

3.4 - INTRODUCTION TO THE DERIVATIVE

I. The **Average Rate of Change** from $x = a$ to $x = a + h$ is

$$\frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}, \quad h \neq 0 .$$

This equation is called the **difference quotient**.

Example: The revenue from the sale of x plastic planter boxes is given by $R(x) = 20x - 0.02x^2$, $0 \leq x \leq 1,000$. What is the average change in revenue if production is changed from 100 planters to 400 planters?

The **Instantaneous Rate of Change** AT $x = a$ is

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}, \quad \text{if the limit exists .}$$

II. **Slope of the Tangent Line** (also called the slope of the graph)

A GEOMETRIC INTERPRETATION

Recall from geometry that a tangent line to a circle is a line that passes through one and only one point on the circle. But for functions in general, this is not a satisfactory definition.

To define a tangent line for f at a point P :

1. A point P is given on f
2. Pick a point Q on f
3. Draw a line through PQ
(this is the secant line)
4. Let $Q \rightarrow P$
5. $\lim_{Q \rightarrow P} \left(\begin{array}{l} \text{slopes of the} \\ \text{secant lines} \end{array} \right) = m_{\text{tan}}$

(m = slope of the tangent line at P)

Knowing the slope of the tangent line, and the coordinates of the point P , enables us to use the point-slope form of a line to write the equation of the tangent line as follows:

If P has coordinates $(x_1, f(x_1))$, then Q has coordinates $(x_1 + h, f(x_1 + h))$ since Q is some distance h from P . Then

$$m_{\overline{PQ}} = \frac{\Delta y}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{(x_1 + h) - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

Now, let the distance from Q to $P \rightarrow 0$. In other words, let $h \rightarrow 0$.

THUS,

$$\lim_{h \rightarrow 0} \frac{f(x_1 + h) - f(x_1)}{h} = m_{\text{tan}} \quad (\text{the slope of the tangent line to } f \text{ at } P)$$

(also called the slope of the graph of f at P , the instantaneous rate of change, velocity and the DERIVATIVE).

III. The Derivative Function

For $y = f(x)$, we define the derivative of f at x , denoted by $f'(x)$, to be

$$f'(x) = \text{rate of change of } f \text{ at } x = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}, \text{ if the limit exists,}$$

$$(\text{Alternate notations: } f'(x) \approx f' \approx y' \approx \frac{dy}{dx} \approx D_x).$$

The notation $\frac{dy}{dx}$ reminds us that the derivative is a rate of change.

The derivative of a function f' is a new function whose domain is a subset of the domain of f .

Example: Find the equation of the tangent line to $f(x) = 8x - 2x^2$ at $x = 1$.

Example:

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The derivative of a function at a point ($x = a$) tells you the rate of change at which the value of the function is changing at that point. We say that f is differentiable at $x = a$. However, if $f'(x)$ exists for each x in the open interval (a, b) , then f is said to be differentiable over the interval (a, b) .

But, when does the derivative NOT exist?

III. Nonexistence of the Derivative - If the limit d.n.e. at $x = a$, then f is nondifferentiable at $x = a$, or $f'(a)$ d.n.e.

1. If the graph of f has a sharp corner at $x = a$, then $f'(a)$ d.n.e. and has no tangent line at $x=a$.
2. If the graph of f has a vertical tangent line at $x=a$, then $f'(a)$ d.n.e. since slope is undefined.
3. If the graph of f is broken at $x = a$ (not continuous at $x = a$), then $f'(a)$ d.n.e.

NOTE : If f is differentiable, then f is continuous. But f continuous DOES NOT IMPLY f differentiable!