Inverse Scattering for Materials with a Conductive Boundary

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Joint work with: O. Bondarenko and A. Kleefeld

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The Direct Scattering Problem
We consider the time-harmonic scattering problem in $\mathbb{R}^3$ or $\mathbb{R}^2$

1. $D \subset \mathbb{R}^m$ be a bounded open region with $\partial D$-Smooth
2. The function $n \in L^\infty(D)$ is the refractive index
3. The function $\eta \in L^\infty(\partial D)$ is conductivity parameter
The Far Field Pattern

The scattered field, has the following asymptotic expansion

$$u^s(x, d; k) = \frac{e^{ik|x|}}{4\pi|x|} \left\{ u^\infty(\hat{x}, d; k) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\} \text{ as } |x| \to \infty.$$ 

Using Green’s Representation Theorem one can show that the far field pattern is given by

$$u^\infty(\hat{x}, d; k) = \int_C u^s(y, d; k)\partial_\nu e^{-ik\hat{x} \cdot y} - \partial_\nu u^s(y, d; k)e^{-ik\hat{x} \cdot y} \, ds(y)$$

where $\hat{x}, d \in S = \{x \in \mathbb{R}^3 : |x| = 1\}$. 
Reconstruction via the Far Field Operator

We now define the far field operator as \( F : L^2(\mathbb{S}) \mapsto L^2(\mathbb{S}) \)

\[(Fg)(\hat{x}) = \int_{\mathbb{S}} u^\infty(\hat{x}, d; k)g(d) \, ds(d)\]

The **inverse problem** reads: given the far field operator \( F \) for a range of wave numbers \( k \in [k_{\text{min}}, k_{\text{max}}] \) obtain information about the scatterer \( D \) and it’s material parameters \( n \) and \( \eta \).
Reconstruction Methods

1. **Optimization Methods:** Solve nonlinear model using iterative scheme to reconstruct the material parameters, which requires **a priori** information, and can be computationally expensive. For matrix valued coefficient the inverse problem lacks uniqueness.

2. **Qualitative Methods:** Using nonlinear model to reconstruct limited information with **no a priori** information, such as the support of the region in a computationally simple manner (i.e. solving a linear integral equations).
Reconstruction of $D$ via the LSM

Let $\Phi^\infty(\hat{x}, z) = e^{-ikz \cdot \hat{x}}$ be the far field pattern for the fundamental solution of the Helmholtz equation.

The Linear Sampling Method (LSM)

The ‘Far-Field equation’ in $\mathbb{R}^3$ is given by (fix the wave number $k$)

$$(Fg_z)(\hat{x}) = \Phi^\infty(\hat{x}, z) \quad \text{for a } z \in \mathbb{R}^3,$$

then the regularized solution of the far-field equation $g_{z,\delta}$ satisfies

- $\|g_{z,\delta}\|_{L^2(\mathbb{S})}$ is bounded as $\delta \to 0$ provided that $z \in D$,
- $\|g_{z,\delta}\|_{L^2(\mathbb{S})}$ is unbounded as $\delta \to 0$ provided that $z \not\in D$.

Assuming that $\|Fg_{z,\delta} - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} \to 0$ as $\delta \to 0$. 

Inverse Scattering for Conductive Boundary
Numerical Reconstruction i.e. plot $z \mapsto 1/\|g_{z,\delta}\|$.

F. Cakoni, David Colton and Peter Monk,
TE-Problem for a Material with a Conductive Boundary

O. Bondarenko, I. Harris, and A. Kleefeld


I. Harris and A. Kleefeld

Motivation

- The Transmission eigenvalue problem is a non-selfadjoint and non-linear eigenvalue problem
- The linear sampling method fails at transmission eigenvalue
- The Transmission eigenvalues can be used to determine/estimate the material properties.
Reconstructing the Real TEs

Theorem (I. Harris, and A. Kleefeld)
Recall the far-field equation for a fixed $z \in D$

$$ (F g_z)(\hat{x}) = \Phi^\infty(\hat{x}, z) \quad \text{for} \quad k \in [k_{\text{min}}, k_{\text{max}}]. $$

then the regularized solution of the far-field equation $g_{z,\delta}$ satisfies

- if $k$ is not a TE then $\|g_{z,\delta}\|_{L^2(\mathbb{S})}$ is bounded as $\delta \to 0$
- if $k$ is a TE then $\|g_{z,\delta}\|_{L^2(\mathbb{S})}$ is unbounded as $\delta \to 0$.

Assuming that $\|F g_{z,\delta} - \Phi^\infty(\cdot, z)\|_{L^2(\mathbb{S})} \to 0$ as $\delta \to 0$.

This result implies that Transmission eigenvalues can be determined without any prior knowledge of the coefficients!!!
Numerical Examples i.e. plot $k \mapsto \|g_{z,\delta}\|

Figure: Plot of the average $\|g_{z,\delta}\|_{L^2(S)}$ for 25 points $z \in D$

### I. Harris, F. Cakoni, and J. Sun

Values of $k \in \mathbb{C}$ where $\exists$ nontrivial $(w, v) \in L^2(D) \times L^2(D)$ such that $w - v \in H^2(D) \cap H^1_0(D)$ satisfies

$$\Delta w + k^2 nw = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in} \quad D \quad \text{in} \quad D$$

$$w - v = 0 \quad \text{and} \quad \partial_\nu w - \partial_\nu v = \eta v \quad \text{on} \quad \partial D.$$

Where we assume that $n > 0$ and $\eta > 0$. Equivalently there exists $u = w - v \in H^2(D) \cap H^1_0(D)$ satisfying

$$(\Delta + k^2) \frac{1}{n - 1} (\Delta + k^2 n) u = 0 \quad \text{in} \quad D$$

$$u = 0 \quad \text{and} \quad \partial_\nu u = -\frac{\eta}{k^2(n - 1)} (\Delta + k^2 n) u \quad \text{on} \quad \partial D.$$
The Variational Formulation

By Green’s 2nd Theorem

\[(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)u = 0 \quad \text{in} \quad D\]

\[u = 0 \quad \text{and} \quad \partial_\nu u = -\frac{\eta}{k^2(n-1)}(\Delta + k^2 n)u \quad \text{on} \quad \partial D\]

has the variational form

\[\int_{\partial D} \frac{k^2}{\eta} \partial_\nu u \partial_\nu \varphi \, ds + \int_D \frac{1}{n-1}(\Delta u + k^2 nu)(\Delta \varphi + k^2 \varphi) \, dx = 0,\]

for all \(\varphi \in H^2(D) \cap H^1_0(D)\).
Theorem (O. Bondarenko, I. Harris, and A. Kleefeld)
Assume that either
\[
\inf_{x \in D} n(x) > 1 \quad \text{or} \quad \sup_{x \in D} n(x) < 1 \quad \text{and} \quad \inf_{x \in \partial D} \eta(x) > 0,
\]
then there exists infinitely many real transmission eigenvalues. Moreover, the set of transmission eigenvalues is discrete with no finite accumulation points.

We appeal to the Analytic Fredholm Theorem for discreteness and to the results in the following paper for existence.

F. Cakoni and H. Haddar
Theorem (O. Bondarenko, I. Harris, and A. Kleefeld)

Assume that $k$ is a transmission eigenvalue

1. if $1 < n$, then $k$ is decreasing with respect to $n$ and $\eta$.
2. if $0 < n < 1$, then $k$ is increasing with respect to $n$ and $\eta$.

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<th>2.</th>
<th>3.</th>
<th>4.</th>
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Table: The first four interior transmission eigenvalues for a unit sphere using the index of refraction $n = 4$ and various choices of $\eta$. 
Numerical Results

The monotonicity of the first interior transmission eigenvalues for the peanut-shaped obstacle using $n = 4$ for increasing $\eta$. 
Theorem (I. Harris, and A. Kleefeld)

As \( \|\eta\|_\infty \to 0 \) the transmission eigenvalues/functions \( k_\eta \) and \( u_\eta \) have subsequences that converge to the classical transmission eigenvalues/functions for \( \eta = 0 \).

<table>
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<th>( \eta )</th>
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<td>1.006</td>
<td>( 5.647 \times 10^{-3} )</td>
<td>1.001</td>
</tr>
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</table>

**Table:** The estimated rate of convergence for the 2nd and 4th transmission eigenvalue for a unit sphere using \( n = 4 \) as \( \eta \to 0 \).
Theorem (I. Harris, and A. Kleefeld)

There exists infinitely many complex transmission eigenvalues for the unit sphere/circle provided that $n$ and $\eta \ll 1$ are constants and $\sqrt{n}$ is not an integer or a reciprocal of an integer (for the $\mathbb{R}^3$ case).

$$kJ_0(k\sqrt{n})J_1(k) - k\sqrt{n}J_0(k)J_1(k\sqrt{n}) - \eta J_0(k\sqrt{n})J_0(k) = 0$$
Some Answered Questions:

- Discreteness and Existence
- Monotonicity of the transmission eigenvalues
- Reconstruction from Scattering Data
- Existence of Complex eigenvalues
- Convergence as $\eta \to 0$

Some ‘Unanswered’ Questions:

- The inverse spectral problem of reconstructing $n$ and/or $\eta$
- Asymptotic expansion as $\eta \to 0$
Figure: Questions?

"That's all folks!"