

MAJORANT SERIES

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ABSTRACT. This article discusses questions in one and several complex variables about the size of the sum of the moduli of the terms of the series expansion of a bounded holomorphic function. Although the article is partly expository, it also includes some previously unpublished results with complete proofs.

1. Introduction

What properties of a holomorphic function can be detected from the *moduli* of its Maclaurin series coefficients?

To make this question precise, fix a holomorphic function f such that $f(z) = \sum_k c_k z^k$, and consider the class \mathcal{F} of all power series expansions $\sum_k b_k z^k$ with the property that $|b_k| = |c_k|$ for all k . Which properties of f are inherited by all functions in the class \mathcal{F} ?

Here are some examples of properties that hold for all members of \mathcal{F} if they hold for one member of \mathcal{F} .

- The radius of convergence of the Maclaurin series equals 1 (since Hadamard's formula for the radius of convergence depends only on the moduli of the coefficients).
- The function belongs to the Hardy space H^2 of the unit disk (since the Hardy space norm equals the square root of the sum of the squares of the moduli of the Maclaurin series coefficients).

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- The function is univalent in a neighborhood of the origin (since this property holds if and only if $c_1 \neq 0$.)

On the other hand, here are some examples of properties that may hold for some members of \mathcal{F} but not for others.

- The function is holomorphic and univalent in the whole unit disk.
For instance, the linear fractional transformation $z \mapsto \frac{z}{1-z}$ is univalent in the unit disk, but changing the sign of the first term of the Maclaurin series yields the function $z \mapsto -2z + \frac{z}{1-z}$, which maps the points 0 and $1/2$ both to 0.
- The function has zeroes at prescribed locations.

Indeed, composing a member of \mathcal{F} with the reflection $z \mapsto -z$ yields a new member of \mathcal{F} with zeroes at different locations.

- The function is holomorphic and bounded in the unit disk.
For instance, for almost every choice of plus and minus signs, the series $\sum_{k=1}^{\infty} \pm z^k/k$ is continuous when $|z| \leq 1$ (see, for example, [15, Chapter V, Theorem 8.34]); but if all plus signs are taken, the series is unbounded in the unit disk.

2. Bohr's theorem

More generally, it might happen that if f has a certain property, then every function in the associated class \mathcal{F} has some related property. For example, an old theorem of Harald Bohr ([4]) implies that if f is in the unit ball of H^∞ of the unit disk, then every element of the class \mathcal{F} is in the unit ball of H^∞ of the disk of radius $1/3$.

THEOREM 1. ([Bohr, 1914]) *If $|\sum_{k=0}^{\infty} c_k z^k| < 1$ when $|z| < 1$, then $\sum_{k=0}^{\infty} |c_k z^k| < 1$ when $|z| < 1/3$. Moreover, the radius $1/3$ is the best possible.*

This surprising theorem has been largely forgotten. The following proof, based on a classical inequality of Carathéodory, is due to Edmund Landau ([8]).

LEMMA. ([Carathéodory's inequality]) *If g is a holomorphic function with positive real part in the unit disk, and $g(z) = \sum_{k=0}^{\infty} b_k z^k$, then $|b_k| \leq 2 \operatorname{Re} b_0$ when $k \geq 1$.*

Proof. ([Proof of Bohr's theorem]) Let $f(z)$ denote the series $\sum_{k=0}^{\infty} c_k z^k$. Let φ be an arbitrary real number, and set the function g in Carathéodory's

inequality equal to $1 - e^{i\varphi}f$ to deduce that $|c_k| \leq 2 \operatorname{Re}(1 - e^{i\varphi}c_0)$ when $k \geq 1$. Since φ is arbitrary, it follows that $|c_k| \leq 2(1 - |c_0|)$ when $k \geq 1$.

If f is a constant function, then the conclusion of the theorem is trivial. For nonconstant f , the preceding inequality shows that if $|z| < 1/3$, then

$$\sum_{k=0}^{\infty} |c_k z^k| < |c_0| + 2(1 - |c_0|) \sum_{k=1}^{\infty} (1/3)^k = 1.$$

To see that the radius $1/3$ in Bohr's theorem is the best possible, consider the linear fractional transformation f_a defined by $f_a(z) = \frac{z - a}{1 - az}$ when $0 < a < 1$. Writing $f_a(z) = \sum_{k=0}^{\infty} c_k(a)z^k$, one easily computes that $\sum_{k=0}^{\infty} |c_k(a)z^k| = 2a + f_a(|z|)$. Simple algebra shows that $2a + f_a(|z|) > 1$ when $|z| > 1/(1 + 2a)$. Since a can approach 1 from below, it follows that the radius $1/3$ in Bohr's theorem cannot be increased. \square

Proof. ([Proof of Carathéodory's inequality]) A common proof of Carathéodory's inequality (see [5, page 41], for example) uses the Herglotz representation for positive harmonic functions. I learned the following even simpler proof of the inequality from [2].

By considering $g(rz)$ and letting r increase toward 1, we may assume without loss of generality that g is holomorphic in a neighborhood of the closed unit disk. When $k \geq 1$, orthogonality implies that

$$b_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} g(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \left(g(e^{i\theta}) + \overline{g(e^{i\theta})} \right) d\theta.$$

Consequently,

$$|b_k| \leq \frac{1}{2\pi} \int_0^{2\pi} |2 \operatorname{Re} g(e^{i\theta})| d\theta = 2 \operatorname{Re} b_0,$$

where the last step follows because $2 \operatorname{Re} g$ is a positive function with the mean-value property. \square

A natural way to study the class \mathcal{F} associated to a holomorphic function f is to single out a canonical representative of the class. If $f(z) = \sum_{k=0}^{\infty} c_k z^k$, then the *majorant function* $\mathcal{M}f$ is defined by $\mathcal{M}f(z) = \sum_{k=0}^{\infty} |c_k| z^k$.

Evidently $\sup_{|z| < r} |f(z)| \leq \sup_{|z| < r} |\mathcal{M}f(z)| = \mathcal{M}f(r)$, which justifies the name "majorant function". Bohr's theorem may be restated as the

inequality $\sup_{|z|<r} |\mathcal{M}f(z)| \leq \sup_{|z|<3r} |f(z)|$ for every r for which the right-hand side makes sense.

3. Wintner's problem

In 1956, Aurel Wintner ([14]) raised the following question related to Bohr's theorem on majorant series:

For the class of holomorphic functions f in the unit disk with modulus bounded by 1, find the best upper bound on $\inf_{0<|z|<1} \left| \frac{\mathcal{M}f(z)}{z} \right|$ or on $\inf_{0<r<1} \frac{\mathcal{M}f(r)}{r}$.

It follows by taking r equal to $1/3$ that the value 3 is an upper bound. Wintner claimed—incorrectly—that the bound 3 cannot be improved.

In a subsequent paper [13] whose title quotes Wintner's title, Günther Schlenstedt pointed out that Wintner made a blunder in high-school algebra, and that it is easy to see from the maximum principle that the best bound on $\inf_{0<|z|<1} \left| \frac{\mathcal{M}f(z)}{z} \right|$ is actually 1. It is amusing to note that the address printed at the end of Schlenstedt's paper is "Carl-Friedrich-von-Siemens-Schule, Berlin", so Schlenstedt was presumably either a student or a teacher at a German high school.

Perhaps one should not be too critical of Wintner for this error, for he published 26 papers in 1956. Moreover, even the mistakes of a good mathematician are interesting.

In my view, the really interesting mistake in Wintner's paper is one that Schlenstedt did not address. Namely, Wintner claimed to study $\inf_{0<|z|<1} \left| \frac{\mathcal{M}f(z)}{z} \right|$, but he actually studied $\inf_{0<r<1} \frac{\mathcal{M}f(r)}{r}$, apparently thinking that these quantities admit the same optimal bound. In fact, the best bound on the second quantity is neither 3 nor 1, but 2, as I shall now demonstrate.

THEOREM 2. *If f is a holomorphic function such that $|f(z)| < 1$ when $|z| < 1$, then the majorant function $\mathcal{M}f$ satisfies the inequality*

$$\inf_{0<r<1} \frac{\mathcal{M}f(r)}{r} \leq 2,$$

and the bound 2 cannot be replaced by any smaller number.

Proof. I shall prove somewhat more than is stated in the theorem. Namely, I do not need the function f to belong to the unit ball of H^∞ ; all I will use is that f belongs to the unit ball of the Hardy space H^2 . In other words, a sufficient hypothesis on f is that

$$\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq 1.$$

By Parseval's theorem, this hypothesis implies that if $f(z)$ has the series expansion $\sum_{k=0}^{\infty} c_k z^k$ when $|z| < 1$, then $\sum_{k=0}^{\infty} |c_k|^2 \leq 1$. Consequently, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{\mathcal{M}f(r)}{r} &= \frac{|c_0|}{r} + \sum_{k=1}^{\infty} |c_k| r^{k-1} \leq \frac{|c_0|}{r} + \left(\sum_{k=1}^{\infty} |c_k|^2 \right)^{1/2} \frac{1}{\sqrt{1-r^2}} \\ &\leq \frac{|c_0|}{r} + \frac{\sqrt{1-|c_0|^2}}{\sqrt{1-r^2}}. \end{aligned}$$

Let $r \rightarrow |c_0|$ to deduce that $\inf_{0 < r < 1} \frac{\mathcal{M}f(r)}{r} \leq 2$.

For the particular function f such that $f(z) = \frac{z - \frac{1}{\sqrt{2}}}{1 - \frac{z}{\sqrt{2}}}$, a straightfor-

ward computation shows that $\inf_{0 < r < 1} \frac{\mathcal{M}f(r)}{r} = 2$, so the bound 2 cannot be improved. \square

4. A multi-dimensional analogue of Bohr's theorem

Some researchers in several complex variables feel that power series are boring, because the really interesting parts of multi-dimensional complex analysis are the parts that differ from the one-variable theory, while the elementary theory of power series superficially appears the same in all dimensions. My goal here is to exhibit some interesting and accessible problems about multi-variable power series in which the dependence on the dimension is the key issue.

I shall use multi-index notation to write an n -variable power series as $\sum_{\alpha} c_{\alpha} z^{\alpha}$, where α denotes an n -tuple $(\alpha_1, \dots, \alpha_n)$ of non-negative integers, and z^{α} denotes the product $z_1^{\alpha_1} \dots z_n^{\alpha_n}$. It is standard (see [12, pages 78–80], for example) that such a power series converges in a logarithmically convex complete Reinhardt domain. To each such domain G

corresponds a *Bohr radius* $K(G)$: the largest r such that whenever $|\sum_{\alpha} c_{\alpha} z^{\alpha}| \leq 1$ for z in G , it follows that $\sum_{\alpha} |c_{\alpha} z^{\alpha}| \leq 1$ for z in the scaled domain rG .

The terminology “Bohr radius” is somewhat whimsical, for physicists consider the Bohr radius a_0 of the hydrogen atom to be a fundamental constant: its value is $4\pi\epsilon_0\hbar^2/m_e e^2$, or about 0.529 Å. The physicists’ Bohr radius is named for Niels Bohr, a founder of the quantum theory and the 1922 recipient of the Nobel Prize for physics. Since Niels was the elder brother of Harald Bohr, adopting the term “Bohr radius” for mathematical purposes keeps the honor within the family.

In contrast to the situation in one dimension, there is no higher-dimensional bounded domain G whose Bohr radius $K(G)$ is known *exactly*. However, reasonably good bounds for the Bohr radius are known for special domains like the ball and the polydisc.

More generally, let B_p^n denote $\{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^p < 1\}$, which is the unit ball of the complex Banach space ℓ_p^n whose norm is defined by $\|z\|_{\ell_p^n} := (\sum_{j=1}^n |z_j|^p)^{1/p}$. The ball B_2^n corresponding to the case that $p = 2$ is the usual Euclidean unit ball in \mathbb{C}^n . The ball B_{∞}^n is to be interpreted as the unit polydisc in \mathbb{C}^n . The following theorem quantifies the rate at which the Bohr radius of B_p^n decays as the dimension n increases.

THEOREM 3. *When $n > 1$, the Bohr radius $K(B_p^n)$ of the ℓ_p^n unit ball in \mathbb{C}^n admits the following bounds.*

- If $1 \leq p \leq 2$, then

$$\frac{1}{3\sqrt[3]{e}} \cdot \left(\frac{1}{n}\right)^{1-\frac{1}{p}} \leq K(B_p^n) < 3 \cdot \left(\frac{\log n}{n}\right)^{1-\frac{1}{p}}.$$

- If $2 \leq p \leq \infty$, then

$$\frac{1}{3} \cdot \sqrt{\frac{1}{n}} \leq K(B_p^n) < 2 \cdot \sqrt{\frac{\log n}{n}}.$$

The theorem does not quite fix the sharp decay rate of the Bohr radius with the dimension, for the upper bounds contain a logarithmic factor not present in the lower bounds. This logarithmic factor, an artifact of the proof, presumably should not really be present.

The numerical values of the constants in the theorem can be improved. For example, the constants in the two parts of the theorem could be made to agree when $p = 2$. I have chosen to write simple constants for clarity,

since the main point of the theorem is the dependence of the Bohr radius on the dimension n .

The lower bound in the theorem is due to Lev Aizenberg ([1]) when $p = 1$. The lower bound when $2 \leq p \leq \infty$ and the upper bound when $p = \infty$ are due jointly to Dmitry Khavinson and myself ([3]). The other parts of the theorem are new.

5. Proof of the estimates for the Bohr radius

5.1. The lower bound

Lower bounds on the Bohr radius follow from upper bounds on the Maclaurin series coefficients of a bounded holomorphic function. In the writings of Bohr ([4]) and Landau ([8]), one finds a useful trick employed by F. Wiener in the context of coefficient bounds for the one-dimensional Bohr theorem.

LEMMA. ([after F. Wiener]) *Let G be a complete Reinhardt domain, and let \mathcal{F} be the set of holomorphic functions on G with modulus bounded by 1. Fix a multi-index α other than $(0, \dots, 0)$, and suppose that the positive real number b is an upper bound for the modulus of the derivative $f^{(\alpha)}(0)$ for every function f in \mathcal{F} . Then $|f^{(\alpha)}(0)| \leq (1 - |f(0)|^2)b$ for every function f in \mathcal{F} .*

Proof. Pick an index j for which $\alpha_j \neq 0$, and let ω_j denote a primitive α_j th root of unity. Consider the function obtained by averaging f :

$$(z_1, \dots, z_n) \mapsto \frac{1}{\alpha_j} \sum_{k=1}^{\alpha_j} f(z_1, \dots, z_{j-1}, \omega_j^k z_j, z_{j+1}, \dots, z_n).$$

Iterate this averaging procedure for each non-zero component of the multi-index α . The resulting function h is still in the class \mathcal{F} , and its Maclaurin series starts out $f(0) + f^{(\alpha)}(0)z^\alpha/\alpha! + \dots$, where $\alpha!$ denotes the product $\alpha_1! \cdots \alpha_n!$. Composing h with a linear fractional transformation of the unit disk that maps $f(0)$ to 0 gives a new function in the class \mathcal{F} whose Maclaurin series begins $f^{(\alpha)}(0)z^\alpha/(1 - |f(0)|^2)\alpha! + \dots$, and the conclusion of the lemma follows. \square

To prove the lower bound in Theorem 3 when $1 \leq p \leq 2$, I will follow the argument used by Aizenberg ([1]) for the case when $p = 1$. The

method actually yields a bound when $1 \leq p \leq \infty$, but the bound is interesting only when $1 \leq p \leq 2$.

Let $|\alpha|$ denote the sum $\alpha_1 + \cdots + \alpha_n$, and let α^α denote the product $\alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}$ (where 0^0 is interpreted as 1). A straightforward calculation using Lagrange multipliers shows that

$$(1) \quad \sup\{|z^\alpha| : z \in B_p^n\} = \left(\frac{\alpha^\alpha}{|\alpha|^{|\alpha|}}\right)^{1/p}.$$

By Cauchy's estimates, it follows that if $|\sum_\alpha c_\alpha z^\alpha| \leq 1$ when $z \in B_p^n$, then $|c_\alpha| \leq (|\alpha|^{|\alpha|}/\alpha^\alpha)^{1/p} \leq |\alpha|^{|\alpha|}/\alpha^\alpha$. Wiener's lemma improves this estimate by a factor of $(1 - |c_0|^2)$. Hence, we have when $k > 0$ that

$$\sum_{|\alpha|=k} |c_\alpha z^\alpha| \leq (1 - |c_0|^2) \sum_{|\alpha|=k} \frac{k^k}{\alpha^\alpha} |z^\alpha|.$$

Multiplying and dividing by the multinomial coefficient $\binom{k}{\alpha}$, which equals $k!/|\alpha|!$, observing that $\sum_{|\alpha|=k} \binom{k}{\alpha} |z^\alpha| = \|z\|_{\ell_1^n}^k$, and applying Hölder's inequality to bound $\|z\|_{\ell_1^n}$ above by $n^{1-\frac{1}{p}} \cdot \|z\|_{\ell_p^n}$, we deduce that

$$\sum_\alpha |c_\alpha z^\alpha| \leq |c_0| + (1 - |c_0|^2) \sum_{k=1}^{\infty} \frac{k^k}{k!} \left(n^{1-\frac{1}{p}} \cdot \|z\|_{\ell_p^n}\right)^k.$$

Because the real quadratic function $t \mapsto t + (1 - t^2)/2$ never exceeds 1, it follows that if x is the unique positive number such that

$$(2) \quad \sum_{k=1}^{\infty} \frac{k^k}{k!} x^k = \frac{1}{2},$$

then the Bohr radius $K(B_p^n)$ is at least as big as $x/n^{1-\frac{1}{p}}$.

In combinatorics, one encounters the tree function T (see, for example, [7, p. 395]), which satisfies the functional equation $T(x)e^{-T(x)} = x$ and has the series expansion $T(x) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} x^k$. The equation (2) says that $xT'(x) = 1/2$. Moreover, the functional equation implies that $xT'(x) = T(x)/(1 - T(x))$, so (2) yields $T(x)/(1 - T(x)) = 1/2$, or $T(x) = 1/3$. The functional equation gives the solution $x = 1/(3\sqrt[3]{e})$. One can also read off the solution of (2) from [11, p. 707], a source brought to my attention by Lev Aizenberg. This completes the proof of the lower bound in Theorem 3 when $1 \leq p \leq 2$.

A different argument shows that a better lower bound for the Bohr radius is available when $2 < p$. I reproduce the proof from [3].

If $|\sum_{\alpha} c_{\alpha} z^{\alpha}| \leq 1$ when $z \in B_p^n$, then applying the one-dimensional version of Wiener's lemma to the one-variable function $\zeta \mapsto \sum_{\alpha} c_{\alpha} \zeta^{|\alpha|} z^{\alpha}$ shows that $|\sum_{|\alpha|=k} c_{\alpha} z^{\alpha}| \leq 1 - |c_0|^2$ when $k \geq 1$. Integrating the square of the left-hand side of this inequality over a torus and using the orthogonality of the monomials z^{α} shows that $(\sum_{|\alpha|=k} |c_{\alpha}|^2 |w^{\alpha}|^2)^{1/2} \leq 1 - |c_0|^2$ for every point w in the unit ball of ℓ_p^n . Now if z lies in the ℓ_p^n ball of radius $1/(3\sqrt{n})$, then applying the Cauchy-Schwarz inequality with w equal to $3\sqrt{n}z$ shows that

$$\begin{aligned} \sum_{|\alpha|=k} |c_{\alpha} z^{\alpha}| &\leq \left(\sum_{|\alpha|=k} |c_{\alpha}|^2 |w^{\alpha}|^2 \right)^{1/2} \left(\sum_{|\alpha|=k} \left(\frac{1}{3\sqrt{n}} \right)^{2k} \right)^{1/2} \\ &\leq (1 - |c_0|^2)/3^k, \end{aligned}$$

since $\sum_{|\alpha|=k} 1 \leq n^k$. Consequently, we have for such points z that

$$\sum_{\alpha} |c_{\alpha} z^{\alpha}| \leq |c_0| + (1 - |c_0|^2) \sum_{k=1}^{\infty} 3^{-k} = |c_0| + (1 - |c_0|^2)/2 \leq 1.$$

This proves the lower bound in Theorem 3 when $2 \leq p \leq \infty$.

5.2. The upper bound

The case of the polydisc, which corresponds to p equal to ∞ , is considered in [3]. The idea is to use probabilistic methods to construct a homogeneous polynomial having relatively small supremum but relatively large majorant function. The argument in [3] applies a result from [6] on random trigonometric polynomials.

To handle the case of general p , I will use a related random technique from [9]. Unfortunately, the estimate I require is not stated explicitly in that paper: for their purposes, the authors did not need to keep track of the constants in the proof, and besides their argument has to be modified for the case of random tensors that are symmetric. Consequently, I repeat here the argument from [9] with appropriate modifications and amplifications.

Although the goal is to show the existence of a homogeneous polynomial of degree d in n variables that satisfies suitable estimates, technical considerations suggest proving a more general result for symmetric multi-linear functions mapping $(\mathbb{C}^n)^d$ into \mathbb{C} .

THEOREM 4. *If $1 \leq p \leq \infty$, and if n and d are integers larger than 1, then there exists a symmetric multi-linear function $F : (\mathbb{C}^n)^d \rightarrow \mathbb{C}$ of the form*

$$F(Z_1, \dots, Z_d) = \sum_{J_1=1}^n \cdots \sum_{J_d=1}^n \pm Z_{1J_1} \cdots Z_{dJ_d}$$

such that the supremum of $|F(Z_1, \dots, Z_d)|$ when every n -vector Z_k lies in the unit ball of ℓ_p^n is at most

$$(3) \quad \sqrt{32d \log(6d)} \times \begin{cases} n^{\frac{1}{2}}(d!)^{1-\frac{1}{p}}, & \text{if } 1 \leq p \leq 2; \\ n^{\frac{1}{2}+(\frac{1}{2}-\frac{1}{p})d}(d!)^{\frac{1}{2}}, & \text{if } 2 \leq p \leq \infty. \end{cases}$$

In the theorem, the plus and minus signs are chosen independently for each set of indices J_1, \dots, J_d , with the proviso that the same choice is made for every permutation of a given set of indices. In other words, F is the sum of the n^d monomials of degree d that can be formed from the components of the vectors Z_1, \dots, Z_d , with plus and minus signs chosen to enforce symmetry under permutations of the vectors.

When all the vectors Z_k are equal, the theorem provides a special homogeneous polynomial. Observe that the number of d -tuples J_1, \dots, J_d that are permutations of a given list of indices is typically less than $d!$ because some indices in the list will be repeated. Indeed, if α_k denotes the number of times the integer k appears in the list J_1, \dots, J_d , then the number of d -tuples that are permutations of the list is the multinomial coefficient $\binom{d}{\alpha}$. Consequently, we have the following corollary of the theorem.

COROLLARY. *If $1 \leq p \leq \infty$, and if n and d are integers larger than 1, then there exists a homogeneous polynomial of degree d in the variable z in \mathbb{C}^n of the form*

$$(4) \quad \sum_{|\alpha|=d} \pm \binom{d}{\alpha} z^\alpha$$

such that the supremum of the modulus of the polynomial when z lies in the unit ball of ℓ_p^n is no greater than the above bound (3).

To prove the upper bound in Theorem 3, consider the homogeneous polynomial of the corollary when z is the vector $(1, \dots, 1)$ scaled by the factor $K(B_p^n)/n^{1/p}$. To write formulas that work for both ranges of p

simultaneously, I will write $m(p)$ for $\min(p, 2)$ and $M(p)$ for $\max(p, 2)$. The definition of the Bohr radius implies that

$$(5) \quad \sum_{|\alpha|=d} \binom{d}{\alpha} (K(B_p^n)/n^{1/p})^d \leq n^{\frac{1}{2}+(\frac{1}{2}-\frac{1}{M(p)})d} (d!)^{1-\frac{1}{m(p)}} \sqrt{32d \log(6d)}.$$

Since $\sum_{|\alpha|=d} \binom{d}{\alpha} = n^d$, it follows that

$$(6) \quad K(B_p^n) \leq \left(\frac{(d!)^{1/d}}{n} \right)^{1-\frac{1}{m(p)}} (32nd \log(6d))^{\frac{1}{2d}}.$$

Stirling's formula implies that if $d \approx \log n$, then the right-hand side of (6) yields the upper bound in Theorem 3 for sufficiently large n . Numerical calculations show that if $d = 2 + \lfloor \log n \rfloor$, then n greater than 148 is sufficiently large. On the other hand, the upper bound in Theorem 3 holds automatically for smaller values of the dimension n , because the multi-dimensional Bohr radius does not exceed the one-dimensional Bohr radius of $1/3$, while $2\sqrt{\log n}/\sqrt{n} > 1/3$ when $1 < n < 189$. Thus the upper bound in Theorem 3 is a consequence of the corollary of Theorem 4.

Proof of Theorem 4. The proof consists of a probabilistic estimate and a covering argument. Three parameters are fixed throughout the proof: the dimension n , the degree d , and the exponent p .

I will use the following notation: Z denotes a d -tuple Z_1, \dots, Z_d of n -vectors, J denotes a d -tuple J_1, \dots, J_d of integers between 1 and n , sums and products run over all such d -tuples of integers, a prime on a sum or product means that it is restricted to d -tuples of integers that are arranged in non-decreasing order, and $J \sim K$ indicates that the d -tuples J and K are permutations of each other. The symbol Z_J is shorthand for the monomial $Z_{1J_1} \cdots Z_{dJ_d}$. Thus, the function in the statement of Theorem 4 can be written in the form

$$F(Z) = \sum_K' \left(\pm \sum_{J \sim K} Z_J \right),$$

and in this expression, all of the plus and minus signs are independent of each other.

To begin the probabilistic argument, fix a point Z in $(\mathbb{C}^n)^d$ such that every Z_k lies in the ℓ_p^n unit ball B_p^n . For each d -tuple K in non-decreasing order, choose a different Rademacher function r_K . (Recall that the

Rademacher functions are independent functions each taking the values $+1$ and -1 with probability $1/2$.) Consider the random sum

$$(7) \quad F(t, Z) := \sum_K' \left(r_K(t) \sum_{J \sim K} Z_J \right),$$

where t lies in the interval $[0, 1]$. The immediate goal is to make an upper estimate on the probability that this sum has large modulus.

Let λ be an arbitrary positive real number; in (13) below, I will specify a value for λ in terms of n , d , and p . Invoking the independence of the Rademacher functions, we can compute the expectation (that is, the integral with respect to t) of the exponential of the real part of $\lambda F(t, Z)$ by computing the product over non-decreasing d -tuples K of the expectations of $\exp(\lambda r_K(t) \sum_{J \sim K} \operatorname{Re} Z_J)$: namely,

$$(8) \quad \prod_K' \cosh \left(\lambda \sum_{J \sim K} \operatorname{Re} Z_J \right).$$

In view of the inequality $\cosh x \leq \exp(x^2/2)$, this expectation is bounded above by

$$(9) \quad \exp \left(\frac{1}{2} \lambda^2 \sum_K' \left(\sum_{J \sim K} \operatorname{Re} Z_J \right)^2 \right).$$

I will again write $m(p)$ for $\min(p, 2)$ and $M(p)$ for $\max(p, 2)$ in order to analyze the cases $1 \leq p \leq 2$ and $2 \leq p \leq \infty$ simultaneously. Hölder's inequality implies that

$$\left(\sum_{J \sim K} \operatorname{Re} Z_J \right)^2 \leq (d!)^{2(1-1/m(p))} \left(\sum_{J \sim K} |Z_J|^{m(p)} \right)^{2/m(p)}.$$

The exponent $2/m(p)$ is equal to 1 when $2 \leq p \leq \infty$, and replacing it by 1 when $1 \leq p < 2$ can only increase the right-hand side when each n -vector Z_k lies in B_p^n . Consequently, (9) is bounded above by

$$(10) \quad \exp \left(\frac{1}{2} \lambda^2 (d!)^{2(1-1/m(p))} \|Z_1\|_{\ell_{m(p)}^n}^{m(p)} \cdots \|Z_d\|_{\ell_{m(p)}^n}^{m(p)} \right).$$

Since $Z_k \in B_p^n$, Hölder's inequality implies that $\|Z_k\|_{\ell_{m(p)}^n}^{m(p)}$ is bounded above by $n^{1-\frac{2}{M(p)}}$. Therefore (10) is bounded above by

$$(11) \quad \exp \left(\frac{1}{2} \lambda^2 (d!)^{2(1-1/m(p))} n^{(1-2/M(p))d} \right).$$

Let R be an arbitrary positive real number; in (13) below, I will specify a value for R in terms of n , d , and p . By Chebyshev's inequality, the upper bound (11) on the expectation of $\exp(\lambda \operatorname{Re} F(t, Z))$ implies that the measure of the set of points t for which $\operatorname{Re} F(t, Z)$ exceeds R is at most

$$(12) \quad \exp\left(-R\lambda + \frac{1}{2}\lambda^2(d!)^{2(1-1/m(p))}n^{(1-2/M(p))d}\right).$$

Symmetric reasoning gives the same estimate for the probability that $\operatorname{Re} F(t, Z)$ is less than $-R$, so the probability that $|\operatorname{Re} F(t, Z)|$ exceeds R is at most 2 times (12). The same argument applies to the imaginary part of $F(t, Z)$. Consequently, the probability that $|F(t, Z)|$ exceeds $\sqrt{2}R$ is at most 4 times (12).

This probabilistic estimate holds for an arbitrary but fixed Z . The second part of the proof is a covering argument to produce an estimate that is uniform in Z . The following lemma is well known (see [10, p. 7], for example). The exponent differs from the statement in [9] because I am considering the complex ℓ_p^n space.

LEMMA. *If ϵ is a positive real number, then the unit ball of the complex space ℓ_p^n can be covered by a collection of open ℓ_p^n balls of radius ϵ , the number of balls in the collection not exceeding $(1 + 2\epsilon^{-1})^{2n}$ and the centers of the balls lying in the closed unit ball of ℓ_p^n .*

Proof. Place arbitrarily in the open ℓ_p^n ball of radius $(1 + \frac{\epsilon}{2})$ a collection of disjoint open ℓ_p^n balls of radius $\epsilon/2$. Since the Euclidean volume of a ball in ℓ_p^n scales like the $(2n)$ th power of its radius, an obvious volume comparison shows that the number of disjoint balls cannot exceed the $(2n)$ th power of the ratio of radii $(1 + \frac{\epsilon}{2})/(\epsilon/2)$: namely, $(1 + 2\epsilon^{-1})^{2n}$.

If the collection of disjoint balls is made maximal, then every point of the closed ℓ_p^n unit ball must lie within $\epsilon/2$ of some point of one of the balls in the collection. Consequently, the balls with the same centers but with radius ϵ must cover the closed unit ball. \square

Next we need a simple Lipschitz estimate for $F(t, Z)$. Suppose that Z and W are points of $(\mathbb{C}^n)^d$ such that all of the component n -vectors Z_1, \dots, Z_d and W_1, \dots, W_d lie in B_p^n , and $\|Z_k - W_k\|_{\ell_p^n} \leq \epsilon$ for every k .

The multi-linearity of F implies that

$$\begin{aligned} F(t, Z_1, \dots, Z_d) &= F(t, Z_1 - W_1, Z_2, \dots, Z_d) \\ &\quad + F(t, W_1, Z_2 - W_2, Z_3, \dots, Z_d) \\ &\quad + F(t, W_1, W_2, Z_3 - W_3, Z_4, \dots, Z_d) + \dots \\ &\quad + F(t, W_1, W_2, \dots, W_{d-1}, Z_d - W_d) + F(t, W_1, \dots, W_d). \end{aligned}$$

Consequently, $|F(t, Z) - F(t, W)|$ is at most d times ϵ times the supremum of $|F(t, \cdot)|$ over $(B_p^n)^d$. Taking ϵ to be $1/(2d)$, we see by the lemma that there is a collection of at most $(1 + 4d)^{2nd}$ points of $(B_p^n)^d$ such that the supremum of $|F(t, \cdot)|$ over $(B_p^n)^d$ is no more than twice the supremum of $|F(t, \cdot)|$ over this finite collection of points.

Applying the preceding probabilistic estimate to each member of the finite collection of points shows that the probability that the supremum of $|F(t, \cdot)|$ over $(B_p^n)^d$ exceeds $2\sqrt{2}R$ is at most

$$4(1 + 4d)^{2nd} \exp\left(-R\lambda + \frac{1}{2}\lambda^2(d!)^{2(1-1/m(p))}n^{(1-2/M(p))d}\right).$$

Now take the following values for the parameters R and λ :

$$(13) \quad \begin{aligned} R &:= \left(2(d!)^{2(1-1/m(p))}n^{(1-2/M(p))d} \log(8(1 + 4d)^{2nd})\right)^{1/2}, \\ \lambda &:= \frac{R}{(d!)^{2(1-1/m(p))}n^{(1-2/M(p))d}}. \end{aligned}$$

With these choices, we find that the probability that the supremum of $|F(t, \cdot)|$ over $(B_p^n)^d$ exceeds $2\sqrt{2}R$ is at most $1/2$, so we are sure that there exists a particular value of t such that the supremum of $|F(t, \cdot)|$ over $(B_p^n)^d$ is no more than

$$(14) \quad \left(16(d!)^{2(1-1/m(p))}n^{(1-2/M(p))d} \log(8(1 + 4d)^{2nd})\right)^{1/2}.$$

The values of the Rademacher functions at this particular value of t produce the pattern of plus and minus signs indicated in the statement of Theorem 4. Moreover, $8(1 + 4d)^{2nd} < (6d)^{2nd}$ when n and d are both at least 2, so the upper bound (14) is even smaller than the bound stated in Theorem 4. \square

6. Another analogue of Bohr's theorem

In [1], Aizenberg introduced a second Bohr radius $B(G)$: namely, the largest r such that whenever $|\sum_{\alpha} c_{\alpha} z^{\alpha}| \leq 1$ for $z \in G$, it follows that $\sum_{\alpha} \sup_{z \in rG} |c_{\alpha} z^{\alpha}| \leq 1$. Clearly $B(G) \leq K(G)$, because the supremum of a sum is no bigger than the sum of the suprema. Equality holds for polydiscs, because points on the torus have the property that they maximize $|z^{\alpha}|$ for all indices α simultaneously.

The following statement is an analogue of Theorem 3 for the second Bohr radius.

THEOREM 5. *When $n > 1$, the second Bohr radius $B(B_p^n)$ of the ℓ_p^n unit ball in \mathbb{C}^n admits the following bounds.*

- If $1 \leq p \leq 2$, then

$$\frac{1}{3} \cdot \frac{1}{n} < 1 - \left(\frac{2}{3}\right)^{\frac{1}{n}} \leq B(B_p^n) < 4 \cdot \frac{\log n}{n}.$$

- If $2 \leq p \leq \infty$, then

$$\frac{1}{3} \cdot \left(\frac{1}{n}\right)^{\frac{1}{2} + \frac{1}{p}} \leq B(B_p^n) < 4 \cdot \left(\frac{\log n}{n}\right)^{\frac{1}{2} + \frac{1}{p}}.$$

Proof. The lower bound is due to Aizenberg ([1]) when $1 \leq p \leq 2$. Namely, Cauchy's estimates imply that if $|\sum_{\alpha} c_{\alpha} z^{\alpha}| \leq 1$ in B_p^n , then $|c_{\alpha}| \leq 1/\sup\{|z^{\alpha}| : z \in B_p^n\}$, and Wiener's lemma provides an extra factor of $1 - |c_0|^2$. Therefore

$$\sum_{\alpha} \sup\{|c_{\alpha} z^{\alpha}| : \|z\|_{\ell_p^n} \leq r\} \leq |c_0| + (1 - |c_0|^2) \sum_{\alpha \neq 0} r^{|\alpha|},$$

and this quantity will be bounded above by 1 if $\sum_{\alpha \neq 0} r^{|\alpha|} \leq 1/2$. Since $\sum_{\alpha \neq 0} r^{|\alpha|} = -1 + 1/(1-r)^n$, we deduce that the second Bohr radius is no smaller than $1 - (2/3)^{1/n}$. It is easy to see that this quantity exceeds $1/(3n)$ when $n > 1$, because the function $n \mapsto n \cdot (1 - (2/3)^{1/n})$ is increasing for positive n and takes the value $1/3$ when $n = 1$.

When $p = \infty$, the lower bound is contained in Theorem 3, since the two Bohr radii are the same for the polydisc. When $2 < p < \infty$, the lower bound follows by observing that an ℓ_{∞}^n ball of radius $1/n^{1/p}$ nests inside B_p^n , and so $B(B_p^n) \geq B(B_{\infty}^n)/n^{1/p}$.

To prove the upper bound in Theorem 5, apply the definition of the second Bohr radius to the homogeneous polynomial of the corollary of Theorem 4. From (1), it follows that

$$\sum_{|\alpha|=d} \binom{d}{\alpha} \left(\frac{\alpha^\alpha}{d^d}\right)^{\frac{1}{p}} B(B_p^n)^d \leq n^{\frac{1}{2} + (\frac{1}{2} - \frac{1}{M(p)})d} (d!)^{1 - \frac{1}{m(p)}} \sqrt{32d \log(6d)}.$$

Since $\alpha^\alpha \geq 1$, and $\sum_{|\alpha|=d} \binom{d}{\alpha} = n^d$, we deduce that

$$B(B_p^n) \leq \frac{1}{n^{\frac{1}{2} + \frac{1}{M(p)}}} d^{\frac{1}{p}} (d!)^{\frac{1}{d}(1 - \frac{1}{m(p)})} (32nd \log(6d))^{\frac{1}{2d}}.$$

When $d = \lfloor \log n \rfloor$, the right-hand side asymptotically gives the desired power of $(\log n)/n$, and numerical calculations show that the right-hand side is smaller than the upper bound stated in Theorem 5 when $n > 28$. On the other hand, the upper bound in the theorem holds automatically for smaller values of n , because $(4 \log n)/n > 1/3$ when $1 < n < 46$. \square

7. Directions for future research

My goal in this article has been not only to demonstrate some interesting results and techniques in the theory of multi-variable power series, but also to suggest some avenues for further investigation. Here are some concrete problems.

1. Construct examples to show that the logarithmic term is not needed in the upper bound in Theorems 3 and 5.
2. Find the *exact* value of the Bohr radius for the polydisc in dimension 2.
3. Estimate the Bohr radius for domains of the form

$$|z_1|^{p_1} + \cdots + |z_n|^{p_n} < 1.$$

4. Generalize Wintner's problem to higher dimensions.

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