots, \bar{\phi}^N$ are denoted by $\delta_t \phi_{\tau}$ and $\bar{\phi}_{\tau}$ correspondingly. We also denote by c a generic constant that is independent of τ and ϵ but possibly depends on the data, the domain, and the solution.

2.2. High-order artificial compressibility

In [1] we introduced a series of second and third-order schemes based on the following first-order artificial compressibility algorithm, proposed by Shen [2]:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + \mathbb{A} \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1}, \quad \frac{\epsilon}{\tau} (p^{n+1} - p^n) + \nabla \cdot \mathbf{u}^{n+1} = 0, \tag{2.2}$$

where $\epsilon > 0$ is a user-dependent parameter that is usually chosen to be proportional to τ , i.e., $\epsilon = \tau/\chi$ where χ is of order one. One interesting property of this scheme is that it decouples the velocity and the pressure; more precisely, the algorithm can be recast as follows:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + \mathbb{A} \mathbf{u}^{n+1} - \chi \nabla \nabla \cdot \mathbf{u}^{n+1} = \mathbf{f}^{n+1} - \nabla p^n, \quad p^{n+1} = p^n - \chi \nabla \cdot \mathbf{u}^{n+1}. \tag{2.3}$$

The above algorithm has been extended to third-order accuracy in time in [1] by using a defect correction method. Since in case of the full Navier–Stokes equations, the nonlinear term $\mathbf{B} \mathbf{u} := \mathbf{u} \cdot \nabla \mathbf{u}$ can also be handled by the defect correction,

we include it in the algorithm for completeness. The full third-order scheme is as follows:

$$n \geq 0, \quad \begin{cases} \mathbf{n}_0^{n+1} = B\mathbf{u}_0^n, \\ \frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\tau} + \mathbb{A}\mathbf{u}_0^{n+1} - \chi \nabla \nabla \cdot \mathbf{u}_0^{n+1} + \nabla p_0^n = \mathbf{f}^{n+1} - \mathbf{n}_0^{n+1} \\ p_0^{n+1} = p_0^n - \chi \nabla \cdot \mathbf{u}_0^{n+1}, \\ d\mathbf{u}_0^{n+1} = (\mathbf{u}_0^{n+1} - \mathbf{u}_0^n)/\tau, \quad dp_0^{n+1} = (p_0^{n+1} - p_0^n)/\tau \end{cases} \quad (2.4)$$

$$n \geq 1, \quad \begin{cases} d^2\mathbf{u}_0^{n+1} = (d\mathbf{u}_0^{n+1} - d\mathbf{u}_0^n)/\tau, \\ \mathbf{n}_1^n = B(\mathbf{u}_0^n + \tau\mathbf{u}_1^{n-1}), \\ \frac{\mathbf{u}_1^n - \mathbf{u}_1^{n-1}}{\tau} + \mathbb{A}\mathbf{u}_1^n - \chi \nabla \nabla \cdot \mathbf{u}_1^n + \nabla(p_1^{n-1} + dp_0^n) = -\frac{1}{2}d^2\mathbf{u}_0^{n+1} - \frac{\mathbf{n}_1^n - \mathbf{n}_0^n}{\tau}, \\ p_1^n = p_1^{n-1} + dp_0^n - \chi \nabla \cdot \mathbf{u}_1^n, \\ d\mathbf{u}_1^n = (\mathbf{u}_1^n - \mathbf{u}_1^{n-1})/\tau, \quad dp_1^n = (p_1^n - p_1^{n-1})/\tau, \end{cases} \quad (2.5)$$

$$n \geq 2, \quad \begin{cases} d^2\mathbf{u}_1^n = (d\mathbf{u}_1^n - d\mathbf{u}_1^{n-1})/\tau, \quad d^3\mathbf{u}_0^{n+1} = (d^2\mathbf{u}_0^{n+1} - d^2\mathbf{u}_0^n)/\tau, \\ \mathbf{n}_2^{n-1} = B(\mathbf{u}_0^{n-1} + \tau\mathbf{u}_1^{n-1} + \tau^2\mathbf{u}_2^{n-2}) \\ \frac{\mathbf{u}_2^{n-1} - \mathbf{u}_2^{n-2}}{\tau} + \mathbb{A}\mathbf{u}_2^{n-1} - \chi \nabla \nabla \cdot \mathbf{u}_2^{n-1} + \nabla(p_2^{n-2} + dp_1^{n-1}) = -\frac{1}{2}d^2\mathbf{u}_1^n + \frac{1}{6}d^3\mathbf{u}_0^{n+1} - \frac{\mathbf{n}_2^{n-1} - \mathbf{n}_1^{n-1}}{\tau^2} \\ p_2^{n-1} = p_2^{n-2} + dp_1^{n-1} - \chi \nabla \cdot \mathbf{u}_2^{n-1}, \\ \mathbf{u}^{n-1} = \mathbf{u}_0^{n-1} + \tau\mathbf{u}_1^{n-1} + \tau^2\mathbf{u}_2^{n-1}, \quad p^{n-1} = p_0^{n-1} + \tau p_1^{n-1} + \tau^2 p_2^{n-1}. \end{cases} \quad (2.6)$$

The stage (2.4) yields a first-order approximation of the velocity and the pressure, the second stage (2.5) yields a second-order approximation of the velocity and the pressure, and the third stage (2.6) yields a third-order approximation of the velocity and the pressure.

One drawback of the above scheme is the presence of the operator $\nabla \nabla \cdot$ since this operator couples all the Cartesian components of the velocity and can lead to locking if not discretized properly. In the next section we introduce a first-order artificial compressibility scheme that decouples the different components of the velocity, i.e., we develop a decoupled version of the first stage (2.4). We will use this approach later in the paper to modify the subsequent two stages and create a high-order time stepping for the Navier–Stokes equations that requires only the solution of a set of scalar-valued parabolic problems for each Cartesian component of the velocity. Since the proofs of stability of these schemes in two and three dimensions differ somewhat, we will consider these two cases separately in the next section.

3. Coordinate splitting of the grad–div operator

3.1. Splitting of \mathbb{A}

In this section we demonstrate that the velocity components in the grad–div operator can be decoupled by discretizing the mixed derivatives explicitly, while still preserving the unconditional stability of the scheme. To stay general, we assume that the operator \mathbb{A} admits the following decomposition $\mathbb{A}\mathbf{u} = A\mathbf{u} - \nabla(\lambda \nabla \cdot \mathbf{u})$ where λ is a smooth positive scalar field. We assume also that A is block diagonal, \mathbf{H}^1 -coercive and bounded, i.e., we assume that there are constants ν and M such that $\mathbf{A}\mathbf{u} = (A_1u_1, \dots, A_du_d)^\top$, $\int_\Omega uA_iu \, d\mathbf{x} \geq \nu \|u\|_{H^1(\Omega)}^2$ and $|\int_\Omega vA_iu \, d\mathbf{x}| \leq M \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$, for all $u, v \in H_0^1(\Omega)$, where u_1, \dots, u_d are the Cartesian components of \mathbf{u} . We do not need to know how ν depends on λ in the following sections. This decomposition holds for instance when $\mathbb{A}\mathbf{u} = -\nabla \cdot (\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top) + \kappa\nabla \cdot \mathbf{u}\mathbb{I})$ where \mathbb{I} is the $d \times d$ identity matrix. Assuming in this case that μ is constant over Ω , we have $\mathbf{A}\mathbf{u} = -\nabla \cdot (\mu\nabla\mathbf{u})$ and $\lambda = \mu + \kappa$.

The first-order algorithm (2.3) can be rewritten as follows in this new context:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + \mathbf{A}\mathbf{u}^{n+1} - \nabla(\varpi \nabla \cdot \mathbf{u}^{n+1}) = \mathbf{f}^{n+1} - \nabla p^n, \quad p^{n+1} = p^n - \varpi \nabla \cdot \mathbf{u}^{n+1}, \quad (3.1)$$

where ϖ depends on λ and χ is defined by $\varpi := \lambda + \chi$ and we recall that $\chi = \tau/\epsilon$. In the rest of the paper, the presence of the field λ in the decomposition $\mathbb{A}\mathbf{u} = A\mathbf{u} - \nabla(\lambda \nabla \cdot \mathbf{u})$ is encoded in (3.1) through the field ϖ .

Remark 3.1 (Other Boundary Conditions). In line with Remark 2.1, if general boundary conditions are imposed, then the meaning of (3.1) and \mathbf{u}^{n+1} is that there are d closed subspaces $V_1, \dots, V_d \subset H^1(\Omega)$ and d bilinear forms a_1, \dots, a_d such that

$$\int_\Omega \frac{(u_i^{n+1} - u_i^n)v}{\tau} + a_i(u_i^{n+1}, v) - \int_\Omega \varpi \nabla \cdot \mathbf{u}^{n+1} \partial_i v \, d\mathbf{x} = \int_\Omega \mathbf{f}_i^{n+1} v \, d\mathbf{x} + \int_\Omega p^n \partial_i v \, d\mathbf{x},$$

for all $v \in V_i, i = 1, \dots, d$, and $p^{n+1} = -p^n - \varpi \nabla \cdot \mathbf{u}^{n+1}$,

3.2. Gauss–Seidel and Jacobi ansätze in two dimensions

In this section we consider the decoupling of the velocity components for the two-dimensional version of the algorithm (3.1). Let us denote by u_1, u_2 the Cartesian components of \mathbf{u} , i.e., $\mathbf{u} = (u_1, u_2)^\top$. Then consider the following modification of the algorithm:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + A\mathbf{u}^{n+1} - \begin{pmatrix} \partial_{x_1}(\varpi(\partial_{x_1}u_1^{n+1} + \partial_{x_2}u_2^n)) \\ \partial_{x_2}(\varpi(\partial_{x_1}u_1^{n+1} + \partial_{x_2}u_2^{n+1})) \end{pmatrix} = \mathbf{f}^{n+1} - \nabla p^n \tag{3.2}$$

with $p^{n+1} = p^n - \varpi \nabla \cdot \mathbf{u}^{n+1}$. Note that since we assumed that A is block diagonal, meaning that $A\mathbf{u} = (A_1u_1, A_2u_2)$, the Cartesian components of \mathbf{u} are indeed decoupled because the algorithm can be recast as follows:

$$\begin{cases} \frac{u_1^{n+1} - u_1^n}{\tau} + A_1u_1^{n+1} - \partial_{x_1}(\varpi \partial_{x_1}u_1^{n+1}) = f_1^{n+1} - \partial_{x_1}(p^n - \varpi \partial_{x_2}u_2^n) \\ \frac{u_2^{n+1} - u_2^n}{\tau} + A_2u_2^{n+1} - \partial_{x_2}(\varpi \partial_{x_2}u_2^{n+1}) = f_2^{n+1} - \partial_{x_2}(p^n - \varpi \partial_{x_1}u_1^{n+1}). \end{cases} \tag{3.3}$$

These two problems only require to solve classical scalar-valued parabolic equations. Before going through the stability analysis, let us first observe that (3.2) can be rewritten as follows:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + A\mathbf{u}^{n+1} - \nabla(\varpi \nabla \cdot \tilde{\mathbf{u}}^{n+1}) - (\mathbf{0}, \partial_{x_2}(\varpi \partial_{x_2} \delta_t u_2^{n+1}))^\top = \mathbf{f}^{n+1} - \nabla p^n \tag{3.4}$$

where $\tilde{\mathbf{u}}^{n+1} = (u_1^{n+1}, u_2^n)^\top$. We assume that $\mathbf{f} = (f_1, f_2) = \mathbf{0}$ in order to establish the stability of the scheme with respect to the initial data. The case of a non-zero source term can be considered similarly, but since this unnecessarily introduces irrelevant technicalities we will omit the source term in the rest of the paper. The scheme (3.2) is unconditionally stable as stated by the following theorem.

Theorem 3.1. *Under suitable initialization and smoothness assumptions, the algorithm (3.2) is unconditionally stable, i.e., for any finite time interval $(0, T]$ we have:*

$$\begin{aligned} & \|\mathbf{u}_\tau\|_{\ell^\infty(L^2(\Omega))}^2 + \tau \|\varpi^{-\frac{1}{2}} p_\tau\|_{\ell^\infty(L^2(\Omega))}^2 + \tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_{2,\tau}\|_{\ell^\infty(L^2(\Omega))}^2 + \tau^{-1} \|\delta_t \mathbf{u}_\tau\|_{\ell^2(L^2(\Omega))}^2 \\ & + 2\nu \|\mathbf{u}_\tau\|_{\ell^2(\mathbf{H}^1(\Omega))}^2 + \|\varpi^{\frac{1}{2}} \nabla \cdot \tilde{\mathbf{u}}\|_{\ell^2(L^2(\Omega))}^2 \leq c(\|\mathbf{u}^0\|_{L^2(\Omega)}^2 + \tau \|\varpi^{-\frac{1}{2}} p^0\|_{L^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^0\|_{L^2(\Omega)}^2). \end{aligned} \tag{3.5}$$

Proof. We first multiply the momentum equation in (3.4) by $2\tau \mathbf{u}^{n+1}$, then, using the identity $2(a - b, a) = \|a\|^2 + \|a - b\|^2 - \|b\|^2$ and the coerciveness of A in $\mathbf{H}^1(\Omega)$, we obtain:

$$\begin{aligned} & \|\mathbf{u}^{n+1}\|_{L^2(\Omega)}^2 + \|\delta_t \mathbf{u}^{n+1}\|_{L^2(\Omega)}^2 - \|\mathbf{u}^n\|_{L^2(\Omega)}^2 + 2\nu\tau \|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 + 2\tau(\varpi \nabla \cdot \tilde{\mathbf{u}}^{n+1}, \nabla \cdot \mathbf{u}^{n+1}) \\ & + \tau(\|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^{n+1}\|_{L^2(\Omega)}^2 + \|\varpi^{\frac{1}{2}} \partial_{x_2} \delta_t u_2^{n+1}\|_{L^2(\Omega)}^2 - \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^n\|_{L^2(\Omega)}^2) - 2\tau(p^n, \nabla \cdot \mathbf{u}^{n+1}) \leq 0. \end{aligned}$$

Now taking the square of the pressure equation $\varpi^{-\frac{1}{2}} p^{n+1} = \varpi^{-\frac{1}{2}} p^n - \varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^{n+1}$ gives

$$\tau \|\varpi^{-\frac{1}{2}} p^{n+1}\|_{L^2(\Omega)}^2 = \tau \|\varpi^{-\frac{1}{2}} p^n\|_{L^2(\Omega)}^2 - 2\tau(\nabla \cdot \mathbf{u}^{n+1}, p^n) + \tau \|\varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^{n+1}\|_{L^2(\Omega)}^2.$$

Adding the above inequality and equation, we obtain:

$$\begin{aligned} & \|\mathbf{u}^{n+1}\|_{L^2(\Omega)}^2 + \|\delta_t \mathbf{u}^{n+1}\|_{L^2(\Omega)}^2 + 2\nu\tau \|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 + \tau \|\varpi^{-\frac{1}{2}} p^{n+1}\|_{L^2(\Omega)}^2 \\ & + \tau \|\varpi^{\frac{1}{2}} \nabla \cdot \tilde{\mathbf{u}}^{n+1}\|_{L^2(\Omega)}^2 - \tau \|\varpi^{\frac{1}{2}} \nabla \cdot (\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1})\|_{L^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} \partial_{x_2} \delta_t u_2^{n+1}\|_{L^2(\Omega)}^2 - \tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^{n+1}\|_{L^2(\Omega)}^2 \\ & \leq \|\mathbf{u}^n\|_{L^2(\Omega)}^2 + \tau \|\varpi^{-\frac{1}{2}} p^n\|_{L^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^n\|_{L^2(\Omega)}^2. \end{aligned}$$

Note that $\nabla \cdot (\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}) = \partial_{x_2} \delta_t u_2^{n+1}$, i.e., $\|\varpi^{\frac{1}{2}} \nabla \cdot (\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1})\|_{L^2(\Omega)} = \|\varpi^{\frac{1}{2}} \partial_{x_2} \delta_t u_2^{n+1}\|_{L^2(\Omega)}$. Then summing the above inequality for $n = 0, \dots, N - 1$, with $N = \lfloor T/\tau \rfloor$, yields the desired result. \square

The algorithm (3.2) can be thought of as a Gauss–Seidel approximation of (3.1). This observation, then leads us to consider also the Jacobi approximation which consists of replacing $\nabla \nabla \cdot \mathbf{u}^{n+1}$ in (3.1) by $\nabla(\varpi \nabla \cdot \mathbf{u}^n) + (\partial_{x_1}(\varpi \partial_{x_1} \delta_t u_1^{n+1}), \partial_{x_2}(\varpi \partial_{x_2} \delta_t u_2^{n+1}))^\top$, that is to say

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + A\mathbf{u}^{n+1} - \nabla(\varpi \nabla \cdot \mathbf{u}^n) - \begin{pmatrix} \partial_{x_1}(\varpi \partial_{x_1} \delta_t u_1^{n+1}) \\ \partial_{x_2}(\varpi \partial_{x_2} \delta_t u_2^{n+1}) \end{pmatrix} = \mathbf{f}^{n+1} - \nabla p^n \tag{3.6}$$

with $p^{n+1} = p^n - \varpi \nabla \cdot \mathbf{u}^{n+1}$. Let us define $\check{\mathbf{u}}^{n+1} = (u_1^n, u_2^{n+1})^\top$.

Theorem 3.2. Under suitable initialization and smoothness assumptions, the Jacobi algorithm (3.6) is unconditionally stable, i.e., for any finite time interval $(0, T]$ we have:

$$\begin{aligned} & \| \mathbf{u}_\tau \|_{\ell^\infty(L^2(\Omega))}^2 + \tau \| \varpi^{-\frac{1}{2}} p_\tau \|_{\ell^\infty(L^2(\Omega))}^2 + \tau \| \varpi^{\frac{1}{2}} \partial_{x_1} u_{1,\tau} \|_{\ell^\infty(L^2(\Omega))}^2 + \tau \| \varpi^{\frac{1}{2}} \partial_{x_2} u_{2,\tau} \|_{\ell^\infty(L^2(\Omega))}^2 \\ & + \tau^{-1} \| \delta_t \mathbf{u}_\tau \|_{\ell^2(L^2(\Omega))}^2 + 2\nu \| \mathbf{u}_\tau \|_{\ell^2(\mathbf{H}^1(\Omega))}^2 + \| \varpi^{\frac{1}{2}} \nabla \cdot \tilde{\mathbf{u}} \|_{\ell^2(L^2(\Omega))}^2 + \| \varpi^{\frac{1}{2}} \nabla \cdot \check{\mathbf{u}} \|_{\ell^2(L^2(\Omega))}^2 \\ & \leq c (\| \mathbf{u}^0 \|_{L^2(\Omega)}^2 + \tau \| \varpi^{-\frac{1}{2}} p^0 \|_{L^2(\Omega)}^2 + \tau \| \varpi^{\frac{1}{2}} \partial_{x_1} u_1^0 \|_{L^2(\Omega)}^2 + \tau \| \varpi^{\frac{1}{2}} \partial_{x_2} u_2^0 \|_{L^2(\Omega)}^2). \end{aligned} \tag{3.7}$$

3.3. Jacobi ansatz in higher dimensions

It seems natural to consider the multidimensional extension of the Jacobi-like algorithm (3.6). More generally in d dimensions one could think of replacing $\nabla \nabla \cdot \mathbf{u}^{n+1}$ by $\nabla \nabla \cdot \mathbf{u}^n + (\partial_{x_1 x_1} \delta_t u_1^{n+1}, \dots, \partial_{x_d x_d} \delta_t u_d^{n+1})^\top$. This approximation may be stable in dimension three but we did not make attempts to verify this. However, the following alternative perturbation is also first-order consistent $\nabla \nabla \cdot \mathbf{u}^n + d(\partial_{x_1 x_1} \delta_t u_1^{n+1}, \dots, \partial_{x_d x_d} \delta_t u_d^{n+1})^\top$, and we can consider the algorithm

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + \mathbf{A} \mathbf{u}^{n+1} - \nabla (\varpi \nabla \cdot \mathbf{u}^n) - d \begin{pmatrix} \partial_{x_1} (\varpi \partial_{x_1} \delta_t u_1^{n+1}) \\ \dots \\ \partial_{x_d} (\varpi \partial_{x_d} \delta_t u_d^{n+1}) \end{pmatrix} = \mathbf{f}^{n+1} - \nabla p^n \tag{3.8}$$

with $p^{n+1} = p^n - \varpi \nabla \cdot \mathbf{u}^{n+1}$.

Theorem 3.3. Under suitable initialization and smoothness assumptions, the Jacobi algorithm (3.8) is unconditionally stable, i.e., for any finite time interval $(0, T]$ we have:

$$\begin{aligned} & \| \mathbf{u}_\tau \|_{\ell^\infty(L^2(\Omega))}^2 + \tau \| \varpi^{-\frac{1}{2}} p_\tau \|_{\ell^\infty(L^2(\Omega))}^2 + \tau d \sum_{i=1}^d \| \varpi^{\frac{1}{2}} \partial_{x_i} u_{i,\tau} \|_{\ell^\infty(L^2(\Omega))}^2 \\ & + \tau^{-1} \| \delta_t \mathbf{u}_\tau \|_{\ell^2(L^2(\Omega))}^2 + 2\nu \| \mathbf{u}_\tau \|_{\ell^2(\mathbf{H}^1(\Omega))}^2 + \| \varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u} \|_{\ell^2(L^2(\Omega))}^2 \\ & \leq c \left(\| \mathbf{u}^0 \|_{L^2(\Omega)}^2 + \tau \| \varpi^{-\frac{1}{2}} p^0 \|_{L^2(\Omega)}^2 + \tau d \sum_{i=1}^d \| \varpi^{\frac{1}{2}} \partial_{x_i} u_i^0 \|_{L^2(\Omega)}^2 \right). \end{aligned} \tag{3.9}$$

Proof. Proceeding as in the proof of Theorem 3.1, we obtain

$$\begin{aligned} & \| \mathbf{u}^{n+1} \|_{L^2(\Omega)}^2 + \| \delta_t \mathbf{u}^{n+1} \|_{L^2(\Omega)}^2 + 2\nu \tau \| \mathbf{u}^{n+1} \|_{\mathbf{H}^1(\Omega)}^2 + \tau \| \varpi^{-\frac{1}{2}} p^{n+1} \|_{L^2(\Omega)}^2 - \tau \| \varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^{n+1} \|_{L^2(\Omega)}^2 \\ & + 2\tau (\varpi \nabla \cdot \mathbf{u}^n, \nabla \cdot \mathbf{u}^{n+1}) + \tau d \sum_{i=1}^d (\| \varpi^{\frac{1}{2}} \partial_{x_i} u_i^{n+1} \|_{L^2(\Omega)}^2 + \| \varpi^{\frac{1}{2}} \partial_{x_i} \delta_t u_i^{n+1} \|_{L^2(\Omega)}^2) \\ & \leq \| \mathbf{u}^n \|_{L^2(\Omega)}^2 + \tau \| \varpi^{-\frac{1}{2}} p^n \|_{L^2(\Omega)}^2 + \tau d \sum_{i=1}^d \| \varpi^{\frac{1}{2}} \partial_{x_i} u_i^n \|_{L^2(\Omega)}^2. \end{aligned}$$

We now observe that $(\nabla \cdot \delta_t \mathbf{u}^{n+1})^2 \leq d \sum_{i=1}^d (\partial_{x_i} \delta_t u_i)^2$, which in turn implies that

$$\begin{aligned} & - \| \varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^{n+1} \|_{L^2(\Omega)}^2 + 2(\varpi \nabla \cdot \mathbf{u}^n, \nabla \cdot \mathbf{u}^{n+1}) + d \sum_{i=1}^d \| \varpi \partial_{x_i} \delta_t u_i \|_{L^2(\Omega)}^2 \\ & = - \| \varpi^{\frac{1}{2}} \nabla \cdot \delta_t \mathbf{u}^{n+1} \|_{L^2(\Omega)}^2 + \| \varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^n \|_{L^2(\Omega)}^2 + d \sum_{i=1}^d \| \varpi \partial_{x_i} \delta_t u_i \|_{L^2(\Omega)}^2 \geq \| \varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^n \|_{L^2(\Omega)}^2. \end{aligned}$$

The conclusion follows readily. \square

3.4. Gauss–Seidel ansatz in higher dimensions

The two-dimensional Gauss–Seidel scheme (3.2) introduced in the previous section can be directly extended in three space dimensions as follows:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + \mathbf{A} \mathbf{u}^{n+1} - \begin{pmatrix} \partial_{x_1} (\varpi (\partial_{x_1} u_1^{n+1} + \partial_{x_2} u_2^n + \partial_{x_3} u_3^n)) \\ \partial_{x_2} (\varpi (\partial_{x_1} u_1^{n+1} + \partial_{x_2} u_2^{n+1} + \partial_{x_3} u_3^n)) \\ \partial_{x_3} (\varpi (\partial_{x_1} u_1^{n+1} + \partial_{x_2} u_2^{n+1} + \partial_{x_3} u_3^{n+1})) \end{pmatrix} = \mathbf{f}^{n+1} - \nabla p^n \tag{3.10}$$

with $p^{n+1} = p^n - \varpi \nabla \cdot \mathbf{u}^{n+1}$. Then again the three Cartesian components of the velocity are decoupled. Unfortunately, we have not been able to prove the stability of this scheme, but our numerical experiments lead us to conjecture that it is unconditionally stable. We have found though that stability can be proved by adding the first-order perturbation $-(0, \partial_{x_2}(\varpi \partial_{x_2} \delta_t u_2^{n+1}), \partial_{x_3}(\varpi(-\partial_{x_2} \delta_t u_2^{n+1} + \partial_{x_3} \delta_t u_3^{n+1})))^\top$, leading to the alternative following scheme

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + A\mathbf{u}^{n+1} - \begin{pmatrix} \partial_{x_1}(\varpi(\partial_{x_1} u_1^{n+1} + \partial_{x_2} u_2^n + \partial_{x_3} u_3^n)) \\ \partial_{x_2}(\varpi(\partial_{x_1} u_1^{n+1} + \partial_{x_2}(2u_2^{n+1} - u_2^n) + \partial_{x_3} u_3^n)) \\ \partial_{x_2}(\varpi(\partial_{x_1} u_1^{n+1} + \partial_{x_2} u_2^n + \partial_{x_3}(2u_3^{n+1} - u_3^n))) \end{pmatrix} = \mathbf{f}^{n+1} - \nabla p^n, \tag{3.11}$$

and $p^{n+1} = p^n - \varpi \nabla \cdot \mathbf{u}^{n+1}$. Stability will be established by relying on the following result.

Lemma 3.4. *Let a_1, b_1, c_1, b_0, c_0 be three real numbers, then the following identity holds:*

$$\begin{aligned} & 2((a_1 + b_0 + c_0)a_1 + (a_1 + b_1 + c_0)b_1 + (a_1 + b_1 + c_1)c_1) + 2(b_1 - b_0)b_1 - 2(b_1 - b_0)c_1 + 2(c_1 - c_0)c_1 \\ & = (a_1 + b_1 + c_1)^2 + (a_1 + b_0 + c_0)^2 + 2(b_1^2 + c_1^2 - b_0^2 - c_0^2) + (b_1 - b_0 - c_1 + c_0)^2. \end{aligned} \tag{3.12}$$

Theorem 3.5. *Under suitable initialization and smoothness assumptions (assuming that $\mathbf{f} = 0$), the algorithm (3.11) is unconditionally stable, i.e., upon setting $\tilde{\mathbf{u}}^{n+1} = (u_1^{n+1}, u_2^n, u_3^n)^\top$, the following holds for any finite time interval $(0, T]$:*

$$\begin{aligned} & \|\mathbf{u}_\tau\|_{\ell^\infty(\mathbf{L}^2(\Omega))}^2 + \tau \|\varpi^{-\frac{1}{2}} p_\tau\|_{\ell^\infty(\mathbf{L}^2(\Omega))}^2 + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_{2,\tau}\|_{\ell^\infty(\mathbf{L}^2(\Omega))}^2 \\ & + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_3} u_{3,\tau}\|_{\ell^\infty(\mathbf{L}^2(\Omega))}^2 + \tau^{-1} \|\delta_t \mathbf{u}_\tau\|_{\ell^2(\mathbf{L}^2(\Omega))}^2 + 2\nu \|\mathbf{u}_\tau\|_{\ell^2(\mathbf{H}^1(\Omega))}^2 \\ & + \|\varpi^{\frac{1}{2}} \nabla \cdot \tilde{\mathbf{u}}_\tau\|_{\ell^2(\mathbf{L}^2(\Omega))}^2 + \|\varpi^{\frac{1}{2}} (\partial_{x_2} \delta_t u_{2,\tau} - \partial_{x_3} \delta_t u_{3,\tau})\|_{\ell^2(\mathbf{L}^2(\Omega))}^2 \\ & \leq c \left(\|\mathbf{u}^0\|_{\mathbf{L}^2(\Omega)}^2 + \|\varpi^{-\frac{1}{2}} p^0\|_{\mathbf{L}^2(\Omega)}^2 + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^0\|_{\mathbf{L}^2(\Omega)}^2 + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_3} u_3^0\|_{\mathbf{L}^2(\Omega)}^2 \right). \end{aligned} \tag{3.13}$$

Proof. The stability has been established by proceeding as in the two dimensional case. Assuming that $\mathbf{f} = 0$, we first test (3.11) with $2\tau \mathbf{u}^{n+1}$, then using the identity $2(a - b, a) = \|a\|^2 + \|a - b\|^2 - \|b\|^2$ and Lemma 3.4 to handle the $\nabla \nabla \cdot$ term, we have

$$\begin{aligned} & \|\mathbf{u}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + \|\delta_t \mathbf{u}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 - \|\mathbf{u}^n\|_{\mathbf{L}^2(\Omega)}^2 + 2\nu\tau \|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 - 2\tau(p^n, \nabla \cdot \mathbf{u}^{n+1}) \\ & + \tau \|\varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} \nabla \cdot \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + 2\tau \|\partial_{x_3} u_3^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 \\ & - 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^n\|_{\mathbf{L}^2(\Omega)}^2 - 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_3} u_3^n\|_{\mathbf{L}^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} (\partial_{x_2} \delta_t u_2^{n+1} - \partial_{x_3} \delta_t u_3^{n+1})\|_{\mathbf{L}^2(\Omega)}^2 \leq 0. \end{aligned}$$

Then we add the pressure equation

$$\tau \|\varpi^{-\frac{1}{2}} p^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 = \tau \|\varpi^{-\frac{1}{2}} p^n\|_{\mathbf{L}^2(\Omega)}^2 - 2\tau(\nabla \cdot \mathbf{u}^{n+1}, p^n) + \tau \|\varpi^{\frac{1}{2}} \nabla \cdot \mathbf{u}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2,$$

and obtain

$$\begin{aligned} & \|\mathbf{u}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + \|\delta_t \mathbf{u}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + 2\nu\tau \|\mathbf{u}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 + \tau \|\varpi^{-\frac{1}{2}} p^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} \nabla \cdot \tilde{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 \\ & + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + 2\tau \|\partial_{x_3} u_3^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} (\partial_{x_2} \delta_t u_2^{n+1} - \partial_{x_3} \delta_t u_3^{n+1})\|_{\mathbf{L}^2(\Omega)}^2 \\ & \leq \|\mathbf{u}^n\|_{\mathbf{L}^2(\Omega)}^2 + \tau \|\varpi^{-\frac{1}{2}} p^n\|_{\mathbf{L}^2(\Omega)}^2 + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_2} u_2^n\|_{\mathbf{L}^2(\Omega)}^2 + 2\tau \|\varpi^{\frac{1}{2}} \partial_{x_3} u_3^n\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

Finally, the result follows by summing the above inequality for $n = 0, \dots, N - 1$. \square

4. Direction splitting schemes

We show in this section that the coordinate splitting schemes considered so far can be further simplified by splitting the operator A direction-wise, thus reducing the multidimensional scalar parabolic subproblems to a sequence of d one-dimensional scalar problems. It will be demonstrated below in Section 5 that the direction-splitting schemes proposed in the present section can be combined with the defect-correction approach in order to increase their accuracy in time.

Direction splitting algorithms based on the artificial compressibility formulation of the Navier–Stokes equations have been proposed many years ago (see [5, Section 8.3], [6, chapter VI, Section 9.2], [7, chapter III, Section 8.3]), and they have largely been abandoned in the last twenty years. Restricting the discussion to two dimensions for simplicity, all these direction splitting schemes can be considered as discretization of the following set of PDEs formulated in [5] and approximating

the incompressible Navier–Stokes equations with constant viscosity:

$$\begin{cases} \frac{1}{2} \partial_t u_1 + u_1 \partial_{x_1} u_1 + \partial_{x_1} p = \mu \partial_{x_1 x_1} u_1, \\ \frac{1}{2} \partial_t u_2 + u_1 \partial_{x_1} u_2 = \mu \partial_{x_1 x_1} u_2, \\ \epsilon \partial_t p + \epsilon u_1 \partial_{x_1} p + p \partial_{x_1} u_1 = 0, \end{cases} \quad (4.1)$$

in the first half of a given time interval $[t^n, t^n + \frac{1}{2}\tau]$ and

$$\begin{cases} \frac{1}{2} \partial_t u_1 + u_2 \partial_{x_2} u_1 = \mu \partial_{x_2 x_2} u_1, \\ \frac{1}{2} \partial_t u_2 + u_2 \partial_{x_2} u_2 + \partial_{x_2} p = \mu \partial_{x_2 x_2} u_2, \\ \epsilon \partial_t p + \epsilon u_2 \partial_{x_2} p + p \partial_{x_2} u_2 = 0, \end{cases} \quad (4.2)$$

in the second half $[t^n + \frac{1}{2}\tau, t^{n+1}]$. Note that in [6,7] the pressure equations are formulated slightly differently:

$$\frac{\epsilon}{2} \partial_t p + \partial_{x_1} u_1 = 0, \quad \text{in } \left[t^n, t^n + \frac{1}{2}\tau \right], \quad (4.3)$$

$$\frac{\epsilon}{2} \partial_t p + \partial_{x_2} u_2 = 0, \quad \text{in } \left[t^n + \frac{1}{2}\tau, t^{n+1} \right]. \quad (4.4)$$

In the scheme of Yanenko [5] the pressure equations are derived from the compressible mass conservation equation at vanishing Mach number, in [6,7] they are derived from the simpler (but less physical) perturbation of the incompressibility constraint: $\epsilon \partial_t p + \nabla \cdot \mathbf{u} = 0$. Both algorithms are formally first-order accurate in time. However, the actual rate of convergence was not established in the above references, despite that convergence was proven in both cases.

In the present paper we are aiming at the development of artificial compressibility schemes of order two and higher. In [1] we proposed two possible approaches for extending the convergence order. The first one uses a bootstrapping perturbation of the incompressibility constraint combined with a high-order BDF time stepping for the momentum equation. The second approach is based on a defect (or deferred) correction for both, the momentum and the continuity equations. Since we are presently unable to devise a higher-order defect correction scheme based on any of the first-order direction splitting methods discussed above (see e.g., (4.1)–(4.2)), we consider here a scheme that is a first-order perturbation of the formally second-order splitting scheme due to Douglas [8]. For simplicity, we will not consider the nonlinear terms in what follows, however, there is no particular difficulty to extend the scheme to the nonlinear case by using Euler explicit discretization. It is also possible to discretize the nonlinear terms semi-implicitly by proceeding as in [5–7]. Denoting by $p^{n+1/2}$ and \bar{p}_τ the approximation of the pressure at time $t^{n+1/2}$ and the time sequence of pressure values, respectively, the derivation of the scheme starts from the Crank–Nicolson discretization of the momentum equation of the artificial compressibility system that is given by:

$$\begin{cases} \frac{\delta_t \mathbf{u}^{n+1}}{\tau} + \frac{1}{2} A(\mathbf{u}^{n+1} + \mathbf{u}^n) = -\frac{1}{2} \nabla p^{n+1/2} + \mathbf{f}^{n+1/2}, \\ (p^{n+1/2} - p^{n-1/2}) + \frac{\varpi}{2} \nabla \cdot (\mathbf{u}^{n+1} + \mathbf{u}^n) = 0. \end{cases} \quad (4.5)$$

Let us assume that the operators A_1 and A_2 can be split into a sum of two self-adjoint semi-definite positive operators i.e., $A_1 = A_{11} + A_{12}$ and $A_2 = A_{21} + A_{22}$. For example, if $A_1 = -\mu \Delta$ and $A_2 = -\mu \Delta$ then the direction splitting algorithm presumes the splitting $A_{11} = -\mu \partial_{x_1 x_1}$, $A_{12} = -\mu \partial_{x_2 x_2}$, $A_{21} = -\mu \partial_{x_1 x_1}$, $A_{22} = -\mu \partial_{x_2 x_2}$. Let us also assume that ϖ is constant over Ω and let us introduce the operators: $C_{11} = -\varpi \partial_{x_1 x_1}$, $C_{12} = -\varpi \partial_{x_1 x_2}$, $C_{21} = -\varpi \partial_{x_2 x_1}$, $C_{22} = -\varpi \partial_{x_2 x_2}$. Then the direction splitting scheme is given by:

$$\begin{cases} \left(I + \frac{\tau}{2} (A_{11} + C_{11}) \right) \left(I + \frac{\tau}{2} A_{12} \right) \frac{\delta_t u_1^{n+1}}{\tau} + (A_{11} + C_{11} + A_{12}) u_1^n = -\frac{1}{2} C_{12} (u_2^n + u_2^{n-1}) - \partial_{x_1} p^{n-1/2} + f_1^{n+1/2}, \\ \left(I + \frac{\tau}{2} (A_{22} + C_{22}) \right) \left(I + \frac{\tau}{2} A_{21} \right) \frac{\delta_t u_2^{n+1}}{\tau} + (A_{22} + C_{22} + A_{21}) u_2^n = -\frac{1}{2} C_{21} (u_1^{n+1} + u_1^n) - \partial_{x_2} p^{n-1/2} + f_2^{n+1/2}, \\ (p^{n+1/2} - p^{n-1/2}) + \frac{\varpi}{2} \nabla \cdot (\mathbf{u}^{n+1} + \mathbf{u}^n) = 0, \end{cases} \quad (4.6)$$

where I is the identity operator. Note that this is a perturbation of the Crank–Nicolson discretization (4.5) that includes the formally second-order terms $\tau(A_{11} + C_{11})A_{12}\delta_t u_1^{n+1}/4$, $\tau(A_{22} + C_{22})A_{21}\delta_t u_2^{n+1}/4$, and the term $C_{12}(u_2^{n+1} + u_2^n)$ is extrapolated by $C_{12}(u_2^n + u_2^{n-1})$. The last perturbation is first-order accurate of course, but since the perturbation of the incompressibility

constraint is also first-order, it does not change the overall first-order approximation of the unsteady Stokes equations. In the next section we will demonstrate how to correct these first-order defects of the scheme and lift the accuracy to second-order.

Let us now assume that the operator $A_{11} + C_{11}$ commutes with A_{12} , and $A_{22} + C_{22}$ commutes with A_{21} . Such commutativity conditions are satisfied if, for example, the viscosity μ is constant and if the domain boundary consists of straight lines parallel to one of the coordinate axes. Then the operator

$$\begin{aligned} \mathbf{B} &= (B_1, B_2)^T = ((A_{11} + C_{11})A_{12}, (A_{22} + C_{22})A_{21})^T \\ &= \frac{1}{2}((A_{11} + C_{11})A_{12} + A_{12}(A_{11} + C_{11}), (A_{22} + C_{22})A_{21} + A_{21}(A_{22} + C_{22}))^T \end{aligned}$$

is a self-adjoint positive semi-definite operator defining a semi-norm which we denote by $|\cdot|_{\mathbf{B}(\Omega)}$. Under such conditions it is quite straightforward to prove the following theorem providing the stability estimate for the splitting scheme. The stability without the commutativity assumption is significantly more difficult to verify, particularly in 3D, and it is still an open problem (see for example the discussion about splitting schemes for non-commutative operators in [9]).

Theorem 4.1. *Under suitable initialization and smoothness assumptions, if $(A_{11} + C_{11})A_{12} = A_{12}(A_{11} + C_{11})$, $(A_{22} + C_{22})A_{21} = A_{21}(A_{22} + C_{22})$, and if $\mathbf{f} = \mathbf{0}$, the algorithm (4.6) is unconditionally stable, i.e., for any finite time interval $(0, T]$ we have:*

$$\begin{aligned} &\|\mathbf{u}_\tau\|_{\ell^\infty(\mathbf{L}^2(\Omega))}^2 + \tau \|\varpi^{-\frac{1}{2}} \bar{p}_\tau\|_{\ell^\infty(\mathbf{L}^2(\Omega))}^2 + \tau \|\varpi^{\frac{1}{2}} \partial_{x_2} \bar{u}_{2,\tau}\|_{\ell^\infty(\mathbf{L}^2(\Omega))}^2 \\ &\quad + 2\mu \|\bar{\mathbf{u}}_\tau\|_{\ell^2(\mathbf{H}^1(\Omega))}^2 + \frac{\tau^2}{4} \|\mathbf{u}_\tau\|_{\ell^\infty(\mathbf{B}(\Omega))}^2 + \|\varpi^{\frac{1}{2}} \nabla \cdot \tilde{\mathbf{u}}\|_{\ell^2(\mathbf{L}^2(\Omega))}^2 \\ &\leq c \left(\|\mathbf{u}^0\|_{\mathbf{L}^2(\Omega)}^2 + \tau \|\varpi^{-\frac{1}{2}} p^{-1/2}\|_{\mathbf{L}^2(\Omega)}^2 + \tau \|\varpi^{\frac{1}{2}} \partial_{x_2} \bar{u}_2^0\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\tau^2}{4} \|\mathbf{u}^0\|_{\mathbf{B}(\Omega)}^2 \right), \end{aligned}$$

where $\bar{u}_2^0 = u_2^0$, $p^{-1/2} = p^0$, and $\tilde{\mathbf{u}}^{n+1} = (\bar{u}_1^{n+1}, \bar{u}_2^n)^T$.

Proof. We first notice that the momentum equation in (4.6) can be rewritten in a form similar to (3.4):

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} + A\bar{\mathbf{u}}^{n+1} + \frac{\tau^2}{4} B \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\tau} - \nabla(\varpi \nabla \cdot \tilde{\mathbf{u}}^{n+1}) - (\mathbf{0}, \partial_{x_2}(\varpi \partial_{x_2} \delta_t \bar{u}_2^{n+1}))^T = -\nabla p^{n-1/2}$$

where $\tilde{\mathbf{u}}^{n+1} = (\bar{u}_1^{n+1}, \bar{u}_2^n)^T$. Multiplying this equation by $2\tau \bar{\mathbf{u}}^{n+1}$, then using the identities $2(a-b)a = \|a\|^2 + \|a-b\|^2 - \|b\|^2$ and $(a-b)(a+b) = \|a\|^2 - \|b\|^2$, and the coerciveness of A in $\mathbf{H}^1(\Omega)$, we obtain:

$$\begin{aligned} &\|\bar{\mathbf{u}}^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 - \|\mathbf{u}^n\|_{\mathbf{L}^2(\Omega)}^2 + 2\mu\tau \|\bar{\mathbf{u}}^{n+1}\|_{\mathbf{H}^1(\Omega)}^2 + 2\tau(\varpi \nabla \cdot \tilde{\mathbf{u}}^{n+1}, \nabla \cdot \bar{\mathbf{u}}^{n+1}) + \frac{\tau^2}{4} (|\mathbf{u}^{n+1}|_{\mathbf{B}(\Omega)}^2 - |\mathbf{u}^n|_{\mathbf{B}(\Omega)}^2) \\ &\quad + \tau \left(\|\varpi^{\frac{1}{2}} \partial_{x_2} \bar{u}_2^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 + \|\varpi^{\frac{1}{2}} \partial_{x_2} \delta_t \bar{u}_2^{n+1}\|_{\mathbf{L}^2(\Omega)}^2 - \|\varpi^{\frac{1}{2}} \partial_{x_2} \bar{u}_2^n\|_{\mathbf{L}^2(\Omega)}^2 \right) - 2\tau(p^{n-1/2}, \nabla \cdot \bar{\mathbf{u}}^{n+1}) \leq 0. \end{aligned}$$

The rest of the proof follows along the same lines as the proof of Theorem 3.1. \square

Remark 4.1 (*Shape of Ω*). The stability analysis of direction-split schemes is usually based on the assumption that the domain of the problem is a rectangle or parallelepiped. Some of these schemes can be extended to domains of a more complex shape, however, their stability is far more difficult to establish. Such possibilities are considered in [10,9], the later work proving the unconditional stability of such schemes in the case of non-commutative one-dimensional operators.

5. Higher-order extensions

The first-order schemes discussed in the previous two sections can be extended to second-order by at least two possible approaches described in [1]. The resulting schemes are quite efficient if the linear systems are solved by means of iterative solvers. In order to handle the 2D and the 3D case together it is convenient to introduce the following operator corresponding to the mixed second-order derivatives appearing in the formulation:

$$\begin{aligned} C_\Delta &= \begin{bmatrix} 0 & C_{12} \\ 0 & 0 \end{bmatrix} \quad \text{in 2D, and} \\ C_\Delta &= \begin{bmatrix} 0 & C_{12} & C_{13} \\ 0 & 0 & C_{23} \\ 0 & 0 & 0 \end{bmatrix} \quad \text{in 3D,} \end{aligned}$$

with C_{13}, C_{23} being defined similarly to C_{12} i.e. $C_{i3} = \varpi \partial_{x_i x_3}, i = 1, 2$. An example of a 2D second-order BDF bootstrapping procedure based on (3.3) and analogous to the scheme (5.1)–(5.2) of Guermond and Mineev [1] is given by:

$$\begin{cases} \frac{\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n}{\tau} + A\tilde{\mathbf{u}}^{n+1} + \nabla \tilde{p}_1^{n+1} - C_\Delta(\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n) = \mathbf{f}^{n+1}, \\ \tilde{p}^{n+1} - \tilde{p}^n + \varpi \nabla \cdot \tilde{\mathbf{u}}^{n+1} = 0, \\ \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + A\mathbf{u}^{n+1} + \nabla p^{n+1} - C_\Delta(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) = \mathbf{f}^{n+1}, \\ p^{n+1} - p^n - (\tilde{p}^{n+1} - \tilde{p}^n) + \varpi \nabla \cdot \mathbf{u}^{n+1} = 0. \end{cases} \tag{5.1}$$

Note that the only difference with the scheme (5.1)–(5.2) of Guermond and Mineev [1] is the presence of the terms $C_\Delta(\tilde{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^n)$ and $C_\Delta(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})$ in the two momentum equations. Presuming enough smoothness of the exact solution, these terms are of order τ and τ^2 respectively, and their presence is compatible with the overall second-order of consistency of the scheme. In the case of the Navier–Stokes equations the advection terms can be approximated by means of a first and second-order Adams–Bashforth (AB2) schemes in the first and second stages of the bootstrapping procedure in (5.1).

As shown in [1], in the case of the full Navier–Stokes equations, the defect correction schemes have better stability properties than the high-order schemes based on BDF time stepping. Using the third-order approximation to the velocity and pressure $\mathbf{u}_0^n + \tau \mathbf{u}_1^n + \tau^2 \mathbf{u}_2^n, p_0^n + \tau p_1^n + \tau^2 p_2^n$, we can write the third-order scheme with a decoupled grad–div operator, analogous to the scheme (2.4)–(2.6), as:

$$\text{for } n \geq 0, \begin{cases} \mathbf{n}_0^{n+1} = B\mathbf{u}_0^n, \\ \frac{\mathbf{u}_0^{n+1} - \mathbf{u}_0^n}{\tau} + A\mathbf{u}_0^{n+1} - \varpi \nabla \nabla \cdot \mathbf{u}_0^{n+1} + C_\Delta(\mathbf{u}_0^{n+1} - \mathbf{u}_0^n) + \nabla p_0^n = \mathbf{f}^{n+1} - \mathbf{n}_0^{n+1} \\ p_0^{n+1} = p_0^n - \varpi \nabla \cdot \mathbf{u}_0^{n+1}, \\ d\mathbf{u}_0^{n+1} = (\mathbf{u}_0^{n+1} - \mathbf{u}_0^n)/\tau, \quad dp_0^{n+1} = (p_0^{n+1} - p_0^n)/\tau \end{cases} \tag{5.2}$$

$$\text{for } n \geq 1, \begin{cases} d^2 \mathbf{u}_0^{n+1} = (d\mathbf{u}_0^{n+1} - d\mathbf{u}_0^n)/\tau, \\ \mathbf{n}_1^n = B(\mathbf{u}_0^n + \tau \mathbf{u}_1^{n-1}), \\ \frac{\mathbf{u}_1^n - \mathbf{u}_1^{n-1}}{\tau} + A\mathbf{u}_1^n - \varpi \nabla \nabla \cdot \mathbf{u}_1^n + C_\Delta(\mathbf{u}_1^n - \mathbf{u}_1^{n-1}) - C_\Delta(\mathbf{u}_0^n - \mathbf{u}_0^{n-1}) + \\ \nabla(p_1^{n-1} + dp_0^n) = -\frac{1}{2}d^2 \mathbf{u}_0^{n+1} - \frac{\mathbf{n}_1^n - \mathbf{n}_0^n}{\tau} + \varpi C d\mathbf{u}_0^{n+1}, \\ p_1^n = p_1^{n-1} + dp_0^n - \varpi \nabla \cdot \mathbf{u}_1^n, \\ d\mathbf{u}_1^n = (\mathbf{u}_1^n - \mathbf{u}_1^{n-1})/\tau, \quad dp_1^n = (p_1^n - p_1^{n-1})/\tau, \end{cases} \tag{5.3}$$

$$\text{for } n \geq 2, \begin{cases} d^2 \mathbf{u}_1^n = (d\mathbf{u}_1^n - d\mathbf{u}_1^{n-1})/\tau, \quad d^3 \mathbf{u}_0^{n+1} = (d^2 \mathbf{u}_0^{n+1} - d^2 \mathbf{u}_0^n)/\tau, \\ \mathbf{n}_2^{n-1} = B(\mathbf{u}_0^{n-1} + \tau \mathbf{u}_1^{n-1} + \tau^2 \mathbf{u}_2^{n-2}) \\ \frac{\mathbf{u}_2^{n-1} - \mathbf{u}_2^{n-2}}{\tau} + A\mathbf{u}_2^{n-1} - \varpi \nabla \nabla \cdot \mathbf{u}_2^{n-1} + C_\Delta(\mathbf{u}_2^{n-1} - \mathbf{u}_2^{n-2}) - C_\Delta(\mathbf{u}_1^{n-1} - \mathbf{u}_1^{n-2}) + \\ \nabla(p_2^{n-2} + dp_1^{n-1}) = -\frac{1}{2}d^2 \mathbf{u}_1^n + \frac{1}{6}d^3 \mathbf{u}_0^{n+1} - \frac{\mathbf{n}_2^{n-1} - \mathbf{n}_1^{n-1}}{\tau^2} + \varpi C d\mathbf{u}_1^{n+1}, \\ p_2^{n-1} = p_2^{n-2} + dp_1^{n-1} - \varpi \nabla \cdot \mathbf{u}_2^{n-1}, \\ \mathbf{u}^{n-1} = \mathbf{u}_0^{n-1} + \tau \mathbf{u}_1^{n-1} + \tau^2 \mathbf{u}_2^{n-1}, \quad p^{n-1} = p_0^{n-1} + \tau p_1^{n-1} + \tau^2 p_2^{n-1}. \end{cases} \tag{5.4}$$

In 3D, this scheme is the defect correction extension of the scheme (3.10). Although we are presently unable to prove its stability, we use it in the numerical experiments presented below. Our tests show that this scheme is unconditionally stable in the case of the unsteady Stokes equations.

Note that all these schemes require only the solution of problems of the type

$$v - \tau \nabla \cdot (\kappa \nabla v) = r,$$

for each component of the velocity, where κ is a $d \times d$ diagonal matrix. For example, in 2D either

$$\kappa = \begin{bmatrix} \mu + \varpi & 0 \\ 0 & \mu \end{bmatrix} \quad \text{or} \quad \kappa = \begin{bmatrix} \mu & 0 \\ 0 & \mu + \varpi \end{bmatrix},$$

when we solve for the first or the second Cartesian component of the velocity, respectively. The solution process for the incompressible unsteady Navier–Stokes equations is thereby reduced to the solution of a fixed number of classical scalar-valued parabolic problems.

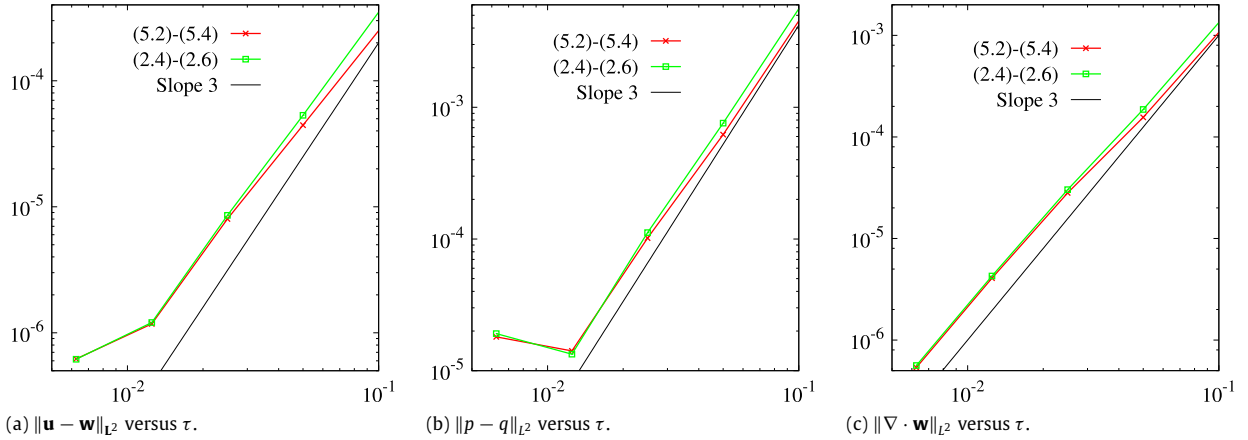


Fig. 1. Log-log plot of the L^2 -norm of the error at $T = 10$ of the 2D unsteady Stokes equations on a 200×200 MAC grid. Continuous lines represent the slope 3, solid lines with \times symbols represent the results with (5.2)–(5.4), solid lines with \square symbols represent the results with (2.4)–(2.6).

Using the defect correction approach of Guermond and Mineev [1] the direction splitting scheme (4.6) can also be extended to second-order as follows:

$$\begin{cases} \left(I + \frac{\tau}{2}(A_{11} + C_{11}) \right) \left(I + \frac{\tau}{2}A_{12} \right) \frac{\tilde{u}_1^{n+1} - \tilde{u}_1^n}{\tau} + (A_{11} + C_{11} + A_{12})\tilde{u}_1^n - \frac{1}{2}C_{12}(\tilde{u}_2^n + \tilde{u}_2^{n-1}) - \partial_{x_1}\tilde{p}^{n-1/2} + f_1^{n+1/2}, \\ \left(I + \frac{\tau}{2}(A_{22} + C_{22}) \right) \left(I + \frac{\tau}{2}A_{21} \right) \frac{\tilde{u}_2^{n+1} - \tilde{u}_2^n}{\tau} + (A_{22} + C_{22} + A_{21})\tilde{u}_2^n \\ = -\frac{1}{2}C_{21}(\tilde{u}_1^{n+1} + \tilde{u}_1^n) - \partial_{x_2}\tilde{p}^{n-1/2} + f_2^{n+1/2}, \\ \tilde{p}^{n+1/2} - \tilde{p}^{n-1/2} + \frac{\overline{\sigma}}{2}\nabla \cdot (\tilde{\mathbf{u}}^{n+1} + \tilde{\mathbf{u}}^n) = 0, \end{cases} \tag{5.5}$$

$$\begin{cases} d\tilde{u}^{n+1} = (\tilde{u}_2^{n+1} - \tilde{u}_2^n)/\tau, \\ \left(I + \frac{\tau}{2}(A_{11} + C_{11}) \right) \left(I + \frac{\tau}{2}A_{12} \right) \frac{u_1^{n+1} - u_1^n}{\tau} + (A_{11} + C_{11} + A_{12})u_1^n \\ = -C_{12} \left(\frac{1}{2}(u_2^n + u_2^{n-1}) + \tau d\tilde{u}^{n+1} \right) - \partial_{x_1}p^{n-1/2} + f_1^{n+1/2}, \\ \left(I + \frac{\tau}{2}(A_{22} + C_{22}) \right) \left(I + \frac{\tau}{2}A_{21} \right) \frac{u_2^{n+1} - u_2^n}{\tau} + (A_{22} + C_{22} + A_{21})u_2^n \\ = -\frac{1}{2}C_{21}(u_1^{n+1} + u_1^n) - \partial_{x_2}p^{n-1/2} + f_2^{n+1/2}, \\ p^{n+1/2} - p^{n-1/2} - (\tilde{p}^{n+1/2} - \tilde{p}^{n-1/2}) + \frac{\overline{\sigma}}{2}\nabla \cdot (\mathbf{u}^{n+1} + \mathbf{u}^n) = 0. \end{cases} \tag{5.6}$$

The nonlinear terms of the Navier–Stokes equations can be included in the above algorithm exactly as in the scheme (5.2)–(5.3).

6. Numerical results

We first present some two dimensional numerical results comparing the performance of the third-order artificial compressibility method in [1], (2.4)–(2.6) and the scheme with the explicit mixed derivatives (5.2)–(5.4). The spatial discretization is done by means of the classical MAC finite volume stencil. The accuracy is tested on the following manufactured solution of the unsteady Stokes equations:

$$\mathbf{u} = (\sin x \sin(y + t), \cos x \cos(y + t))^T, \quad p = \cos x \sin(y + t). \tag{6.1}$$

and the problem is solved in $\Omega = (0, 1) \times (0, 1)$, for $0 \leq t \leq T := 10$ with Dirichlet boundary conditions (given by the pointwise values of the exact solution). The initial condition is the exact solution at $t = 0$. In Fig. 1 we present the L^2 -norm of the errors in the velocity, the pressure, and the divergence for the unsteady Stokes equations as a function of the time step. The spatial grid consists of 200×200 MAC cells and provides enough spatial resolution to avoid any interference with the temporal convergence rate over the range of time steps used in the graph. The formal convergence rate of both schemes is third-order and this theoretical rate seems to be confirmed by the numerical results. The results with both

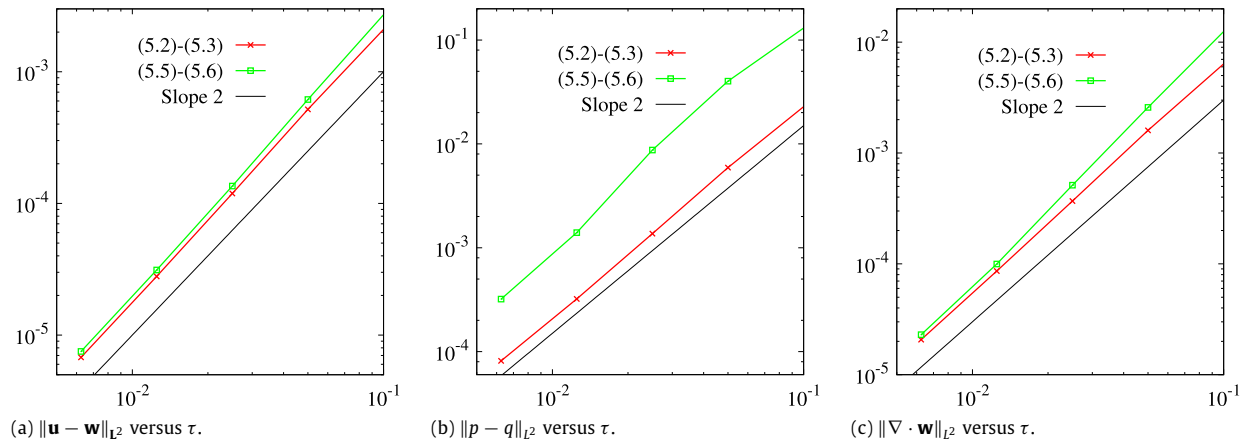


Fig. 2. Log–log plot of the L^2 -norm of the error at $T = 10$ of the 2D unsteady Stokes equations on a 200×200 MAC grid. Continuous lines represent the slope 2, solid lines with \times symbols represent the results with (5.2)–(5.3), solid lines with \square symbols represent the results with (5.5)–(5.6).

schemes are very similar, however, the equations in (5.2)–(5.4) are much easier to solve since all the velocity components are decoupled.

Next, we compare the accuracy of the formally second-order scheme (5.2)–(5.3) and the second-order direction-splitting bootstrapping scheme (5.5)–(5.6) in Fig. 2. We use the same spatial resolution as in the previous case and it again seems to be enough to not significantly affect the temporal convergence over the range of time steps shown in the graphs. Although being slightly less accurate in the pressure, the direction splitting scheme clearly has a good potential since it is less computationally demanding; we recall that this scheme only requires the solution of tridiagonal problems and thus can be massively parallelized as in [11].

Finally we present 3D numerical results that demonstrate the accuracy of (5.2)–(5.3) in the case of the unsteady Stokes and Navier–Stokes problems at $Re = 100$. The 3D manufactured solution is given by: $u_1 = \cos x \sin y \sin(z + t)$, $u_2 = \sin x \cos y \sin(z + t)$, $u_3 = -2 \sin x \sin y \cos(z + t)$, $p = \cos(x + y + z + t)$. The problem is solved in a cube of size one, for $0 \leq t \leq T := 10$. The results on a grid of $20 \times 20 \times 20$ MAC cells are presented in Fig. 3. Although the spatial resolution is much lower than the one used in 2D, it influences the convergence results only at the final time step used in the study $\tau = 0.00625$. This suggests that the accuracy of the overall discretization is mostly dominated by the temporal discretization accuracy and this confirms the need to use higher-order time discretization techniques. Again, the defect correction method (5.2)–(5.3) demonstrates good accuracy and robustness, maintaining stability even at relatively large time steps $\tau = 0.1$.

7. Conclusions

In this paper we have revisited the first-order artificial compressibility method for incompressible flow, proposed by Shen [2], and we have demonstrated that the coupling of the Cartesian components of the velocity, which is due to the presence of the implicit operator $\nabla \nabla \cdot$ can be avoided by discretizing the mixed derivatives explicitly in time. While the original scheme requires the solution at each time step of a coupled vectorial problem of the type $\mathbf{u} - \tau \nabla \nabla \cdot \mathbf{u} = \mathbf{f}$, the modified scheme requires only the solution of a set of d classical scalar parabolic problems of the type: $u_k - \tau \nabla \cdot (\kappa \nabla u_k) = f$, d being the dimensionality of the original problem. Similarly to the approach in [1], the resulting scheme can be combined with the defect correction technique, so that its order in time can be extended to second, third, etc. by performing the necessary number of correction steps. These schemes can also be factorized direction-wise to yield computationally very simple, and yet accurate direction splitting schemes.

When compared to the classical Chorin–Temam-type projection schemes, the algorithms proposed in this paper are computationally more efficient since they require the solution of problems with conditioning scaling like τh^{-2} whereas projection methods require the solution of an elliptic problem for the pressure whose conditioning scales like h^{-2} . In addition, the present approach allows to develop schemes of any order in time unlike the projection methods whose accuracy is limited to second-order.

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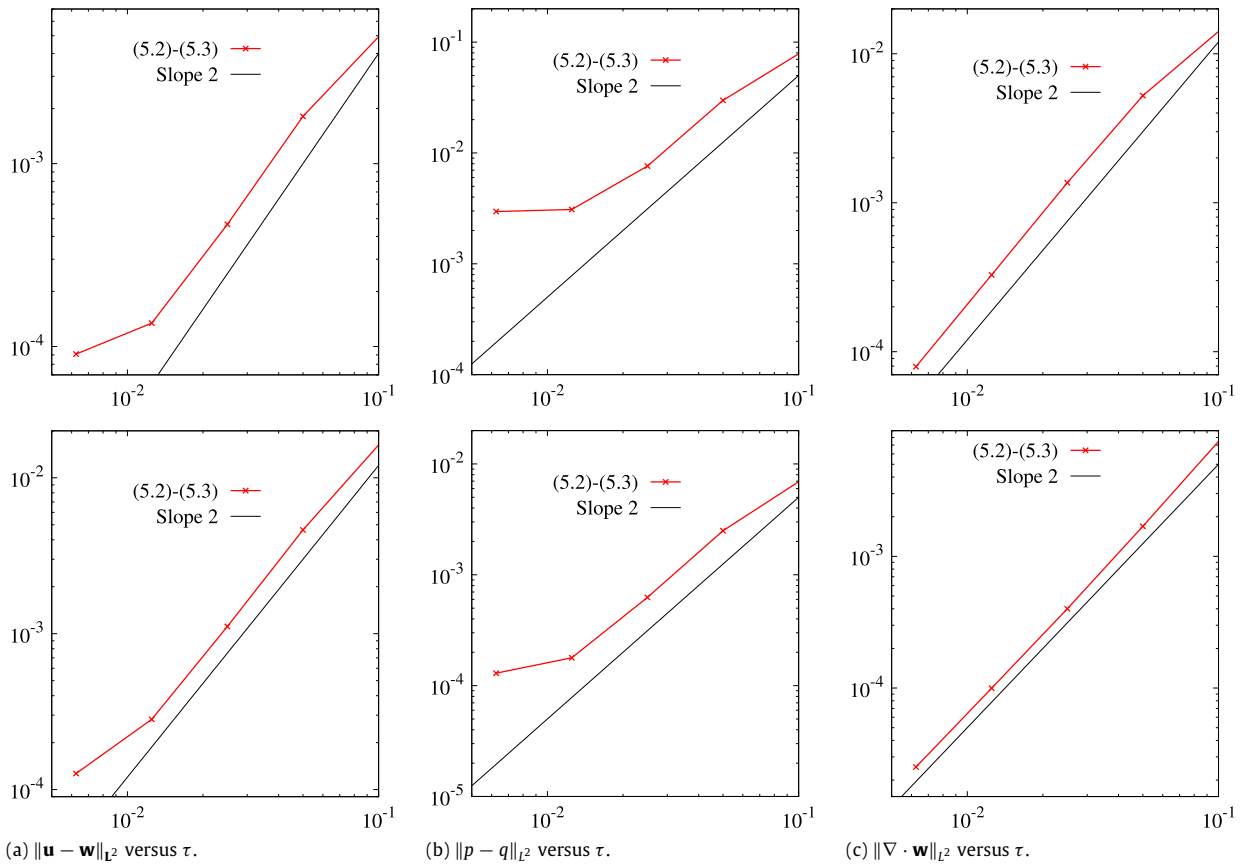


Fig. 3. Log–log plot of the L^2 -norm of the error at $T = 10$ on a $20 \times 20 \times 20$ MAC grid. Top row: 3D unsteady Stokes equations, bottom row: 3D Navier–Stokes equations at $Re = 100$. Continuous lines represent the slope of 2, solid lines with \times symbols represent the results with (5.2)–(5.3).

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