The Factorization Method for Imaging Defects in Anisotropic Materials

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The Physical Model
We consider the time harmonic Acoustic scattering in $\mathbb{R}^3$ or Electromagnetic scattering in $\mathbb{R}^2$ (scalar) TE-polarization case.

Inverse Problem
Reconstruct the support of the defective region $D_0$ in a known Anisotropic Material $D$ from the measured scattered field.
Reconstruction Methods

1. **Optimization Methods**: Solve nonlinear model using iterative scheme to reconstruct the material parameters, which requires a priori information, can be computationally expensive, and lacks uniqueness for matrix coefficients.

2. **Qualitative Methods**: Using nonlinear model to reconstruct limited information with no a priori information, such as the support of the region in a computationally simple manner.
The Scattered Field $u^s_b$

We consider the scattering by a healthy material where we find the scattered field $u^s_b \in H^1_{loc}(\mathbb{R}^m)$ with $u^i = \exp(ikx \cdot d)$.

$$\nabla \cdot \tilde{A} \nabla u^s_b + k^2 \tilde{n} u^s_b = \nabla \cdot (I - \tilde{A}) \nabla u^i + k^2 (1 - \tilde{n}) u^i + \text{SRC} \quad \text{in } \mathbb{R}^m$$

$D \subset \mathbb{R}^m$ be a bounded simply connected open region

1. $\tilde{A} := A\chi_D + I(1 - \chi_D)$ and $\tilde{n} := n\chi_D + 1(1 - \chi_D)$

2. The matrix $A \in C^1(D, \mathbb{R}^{m \times m})$ is symmetric-positive definite

3. The function $n \in L^\infty(D)$ such that $n(x) > 0$ for a.e. $x \in D$. 
The Scattered Field $u^s_0$

We consider the scattering by a defective material where we find the scattered field $u^s_0 \in H^1_{\text{loc}}(\mathbb{R}^m)$ with $u^i = \exp(ikx \cdot d)$.

$$\nabla \cdot \tilde{A}_0 \nabla u^s_0 + k^2 \tilde{n}_0 u^s_0 = \nabla \cdot (I - \tilde{A}_0) \nabla u^i + k^2 (1 - \tilde{n}_0) u^i + \text{SRC \ in \ } \mathbb{R}^m$$

Now $D_0 \subset D$ be possibly multiple connected open set

1. $\tilde{A}_0 := A_0 \chi_{D_0} + \tilde{A}(1 - \chi_{D_0})$ and $\tilde{n}_0 := n_0 \chi_{D_0} + \tilde{n}(1 - \chi_{D_0})$

2. The matrix $A_0 \in C^1(D_0, \mathbb{C}^{m \times m})$ is symmetric

3. The function (possibly complex valued) $n_0 \in L^\infty(D_0)$. 

Far-Field Patterns

It is known that the radiating scattered fields which depends on the direction \( d \), has the following asymptotic expansion

\[
u_s^b(x, d) = \frac{e^{ik|x|}}{|x|^{m-1}} \left\{ u_{\infty}^b(\hat{x}, d) + O\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \to \infty
\]

\[
u_s^0(x, d) = \frac{e^{ik|x|}}{|x|^{m-1}} \left\{ u_{\infty}^0(\hat{x}, d) + O\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \to \infty
\]

where the observation direction \( \hat{x} \) and incident direction \( d \) are in the the unit circle or sphere \( \mathbb{S} = \{ x \in \mathbb{R}^m : |x| = 1 \} \).
Some Related Work

O. Bondarenko, A. Kirsch and X. Liu

Y. Grisel, V. Mouysset, P.A. Mazet and J.P. Raymond

A. Kirsch and N. Grinberg
The Factorization Method for Inverse Problems.
*Oxford University Press, Oxford 2008.*
The Relative Far-Field Operator

\[(Fg)(\hat{x}) := \int_{S} [u_{0}^{\infty}(\hat{x}, d) - u_{b}^{\infty}(\hat{x}, d)] g(d) ds(d)\]

It can be shown that the operator \( F \) is the far-field operator corresponding to only the defective region.

The Far-Field Data

1. \( u_{0}^{\infty}(\hat{x}, d) \) - is known from physical measurements
2. \( u_{b}^{\infty}(\hat{x}, d) \) - can be computed directly since we assume \( A(x) \) and \( n(x) \) along with \( D \) are known
The Scattering Operator

Just as in the two other applications of the factorization method for imaging inside of an object we see that the Scattering operator for the ‘healthy’ material will be used in the factorization.

The Operator $S$

$S : L^2(\mathbb{S}) \mapsto L^2(\mathbb{S})$ for the healthy background where the constant $\gamma$, is given by $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ in $\mathbb{R}^2$ and $\gamma = \frac{1}{4\pi}$ in $\mathbb{R}^3$

$S = I + 2ik\gamma F_b$ where $(F_b g)(\hat{x}) := \int_{\mathbb{S}} u_b^\infty(\hat{x}, d) g(d) \, ds(d)$.

Since $A$ and $n$ are real valued, the scattering operator is unitary, i.e. $SS^* = S^*S = I$. 
Modified Herglotz Operator

The radiating Green’s function of the background media,

\[ \nabla \cdot \tilde{\mathbf{A}} \nabla \mathbf{G}(\cdot, z) + k^2 \tilde{n} \mathbf{G}(\cdot, z) = -\delta(\cdot - z) \text{ in } \mathbb{R}^m \setminus \{z\} + \text{SRC} \]

Define the operator \( H : L^2(\mathbb{S}) \hookrightarrow H^1(D_0) \), by

\[ Hg := \int_{\mathbb{S}} [\mathbf{u}_b^S(\mathbf{x}, d) + \exp(ik\mathbf{x} \cdot d)] g(d) \, ds(d). \]

Theorem (F. Cakoni and I. Harris)

The operator \( H^* : H^1(D_0) \hookrightarrow L^2(\mathbb{S}) \) satisfies the following:

1. \( H^* \) is compact with dense range.
2. \( S^* \mathbf{G}^\infty(\cdot, z) \in \text{Range}(H^*) \) if and only if \( z \in D_0 \).
A Factorization Emerges

Given \( v \in H^1(D_0) \) we can construct \( u \in H^1_{loc}(\mathbb{R}^m) \) that satisfies

\[
\nabla \cdot \tilde{A} \nabla u + k^2 \tilde{n} u = \nabla \cdot (\tilde{A} - \tilde{A}_0) \nabla v + k^2 (\tilde{n} - \tilde{n}_0) v + \text{SRC in } \mathbb{R}^m
\]

Define the operator \( T : H^1(D_0) \leftrightarrow H^1(D_0) \) such that

\[
(Tv, \varphi)_{H^1(D_0)} = - \int_{D_0} (A - A_0) \nabla (v + u) \cdot \nabla \varphi - k^2 (n - n_0) (v + u) \varphi \, dx.
\]

Theorem (F. Cakoni and I. Harris)
The far field operator \( F : L^2(\mathbb{S}) \leftrightarrow L^2(\mathbb{S}) \) associated with the defect can be factorized as \( S^* F = - \gamma H^* TH \).
Analysis of the Middle Operator

We define the real and imaginary part of an operators by

\[ \Re(T) = \frac{T + T^*}{2} \quad \text{and} \quad \Im(T) = \frac{T - T^*}{2i}. \]

For simplicity lets assume we have that \( A_0 \) and \( n_0 \) are real valued.

Theorem (F. Cakoni and I. Harris)

The operator \( T : H^1(D_0) \leftrightarrow H^1(D_0) \) satisfies the following:

1. \( \pm \Re(T) \) is the compact perturbation of a coercive operator
2. \( \Im(T) \) is non-positive and compact

Provided that \( A - A_0 \) is either positive or negative definite \( D_0 \).
Main Result (for Non-Absorbing Defects)

We define the selfadjoint compact operator

\[ \mathbf{F}_\# := |\Re(\gamma^{-1}S^* F)| + |\Im(\gamma^{-1}S^* F)| : L^2(\mathbb{S}) \rightarrow L^2(\mathbb{S}). \]

Theorem (F. Cakoni and I. Harris)

If \( A - A_0 \) is either positive or negative definite in \( D_0 \), and let \((\lambda_i, \psi_i) \in \mathbb{R}^+ \times L^2(\mathbb{S})\) be an orthonormal eigensystem of \( \mathbf{F}_\# \). Then

\[ z \in D_0 \iff W(z) = \left[ \sum_{i=1}^{\infty} \frac{|(S^* G_{\infty}(\hat{x}, z), \psi_i)|^2}{\lambda_i} \right] ^{-1} > 0. \]

For the results on absorbing defects see:

F. Cakoni and I. Harris

Numerical Reconstructions i.e. plot $z \mapsto \mathcal{W}(z)$

**Example 1.** We consider $D = [-2, 2]^2$ where the defective region is a void $D_0$ (i.e. $A_0 = I$ and $n_0 = 1$ in $D_0$) embedded in isotropic media. The coefficients in $D$ are given by $A = 0.5I$ and $n = 3$.

**Figure:** On the left is the reconstruction of the 2 circular voids. While on the right is the reconstruction of the a circular void of radius 1. Where the wavenumber is $k = 1$. 

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**Example 2.** For this example we now reconstruct voids in an anisotropic square scatterer $D = [-2, 2]^2$. The coefficients in $D$ are chosen to be given by $n = 3$ and

$$A = \begin{pmatrix} 0.6022 & 0.1591 \\ 0.1591 & 0.7478 \end{pmatrix}$$

**Figure:** On the left is the reconstruction of the 2 circular voids. While on the right is the reconstruction of the a circular void of radius 1. Where the wavenumber is $k = 1$. 
Figure: Questions?

"That's all folks!"