Reflections on the Arbelos

Harold P. Boas

1. INTRODUCTION. Written in block capitals on lined paper, the letter bore a postmark from a Northeastern seaport. "I have a problem that I would like to solve," the letter began, "but unfortunately I cannot. I dropped out from 2nd year high school, and this problem is too tough for me." There followed the diagram shown in Figure 1 along with the statement of the problem: to prove that the line segment DE has the same length as the line segment AB.

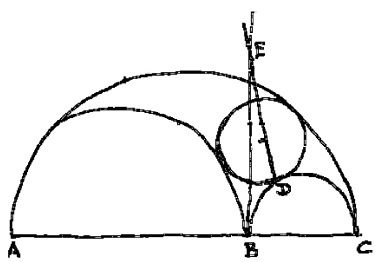

Figure 1. A correspondent's hand-drawn puzzle.

"I tried this and went even to the library for information about mathematics," the letter continued, "but I have not succeeded. Would you be so kind to give me a full explanation?"

Mindful of the romantic story [**20**] of G. H. Hardy's discovery of the Indian genius Ramanujan, a mathematician who receives such a letter wants first to rule out the remote possibility that the writer is some great unknown talent. Next, the question arises of whether the correspondent falls into the category of eccentrics whom Underwood Dudley terms "mathematical cranks" [**16**], for one ought not to encourage cranks.

Since this letter claimed no great discovery, but rather asked politely for help, I judged it to come from an enthusiastic mathematical amateur. Rather than file such letters in the oubliette, or fob them off on junior colleagues, I try to reply in a friendly way to communications from coherent amateurs. Since mathematics has a poor image in our society, it seems incumbent on professional mathematicians to seize every opportunity to foster good will with the lay public. Moreover, any teacher worth the name jumps at a chance to enlighten an eager, inquiring mind; besides, a careful investigation of elementary mathematics can be educational even for the professional.

As Doron Zeilberger and his computer collaborator Shalosh B. Ekhad have shown [**17**], the standard theorems of planar geometry can be checked by executing a few lines of Maple code: typically one merely has to verify the vanishing of a polynomial in the coordinates of certain points.[1] Nonetheless, proving the theorems by hand—like solving crossword puzzles—is an entertaining pastime for many, including my high-school-dropout correspondent.

[1] A similar attempt in 1969 failed for lack of computing power [**11**].

Since the letter reached me in the slack period before the start of a semester, I was able to find time to begin examining the problem, which was new to me. I soon discovered that the underlying geometric figure has a long history, and learning some details of that history sent me repeatedly to the interlibrary loan office to puzzle over publications in half-a-dozen languages. In this article, I reflect on both the mathematics and the history.

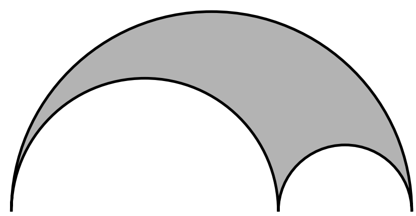

Figure 2. The arbelos.

2. FIRST REFLECTION. The origin of Figure 2, commonly called the *arbelos* (a transliteration of the Greek ἄρβηλος), is lost in the sands of time. The figure shows the region bounded by three semicircles, tangent in pairs, with diameters lying on the same line. The first substantial treatment of the arbelos in modern times (say the last two hundred years) is part of a famous paper [**37**] by Jacob Steiner in the first volume of Crelle's journal in 1826. The arbelos continues to make occasional appearances in journal articles (see [**15**] and its references) and in student theses (for instance [**23**], [**26**], [**44**]); one of Martin Gardner's columns in *Scientific American* discusses it [**19**]; Eric W. Weisstein covers it in his *MathWorld* encyclopedia [**43**]; and there is today a web site `http://www.arbelos.org/`. Yet Victor Thébault's characterization of the arbelos (half a century ago) as "universally known" seems to be an exaggeration.[2]

The fascinating geometric properties of the arbelos range from the elementary to the abstruse. An elementary first proposition is that the length of the lower boundary of the arbelos equals the length of the upper boundary. The proof is immediate from the knowledge that the circumference of a circle is proportional to its diameter; one does not even need to know that the constant of proportionality is π.

A slightly more sophisticated property (see Figure 3) is that the area of the arbelos equals the area of the circle whose diameter CD is the portion inside the arbelos of the common tangent line to the two smaller semicircles at their point C of tangency. This property of the arbelos is Proposition 4 in the ancient Greek *Book of Lemmas* (about which more later).

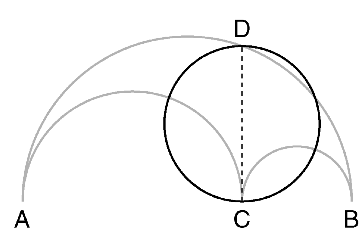

Figure 3. An area property of the arbelos.

[2]The abstract of [**40**] speaks of "cette figure universellement connue."

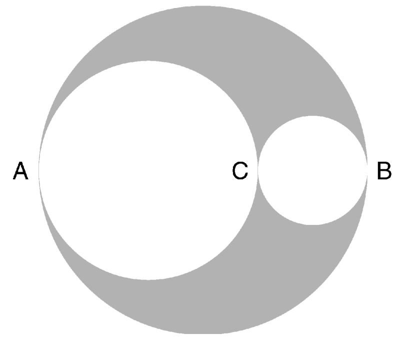

Figure 4. Proof by reflection.

For the proof, reflect in the line through the points A and B (see Figure 4) and observe that twice the area of the arbelos is what remains when the areas of the two smaller circles (with diameters AC and CB) are subtracted from the area of the large circle (with diameter AB). Since the area of a circle is proportional to the square of the diameter (Euclid's *Elements*, Book XII, Proposition 2; we do not need to know that the constant of proportionality is $\pi/4$), the problem reduces to showing that $2(\mathsf{CD})^2 = (\mathsf{AB})^2 - (\mathsf{AC})^2 - (\mathsf{CB})^2$. (To indicate the length of a line segment, I enclose the name of the segment in parentheses.) The length (AB) equals the sum of the lengths (AC) and (CB), so this equation simplifies algebraically to the statement that $(\mathsf{CD})^2 = (\mathsf{AC})(\mathsf{CB})$. Thus the claim is that the length of the segment CD is the geometric mean of the lengths of the segments AC and CB. Now (see Figure 5) the triangle ADB, being inscribed in a semicircle, has a right angle at the point D (Euclid, Book III, Proposition 31), and consequently (CD) is indeed "a mean proportional" between (AC) and (CB) (Euclid, Book VI, Proposition 8, Porism). This proof approximates the ancient Greek argument; one may find the idea implemented as a "proof without words" in [**28**].

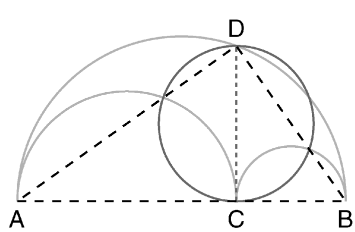

Figure 5. A mean proportional.

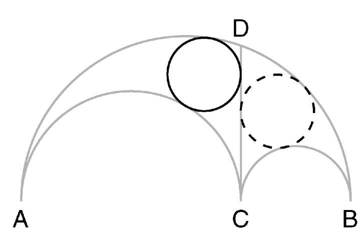

Figure 6. The twin circles.

Proposition 5 of the *Book of Lemmas* is the more arresting statement that if two circles are inscribed in the arbelos tangent to the line segment CD, one on each side as shown in Figure 6, then the two circles are congruent. The proofs that I know proceed by explicitly computing the diameters of the two circles. I invite the reader to attempt the computation via Euclidean methods before reading on for a simple modern argument by reflection.

3. REFLECTION IN A CIRCLE. Speaking at the end of the year 1928, Julian Lowell Coolidge said that the "most notable epoch in all the long history of geometry, the heroic age, was almost exactly a hundred years ago" [**12**, p. 19]. He cited dramatic nineteenth-century advances in all sorts of geometry: synthetic, analytic, projective, hyperbolic, elliptic, and differential. The heroic geometric development of concern here, dating from that period, is the method of reflection in a circle, also known as inversion.

Perhaps the most renowned nineteenth-century user of this tool, who discovered in the year of his twenty-first birthday how to solve problems in electrostatics via inversion [**42**], is William Thomson, subsequently created Baron Kelvin of Largs. The great geometer Steiner, however, usually gets credit for the method of inversion on the basis of notes he wrote on the subject in 1824, the year of Thomson's birth.[3]

The inverse of a point M with respect to a given circle in the plane is the point W determined by two conditions: (i) the points M and W lie on the same ray emanating from the center of the circle, and (ii) the product of the distances of M and W from the center of the circle equals the square of the radius of the circle. (See Figure 7; the reason for the labels M and W is that these glyphs of the standard Latin character set are very nearly reflections of each other.) The points of the circle are fixed under inversion, and the center of the circle corresponds under inversion to the ideal point at infinity.

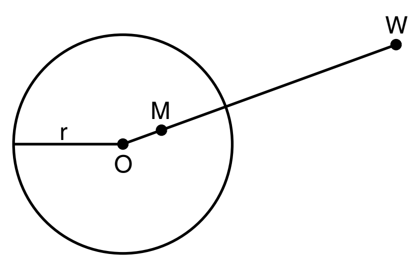

Figure 7. Reflection in a circle: $(\mathsf{OM})(\mathsf{OW}) = \mathsf{r}^2$.

Under inversion in a given circle, both lines and circles behave nicely. For example, the definition of the inverse point immediately implies that a line through the center of the circle of inversion inverts into itself (not pointwise, but as a set). Moreover, either by purely geometric methods or by algebraic calculations, one can show that

- a line not passing through the center of the circle of inversion inverts to a circle;
- a circle not passing through the center of the circle of inversion inverts to a circle;
- a circle passing through the center of the circle of inversion inverts to a line.

The other key property of inversion is anticonformality: the angle at which two oriented curves intersect has the same magnitude as the angle at which the inverted curves intersect, but the opposite sense. For example, two rays starting at the center of the circle of inversion invert into themselves, but with their directions reversed, so the angle between the rays reverses sense. For another example, consider a circle Γ that cuts the circle of inversion at right angles. The inverse of Γ is another circle that meets the circle of inversion orthogonally, and at the same points as Γ does; hence a circle orthogonal to the circle of inversion inverts into itself.

[3]Extracts from Steiner's notes were eventually published in [**10**], but not until half a century after Steiner's death. Priority for the concept of inversion is sometimes claimed on behalf of various other mathematicians active in the first half of the nineteenth century. Nathan Altshiller Court [**13**] went so far as to attribute the idea of inversion to Apollonius of Perga, a contemporary of Archimedes from two millennia earlier!

Inversion is a standard topic both in textbooks on geometry and in textbooks on complex analysis, but the word "inversion" has different meanings in the two subjects. Geometric inversion corresponds to the composition of the complex analytic inversion $z \mapsto 1/z$ with complex conjugation $z \mapsto \overline{z}$.

Returning to Figure 6, consider the effect of inverting the diagram with respect to a circle that is centered at the point A and that orthogonally intersects the left-hand member—call it Γ—of the pair of inscribed circles. Then Γ inverts into itself, and the horizontal line through A and B inverts into itself. The semicircle with diameter AB inverts into a portion of a line that is tangent to the circle Γ and that meets the line segment AB orthogonally at the point B' inverse to the point B (see Figure 8). The semicircle with diameter AC inverts into a portion of a line that is tangent to Γ and that meets the line segment AB orthogonally at the point C' inverse to the point C. Since Γ is fixed by inversion, the points C and C' must be identical, which means that the point C lies on the circle of inversion. (Also shown in the figure—but not needed in the argument—are the image D' of the point D, the circular image of the line segment CD, and the image of the right-hand twin circle.)

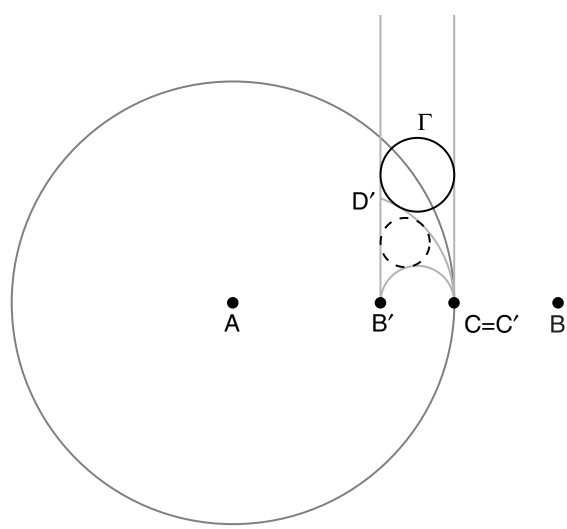

Figure 8. Inversion of Figure 6.

Let d_1 denote the length of the line segment AC, let d_2 denote the length of the line segment CB, and let d denote the unknown diameter of the circle Γ. The points B and B' are inverse points with respect to the circle with center A and radius d_1, so $(d_1 - d)(d_1 + d_2) = d_1^2$, which implies that $d = d_1 d_2/(d_1 + d_2)$. This quantity is symmetric in d_1 and d_2, so repeating the argument for inversion with respect to a suitable circle centered at B will yield the same diameter for the right-hand circle inscribed in the arbelos. Thus the twin circles are indeed congruent: they have equal diameters.

4. LOST IN TRANSLATION. Figure 2 appears frequently in the literature on recreational mathematics with a comment to the effect: "The figure was first studied by Archimedes of Syracuse, who called it the arbelos or shoemaker's knife." Such historical statements, unlike mathematical theorems, are only an approximation to an unknowable truth. The surviving works of Archimedes do not mention the arbelos.

The source for the claim that Archimedes studied and named the arbelos is the *Book of Lemmas*, also known as the *Liber assumptorum* from the title of the seventeenth-century Latin translation of the ninth-century Arabic translation of the lost Greek original. Although this collection of fifteen propositions is included in standard editions of the works of Archimedes [**1**], [**2**], [**3**], [**4**], the editors acknowledge that the author of the *Book of Lemmas* was not Archimedes but rather some anonymous later compiler, who indeed refers to Archimedes in the third person.

Perhaps one day someone will find direct evidence that Archimedes explored the properties of Figure 2. After all, just a century ago the lost *Method* of Archimedes was partially recovered by Johan Ludvig Heiberg from a palimpsest in a library in Constantinople. After the palimpsest was auctioned in 1998 for two million dollars, contemporary scholars were allowed access to recover more of the text using modern technology, thereby gaining new insight into the works of Archimedes (see [**29**]).

Why is the arbelos called "the shoemaker's knife"? The oldest extant source for the Greek word seems to be Nicander's *Theriaca*, a work about venomous creatures that apparently dates from the middle of the second century B.C., about half a century after the death of Archimedes. What I am able to understand from the obscure poetical passage is that Nicander knew the arbelos as a tool used by leather-workers to trim smelly green hides.[4] A scholium from an unknown date and hand glosses "arbelos" as a circular knife used by σκυτοτόμοι, a word that literally means "leather-cutters" but that typically is rendered as "shoemakers." The ancient use of a knife with a curved blade in the manufacture of sandals is indeed attested by Egyptian drawings from 3500 years ago [**35**, p. 18], and a similar tool is still marketed today to leather-crafters (Figure 9).

Figure 9. A modern round knife.

My view, however, is that "shoemaker's knife" is a bad translation. Just as most users of the "Swiss army knife" have no connection with the armed forces of Switzerland, most wielders of the arbelos have nothing to do with the making of shoes. D'Arcy W. Thompson reported more than half a century ago that the round knife was then in use by saddlers but not by shoemakers [**41**]. A century before that, Charles Dickens surely was not thinking of a tool like the one in Figure 9 when he wrote of visiting an imprisoned thief who "would have gladly stabbed me with his shoemaker's knife" [**14**, pp. 248–249].

[4] The original Greek, as given in [**30**, pp. 34–35], reads: Τὸ δ' ἀπὸ χροὸς ἐχθρὸν ἄηται, οἷον ὅτε πλαδόωντα περὶ σκύλα καὶ δέρε' ἵππων γναπτόμενοι μυδόωσιν ὑπ' ἀρβήλοισι λάθαργοι.

A good translation ought to bring a familiar image to the reader's mind. Since most modern readers are unfamiliar with the round knife shown in Figure 9, I suggest renaming the arbelos as "the claw."[5]

5. REFLECTIONS ON PAPPUS. Some half a millennium after Archimedes died during the sack of Syracuse in the second Punic war, Pappus of Alexandria wrote his great *Collection*, a set of eight books (mostly extant) that preserve much Greek mathematics that otherwise would have been lost. The precise span of the life of Pappus is uncertain, but modern histories often represent him as the last bright light of the Greek mathematical tradition in the penumbra of the Dark Ages.

Although Pappus sometimes cites his sources, one cannot always tell when he is improving earlier treatments and when he is being wholly original. His discussion of the arbelos in Book IV of the *Collection* notably does not cite Archimedes or anybody else by name. But Pappus makes clear that he holds no claim to the most famous theorem about the arbelos, a theorem nowadays often attributed to Pappus by default.

This remarkable theorem states (see Figure 10) that if one inscribes a chain of circles in the arbelos, the first circle in the chain being tangent to all three semicircles, and the subsequent circles in the chain being tangent to the preceding circle and to two of the semicircles, then the height of the center of the nth circle above the horizontal line segment AB is n times the diameter of that circle. The direction of the chain is immaterial: the chain can be directed to the left (illustrated in Figure 10), to the right, or downward. The latter two cases are illustrated in Figure 11.

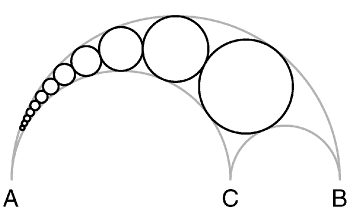

Figure 10. A classical diagram.

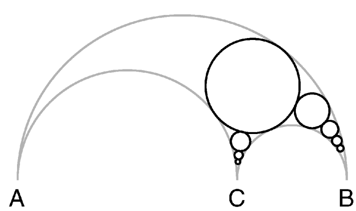

Figure 11. Two variations.

The proposition is sometimes called the "ancient theorem" of Pappus because of the words with which Pappus introduces it in Book IV of the *Collection*. Translated loosely into modern idiom, the phrase that Pappus uses is: "The following classical result is well known."[6]

The proof that Pappus gives is a tour de force of Euclidean geometry: three lemmas and many pages of sophisticated, systematic use of auxiliary lines, similar triangles, and the Pythagorean theorem. For a sketch of the proof of Pappus in contemporary language, see the paper [**5**] by Leon Bankoff, who was by avocation a mathematician but by profession a dentist with an office at an upscale address near Beverly Hills.[7]

The modern proof using inversion is elegantly simple. In Figure 10, invert with respect to a circle centered at A and orthogonal to the nth circle of the inscribed chain. The nth circle inverts into itself, the two semicircles tangent to it invert into vertical

[5]Paul Ver Eecke, writing for a French audience [**32**], evidently felt that "tranchet de cordonnier" was an insufficient translation of "arbelos," for he glossed the term with the description "griffe de félin"—feline's claw.

[6]The Greek is given in [**33**, p. 208] as: Φέρεται ἔν τισιν ἀρχαία πρότασις τοιαύτη.

[7]After the death of Victor Thébault in 1960, Bankoff was the world expert on the arbelos. Clayton W. Dodge is editing a book manuscript on the arbelos by Bankoff and Thébault for eventual publication.

lines, and the preceding $n-1$ circles in the chain invert into circles tangent to those lines. See Figure 12, where it is now obvious that the height of the center of the nth circle is n times the diameter of the circle. (For the variations shown in Figure 11, one can argue similarly by inverting in a circle centered either at C or at B.)

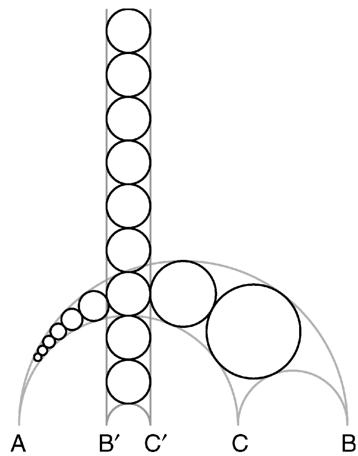

Figure 12. Inversion of Figure 10.

I do not know which nineteenth-century author first published this proof of the theorem by inversion. To see a treatment in an accessible recent book, consult [**8**, p. 14] or [**27**, pp. 136–137].

Let us go a little further with Figure 12. By the defining property of inversion, $(\mathsf{AC})/(\mathsf{AB}') = (\mathsf{AB})/(\mathsf{AC}')$. Dilating the figure by this factor with respect to the center A moves the vertical tower of circles to a vertical tower over the line segment CB. Since dilation preserves ratios of lengths, it follows that the distance from the center of the nth circle of the original chain to the *vertical* line through the point A is equal to a *constant* times the diameter of the nth circle. This observation is due to Steiner (see [**37**, p. 261] or, equivalently, [**38**, p. 49]). The value of the constant can be read off from the circle with diameter CB: namely, the constant equals $(1/2) + (\mathsf{AC})/(\mathsf{CB})$.

Exercise for the reader. What is the analogous statement for a chain of circles converging to the point B? to the point C?

A right triangle is called *Pythagorean* if it is similar to a triangle whose sides have integral lengths. (This means that the original triangle has integral sides with respect to a suitable unit of measurement.) An amusing remark of Steiner ([**37**, p. 265], [**38**, p. 53]) is that if the lengths (AC) and (CB) are commensurable—that is, if the ratio $(\mathsf{AC})/(\mathsf{CB})$, which I will denote by ρ, is a rational number—then in Figure 10 every right triangle with vertices at the midpoint of the line segment AB, at the center of the nth circle, and at the foot of the perpendicular dropped from that center to the line segment AB is a Pythagorean triangle (see Figure 13).

To see why, suppose that the nth circle has radius r_n and center at the point (x_n, y_n), where the midpoint of the line segment AB is taken as the origin of the Cartesian coordinates. The discussion of Figure 12 implies that $y_n = 2nr_n$ and $x_n = (1+2\rho)r_n - \frac{1}{2}(\mathsf{AB})$. Moreover, the length of the hypotenuse of the nth triangle is $\frac{1}{2}(\mathsf{AB}) - r_n$. Con-

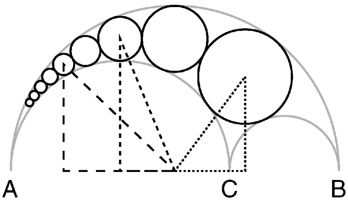

Figure 13. Some Pythagorean triangles.

sequently, what needs to be shown is that the length (AB) is a rational multiple of r_n, for then all three sides of the nth triangle will be commensurate with r_n. Combining the Pythagorean theorem with the explicit expressions for x_n and y_n yields

$$\begin{aligned}(2nr_n)^2 &= \left(\tfrac{1}{2}(\mathsf{AB}) - r_n\right)^2 - \left((1+2\rho)r_n - \tfrac{1}{2}(\mathsf{AB})\right)^2 \\ &= 2\rho r_n\left((\mathsf{AB}) - 2r_n - 2\rho r_n\right).\end{aligned}$$

Thus $(\mathsf{AB}) = 2r_n(\rho + 1 + n^2\rho^{-1})$, so the length (AB) is indeed commensurate with r_n under the hypothesis that ρ is a rational number.

Exercise for the reader. Moving the first vertex of the triangle from the midpoint of the line segment AB to the midpoint of the line segment AC produces another set of Pythagorean triangles.

6. ARCH REFLECTIONS. One of the attractions of planar geometry is that an enthusiastic amateur, like Leon Bankoff, can acquire lore that surpasses the erudition of professionals who ought to know better. I offer as evidence some unrecognized appearances of the arbelos in the lovely book *Geometry Civilized* by J. L. Heilbron [**21**].

In [**21**, Exercise 5.5.15], the author draws the arbelos and states the area theorem associated with Figure 3. He does not name the arbelos, however, and he cites as his source a problem in the *Ladies Diary* of 1808 (as given in [**25**, vol. 4, p. 106]). Certainly it is interesting to know that geometry was a popular pastime in England two hundred years ago, but the author is off by perhaps two thousand kilometers and two thousand years in the provenance of the problem.

Moreover, the author does not mention that geometry was also a popular pastime in Japan of the Edo period. In the case of a symmetric arbelos, the theorem of section 5 about a chain of inscribed circles appeared on a wooden *sangaku*, a geometry tablet hung in a Japanese temple. See Figure 14, which shows also a version with a higher-

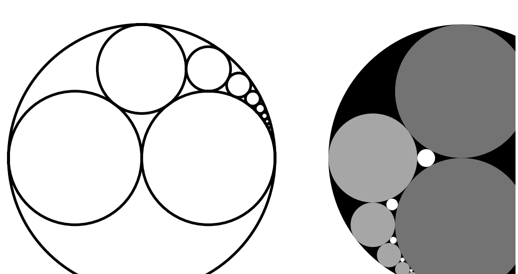

Figure 14. Sangaku problems (adapted from [**18**, p. 18] and [**34**]).

order chain of inscribed circles from a sangaku about two centuries old. Hidetoshi Fukagawa, a high-school mathematics teacher, is the driving force in preserving the history of sangaku [**31**].

An unrecognized appearance of the arbelos occurs more prominently in [**21**] in the penultimate section, which is devoted to geometric designs reminiscent of Gothic windows. The discussion focuses on Figure 15, consisting of a semicircle, arcs of two circles with radius coincident with the diameter of the semicircle, and a sequence of inscribed circles condensing on the right-hand endpoint. The author states and proves "Alison's conjecture" to the effect that the right triangle with vertices at the left-hand endpoint, at the center of the *n*th inscribed circle, and at the foot of the perpendicular dropped from that center to the base is a Pythagorean triangle; and similarly if the first vertex is placed at the midpoint of the base.

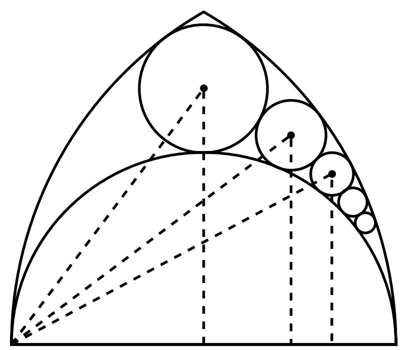

Figure 15. Arch (adapted from [**21**, p. 289]).

One distinguished reviewer of the book was sufficiently taken with this "original result" to reproduce it (including the figure) at the conclusion of the review in this MONTHLY [**36**]. As it happens, Leon Bankoff contributed this very problem to this MONTHLY [**6**] half a century ago! The solution [**9**] was accompanied by the diagram shown in Figure 16, which is effectively the same as Figure 15 after a rotation and a reflection.

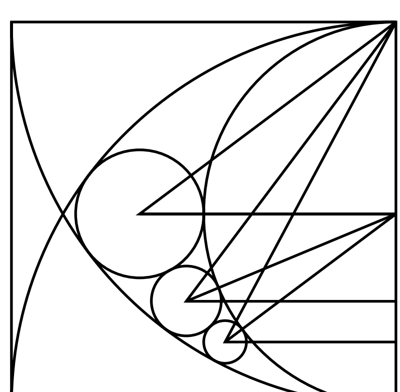

Figure 16. Pythagorean triangles (adapted from [**9**]).

The reader has realized by now, I hope, that the problem about the Gothic arch is nothing more than a special case of Steiner's remark (discussed at the end of section 5) about the Pythagorean triangles lurking in the chain of circles inscribed in the arbelos (see Figure 17). This special case is explicitly written out in Steiner's paper [**37**, p. 266], [**38**, p. 53].

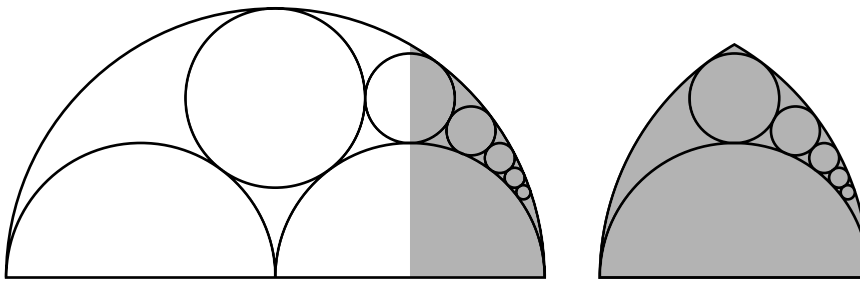

Figure 17. Reflecting the symmetric arbelos into the arch.

7. MOHR ON THE ARBELOS. Since the arbelos is a classical bit of "pure" mathematics, I was startled to learn from [**7**] that the arbelos is well known in textbooks on solid mechanics under the name "Mohr's circles." One of the topics of interest to Otto Mohr (1835–1918), a renowned German civil engineer and professor of mechanics, was how materials react to stress. Because shear stress is a key factor in the failure of materials, Mohr used a diagram that relates the shear force to the normal force. Understanding this diagram involves the following mathematical problem (which Mohr posed and solved).

Let $\mathbf{L}$ be a symmetric 3×3 matrix with real entries, and let $\mathbf{V}$ denote a unit vector in $\mathbf{R}^3$, thought of as representing the normal vector to some surface. Consider the mapping that takes the unit vector $\mathbf{V}$ to the pair of real numbers (x, y), where $x = \mathbf{V} \cdot \mathbf{LV}$ (the dot denotes the standard scalar product, so $x\mathbf{V}$ is the normal component of the vector $\mathbf{LV}$), and $y = \|\mathbf{LV} - x\mathbf{V}\|$ (the length of the tangential component of $\mathbf{LV}$). What is the range of this mapping (in $\mathbf{R}^2$) as $\mathbf{V}$ varies over the unit sphere (in $\mathbf{R}^3$)?

The remarkable answer is that the range is an arbelos. Moreover, the abscissae of the three cusp points are the eigenvalues of the matrix. In Figure 18, the eigenvalues are denoted by λ_1, λ_2, and λ_3 in increasing order. (I assume that the three eigenvalues are all distinct; otherwise the arbelos degenerates.)

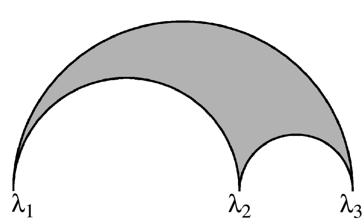

Figure 18. Eigenvalues and Mohr's arbelos.

To see why the range is an arbelos, first observe that since $\mathbf{R}^3$ has an orthonormal basis consisting of eigenvectors of the symmetric matrix $\mathbf{L}$, and since the range in $\mathbf{R}^2$ does not change when $\mathbf{R}^3$ is subjected to an orthogonal transformation, there is no loss of generality in assuming that the matrix $\mathbf{L}$ is diagonal. Let u_j denote the square of the

jth component of the unit vector $\mathbf{V}$. Then $x = \sum_{j=1}^{3} \lambda_j u_j$, and $y^2 = \sum_{j=1}^{3} \lambda_j^2 u_j - x^2$. By using u_1, u_2, and u_3 as the independent variables, one can view the domain of Mohr's mapping as the triangle in $\mathbf{R}^3$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

These vertices evidently map to the three cusp points of the arbelos indicated in Figure 18. I claim that the three edges of the triangular domain map to the three semicircles bounding the arbelos. By symmetry, it suffices to check one edge, say the edge where $u_3 = 0$ and $u_1 + u_2 = 1$. In this case, a simple calculation shows that $y^2 + x^2 - (\lambda_1 + \lambda_2)x = -\lambda_1\lambda_2$. Since

$$-4\lambda_1\lambda_2 = (\lambda_1 - \lambda_2)^2 - (\lambda_1 + \lambda_2)^2 ,$$

it follows that

$$y^2 + \left(x - \frac{\lambda_1 + \lambda_2}{2}\right)^2 = \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 .$$

This equation indeed describes a circle with center at the point midway between the first two eigenvalues and with radius equal to half the distance between those eigenvalues.

To see that the interior of the triangular domain maps precisely to the interior of the arbelos, consider how the ordinate y varies when x is held fixed. Since fixing x corresponds to intersecting the triangular domain with a certain plane, the effective domain becomes a line segment in $\mathbf{R}^3$. On this line segment, y^2 is an affine linear function of the variables u_1, u_2, and u_3, so the values of y^2 (hence the values of the nonnegative quantity y) fill out some interval. The endpoints of that interval correspond to certain boundary points of the domain. As observed in the preceding paragraph, each boundary point of the domain maps to one of the semicircles bounding the arbelos. In other words, the image of Mohr's mapping is made up of vertical line segments each of which connects one of the lower semicircles of the arbelos to the upper semicircle.

8. PARTING REFLECTIONS. This article is an amplified version of my reply to the correspondent who posed the problem stated in the introduction. A full reply would require a book, for the problem has ramifications that lead to recent developments in contemporary mathematical research. The theorem about the chain of circles inscribed in the arbelos has a close affinity with the famous classical problem of Apollonius: to construct a circle tangent to three given circles. Continuing the iterations suggested by the second sangaku in Figure 14 so as to fill up the interstices in the figure with touching circles produces a so-called Apollonian circle packing, a subject that has attracted much recent attention (see [**24**] and [**39**] and their references).

Eventually I discovered that the diagram in Figure 1 is incorrectly drawn. Following the dictum of Arnold Ross to "prove or disprove and salvage if possible," I found the corrected problem shown in Figure 19. The central circle, instead of being tangent to all three semicircles (corresponding to the initial circle in the chain of Figure 10), should be the right-hand twin circle from Figure 6. Having reached the end of this article, readers should know enough about the arbelos to solve the problem themselves. One can look up a published solution [**22**] that uses Euclidean methods, but simpler is to apply inversion.[8]

[8]Hint: It suffices to show that the line segments AF and EF meet at a right angle, for then the triangles AFG and ECG are similar. Invert in a circle centered at A that cuts the right-hand twin circle orthogonally, and prove that the point F is left fixed.

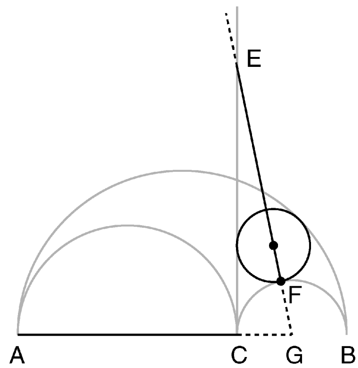

Figure 19. Corrected problem: $\mathsf{AC} = \mathsf{EF}$.

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REFERENCES

1. Archimedes, *Archimède*, vol. 3, C. Mugler, ed., Belles Lettres, Paris, 1971.
2. ———, *Archimedis opera omnia cum commentariis Eutocii*, vol 2, J. L. Heiberg, ed. (with corrections by E. S. Stamatis), B. G. Teubner, Stuttgart, 1972; reprint of the 1913 edition.
3. ———, *Les oeuvres complètes d'Archimède*, P. Ver Eecke, ed., Desclée, de Brouwer, Paris, 1921.
4. ———, *The Works of Archimedes*, T. L. Heath, ed., Dover, Mineola, NY, 2002; reprint of the 1897 edition and the 1912 supplement.
5. L. Bankoff, How did Pappus do it?, in *The Mathematical Gardner*, D. A. Klarner, ed., Prindle, Weber, & Schmidt, Boston, 1981, pp. 112–118.
6. ———, Problem 4600, this MONTHLY **61** (1954) 476.
7. P. Bisegna and P. Podio-Guidugli, Mohr's arbelos, *Meccanica* **30** (1995) 417–424.
8. D. E. Blair, *Inversion Theory and Conformal Mapping*, American Mathematical Society, Providence, 2000.
9. W. J. Blundon, An infinite sequence of Pythagorean triangles, Solution to problem 4600, this MONTHLY **62** (1955) 734–736.
10. F. Bützberger, *Über Bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion*, Teubner, Leipzig, 1913.
11. E. Cerutti and P. J. Davis, FORMAC meets Pappus: Some observations on elementary analytic geometry by computer, this MONTHLY **76** (1969) 895–905.
12. J. L. Coolidge, The heroic age of geometry, *Bull. Amer. Math. Soc.* **35** (1929) 19–37; address read on December 28, 1928.
13. N. A. Court, Notes on inversion, *Math. Teacher* **55** (1962) 655–657.
14. C. Dickens, *American Notes for General Circulation*, Chapman and Hall, London, 1842.
15. C. W. Dodge, T. Schoch, P. Y. Woo, and P. Yiu, Those ubiquitous Archimedean circles, *Math. Mag.* **72** (1999) 202–213.
16. U. Dudley, *Mathematical Cranks*, Mathematical Association of America, Washington, D.C., 1992.
17. S. B. Ekhad and D. Zeilberger, *Plane Geometry: An Elementary Textbook*, *circa* 2050 ("downloaded from the future"); available at `http://www.math.rutgers.edu/~zeilberg/GT.html`.
18. H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, MB, 1989.
19. M. Gardner, Mathematical games: The diverse pleasures of circles that are tangent to one another, *Sci. Amer.* **240** (1979) 18–28; reprinted in *Fractal Music, Hypercards and More...*, W. H. Freeman, 1992, pp. 149–166.
20. G. H. Hardy, The Indian mathematician Ramanujan, this MONTHLY **44** (1937) 137–155.
21. J. L. Heilbron, *Geometry Civilized: History, Culture, and Technique*, Oxford University Press, New York, 1998.
22. T. Hermann, Twin segments in the arbelos, Solution to problem 10895, this MONTHLY **110** (2003) 63–64.
23. R. D. Knight, *Using Laguerre Geometry to Discover Euclidean Theorems*, Ph.D. dissertation, University of California, San Diego, 2000.

24. J. C. Lagarias, C. L. Mallows, and A. R. Wilks, Beyond the Descartes circle theorem, this MONTHLY **109** (2002) 338–361.
25. T. Leybourn, *The Mathematical Questions, Proposed in the Ladies' Diary, and Their Original Answers, together with Some New Solutions, from Its Commencement in the Year 1704 to 1816*, four volumes, J. Mawman, London, 1817.
26. M. C. Miller, *The Arbelos*, Master's thesis, Indiana University, 1932.
27. T. Needham, *Visual Complex Analysis*, Oxford University Press, New York, 1997.
28. R. B. Nelsen, Proof without words: The area of an arbelos, *Math. Mag.* **75** (2002) 144.
29. R. Netz, K. Saito, and N. Tchernetska, A new reading of *Method* Proposition 14: Preliminary evidence from the Archimedes palimpsest. I and II, *SCIAMVS* **2** (2001) 9–29 and **3** (2002) 109–125.
30. Nicander, *Œuvres*, vol. 2, J.-M. Jacques, ed., Belles Lettres, Paris, 2002.
31. D. Normile, "Amateur" proofs blend religion and scholarship in ancient Japan, *Science* **307** (2005) 1715–1716.
32. Pappus, *La collection mathématique*, vol. 1, P. Ver Eecke, ed., Desclée, de Brouwer, Paris, 1933.
33. ———, *Pappi Alexandrini collectionis*, vol. 1, F. O. Hultsch, ed., Adolf M. Hakkert, Amsterdam, 1965; reprint of the 1875 edition.
34. T. Rothman, Japanese temple geometry, *Sci. Amer.* **278** (1998) 84–91.
35. R. A. Salaman, *Dictionary of Leather-Working Tools, c. 1700–1950, and the Tools of Allied Trades*, Macmillan, New York, 1986.
36. D. Schattschneider, Review of *Geometry Civilized*, this MONTHLY **108** (2001) 182–188.
37. J. Steiner, Einige geometrische Betrachtungen, *J. Reine Angew. Math.* **1** (1826) 161–184, 252–288.
38. ———, *Gesammelte Werke*, vol. 1, K. Weierstrass, ed., 2nd ed., Chelsea, New York, 1971; corrected reprint of the 1881 edition.
39. K. Stephenson, Circle packing: A mathematical tale, *Notices Amer. Math. Soc.* **50** (2003) 1376–1388.
40. V. Thébault, A propos du tranchet d'Archimède, *Ann. Soc. Sci. Bruxelles Sér. I* **66** (1952) 41–48.
41. D'A. W. Thompson, Mathematics from Aristarchus to Pappus, *Classical Review* **56** (1942) 75–76; review of Ivor Thomas's *Selections Illustrating the History of Greek Mathematics. II. From Aristarchus to Pappus*.
42. W. Thomson (Lord Kelvin), Extrait d'une lettre de M. William Thomson à M. Liouville, *J. Math. Pures Appl.* **10** (1845) 364–367.
43. E. W. Weisstein, *Arbelos*; available at `http://mathworld.wolfram.com/Arbelos.html`.
44. M. G. Welch, *The Arbelos*, Master's thesis, University of Kansas, 1949.

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