Final Exam, May 6

1. Determine (not identically zero) polynomials p_0, p_1, and p_2 of degrees 0, 1, and 2 respectively that are mutually orthogonal on the interval $[0, 1]$: namely,

$$\int_0^1 p_0(x)p_1(x)\,dx = 0,$$
$$\int_0^1 p_0(x)p_2(x)\,dx = 0,$$
$$\int_0^1 p_1(x)p_2(x)\,dx = 0.$$

Suggestion: You can do this either bare hands, or by applying to the functions 1, x, and x^2 the so-called Gram-Schmidt orthogonalization procedure (that is, subtract from each vector its projection onto the subspace generated by the previous vectors).

2. Solve the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2} \qquad \text{for } 0 < x < 1 \text{ and } 0 < t$$

with the boundary conditions

$$\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned} \qquad \text{for } 0 < t$$

and the initial conditions

$$\begin{aligned} u(x,0) &= 0 \\ \frac{\partial u}{\partial t}(x,0) &= 1 \end{aligned} \qquad \text{for } 0 < x < 1.$$

Reminders: When a and b are real numbers, $e^{a+ib} = (\cos b + i \sin b) \cdot e^a$; and when k and λ are real numbers (not both zero),

$$\int e^{kx} \sin(\lambda x)\,dx = \frac{e^{kx}(k \sin(\lambda x) - \lambda \cos(\lambda x))}{k^2 + \lambda^2}.$$