

Calculus

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

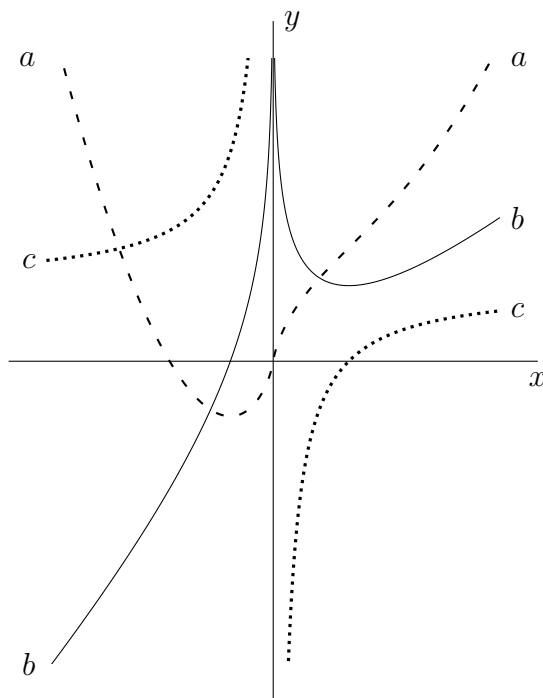
1. On what interval(s) is the function x^2e^x decreasing?

Solution. By the product rule, the derivative equals $x^2e^x + 2xe^x$ or $x(x+2)e^x$. Since e^x is always positive, the derivative is negative only on the interval $(-2, 0)$. This is the interval on which the function is decreasing.

2. Suppose $f(x)$ is the function $\sin(x) + \arcsin(x)$, which is a continuous function whose domain is the closed interval $[-1, 1]$. At which point x in the domain does this function attain its absolute maximum value?

Solution. The function is differentiable on the open interval $(-1, 1)$, and the derivative equals $\cos(x) + (1-x^2)^{-1/2}$. Both terms of the derivative are positive on the interval $(-1, 1)$, so the function is increasing on its domain. The largest value therefore occurs at the right-hand endpoint: namely, at $x = 1$.

3. In the figure below, one graph represents $f(x)$, another graph represents the derivative $f'(x)$, and a third graph represents the indefinite integral (antiderivative) $\int f(x) dx$. Which graph is which?



Solution. The b graph has a local minimum at the same value of x at which the c graph crosses the x -axis, and the a graph has a local minimum at the same value of x at which the b graph crosses the x -axis. Therefore the c graph must represent the derivative of the b graph which in turn represents the derivative of the a graph. Hence the b graph represents f , the c graph represents f' , and the a graph represents the antiderivative of f .

4. The TI-89 calculator says that $\lim_{x \rightarrow 0} \frac{\arctan(2x)}{\sin(3x)} = \frac{2}{3}$. Prove that the calculator is correct.

Solution. Since $\arctan(0) = 0$ and $\sin(0) = 0$, the limit is an indeterminate form of $0/0$ type. Since the derivative of $\arctan(2x)$ equals $\frac{2}{1+4x^2}$, and the derivative of $\sin(3x)$ equals $3 \cos(3x)$, an application of l'Hospital's rule shows that

$$\lim_{x \rightarrow 0} \frac{\arctan(2x)}{\sin(3x)} = \lim_{x \rightarrow 0} \frac{2/(1+4x^2)}{3 \cos(3x)} = \frac{2}{3}.$$

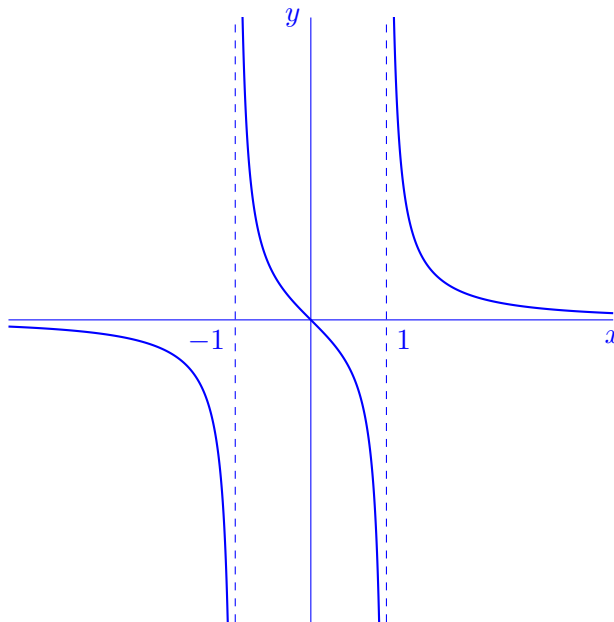
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5. The *error function* $\operatorname{erf}(x)$ is defined to be $\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. Show that the graph of $\operatorname{erf}(x)$ is concave down when $x > 0$.

Solution. An equivalent problem is to show that the second derivative of $\operatorname{erf}(x)$ is negative when $x > 0$. By the fundamental theorem of calculus, the first derivative of $\operatorname{erf}(x)$ equals $\frac{2}{\sqrt{\pi}} e^{-x^2}$. By the chain rule, the second derivative equals $\frac{2}{\sqrt{\pi}} e^{-x^2} \times (-2x)$. Since the first factor $\frac{2}{\sqrt{\pi}} e^{-x^2}$ is always positive, the second derivative evidently is negative when $x > 0$.

6. Sketch the graph of a function f that satisfies the following properties:
- the domain of f is all real numbers except for ± 1 ;
 - the derivative $f'(x) < 0$ everywhere on its domain;
 - $\lim_{x \rightarrow 1^+} f(x) = +\infty$ and $\lim_{x \rightarrow 1^-} f(x) = -\infty$;
 - $\lim_{x \rightarrow -1^+} f(x) = +\infty$ and $\lim_{x \rightarrow -1^-} f(x) = -\infty$;
 - $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow -\infty} f(x) = 0$;
 - the point $(0, 0)$ is an inflection point of the graph of f , and there is no other inflection point.

Solution. The given information indicates that the graph has vertical asymptotes when $x = \pm 1$, and the x -axis is a horizontal asymptote when $x \rightarrow \pm\infty$. In order for the graph to have negative slope everywhere on its domain and to have the origin as its only inflection point, the graph must look something like the following figure.



Although it is not necessary to give a formula for such a function, it is not hard to do so. The graph above corresponds to $f(x) = \frac{\tan^{-1}(x)}{x^2 - 1}$.

7. A farmer has 1,000 linear feet of fence with which to bound a rectangular field. If one side of the field will be formed by a river, and the fence is used for the other three sides, what is the largest possible area of the field?

Solution. Let x denote the length of the side parallel to the river, and let y denote the length of the side perpendicular to the river. The area of the field equals xy , and the constraint is that $x + 2y = 1000$. Using the side condition to eliminate one of the variables, write the area A as a function of (say) y : namely, $A(y) = (1000 - 2y)y = 1000y - 2y^2$.

The domain of the variable y is evidently the interval $(0, 500)$, and $A(y)$ approaches the value 0 at both endpoints. The maximum value of A must therefore occur at a critical point inside the domain. Since $A'(y) = 1000 - 4y$, there is a critical point only when $y = 250$. The corresponding value of the area is $A(250) = (1000 - 500)(250) = (500)(250) = 125000$ in units of square feet. (This area is about 3.44 acres, or about 1.39 hectares.)

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8. If $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \sqrt{\ln \left(1 + \frac{i}{n} \right)} = \int_1^2 f(x) dx$, what is the function $f(x)$?

Solution. If the interval $[1, 2]$ is partitioned into n subintervals, each of width $1/n$, then the partition points have the form $1 + \frac{i}{n}$, and the left-hand side represents the limit of right-hand endpoint Riemann sums for the function $\sqrt{\ln x}$.

9. State **three** of the following theorems:

- (a) l'Hospital's rule for indeterminate forms of type $0/0$
- (b) the extreme value theorem
- (c) Fermat's theorem
- (d) the mean-value theorem
- (e) the first derivative test for local extrema
- (f) the first derivative test for absolute extrema
- (g) the second derivative test for local extrema
- (h) the fundamental theorem of calculus

10. **Optional problem for extra credit**

State the remaining five theorems from problem 9.

Solution. The statements of the theorems can be found in the text-book on the following pages: (a) p. 290; (b) p. 308; (c) p. 309; (d) p. 314; (e) p. 317; (f) p. 334; (g) p. 318; and (h) p. 397.