Math311-J.Zhou. Review for Exam I.

Vector in $\mathbb{R}^n$: basic operations $rv, su+tv$, linear combination (LC) $u = c_1v_1 + ... + c_nv_n$, distance $|x-y| = \sqrt{(x_1-y_1)^2 + .. + (x_n-y_n)^2}$ and length $|v| = \sqrt{v_1^2 + ... + v_n^2}$, equations for lines $x = v_0 + tu$ and planes $x = v_0 + su_1 + tu_2$ or $ax + by + cz = ax_0 + by_0 + cz_0$ where $n = (a, b, c)$ is a normal vector of the plane and $v_0 = (x_0, y_0, z_0)$ is a point on the plane, dot product $x \cdot y = x_1y_1 + ... + x_ny_n$, angles $\theta = \cos^{-1}(\frac{x \cdot y}{|x||y|})$, orthogonality $x \perp y \iff x \cdot y = 0$, triangle (Cauchy-Schwarz) inequality $|x+y| \le |x| + |y|, |x \cdot y| \le |x||y|$, projection $P_u(x) = (x \cdot u)u$ where $|u| = 1$. Distance D from a point v to a plane $ax + by + cz = d$. Set $n = \frac{(a,b,c)}{|(a,b,c)|}$ and find a point v_0 on the plane, then $D = |(v - v_0) \cdot n|$.

Orthogonal decomposition: given unit u, any $x = x_u + x_\perp$ where $x_u = (x \cdot u)u, x_\perp = x - (x \cdot u)u \perp u$.

Cross Product: $u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \perp u$ and v. $|u \times v|$ =area of parallelogram by u, v.

Matrix Operations: $A = (a_{ij})_{m \times n}, B = (b_{ij})_{m \times n},\ sA + tB$.

Given $A_{m \times r}, B_{r \times m}$, then $AB = C_{m \times n} = (c_{ij})_{m \times n}$ with $c_{ij} = \sum_{k=1}^r a_{ik}b_{kj}$.

3 elementary row operations (3ERO): (1) $\alpha(i)$, (2) $(i) \leftrightarrow (j)$, (3) $\alpha(i) + \longrightarrow (j)$

(no change in solutions and linear dependency in columns).

$A = (a_{ij})_{m \times n}, x = (x_1, x_2, ..., x_n)^T, b = (b_1, b_2, ..., b_m)^T$. $[A_1, A_2, ..., A_n]x = x_1A_1 + x_2A_2 + ... + x_nA_n$.

Solve $Ax = b$ by applying the elimination method (3ERO) to the augmented matrix $[A|b]$ for unique, infinitely many or no solutions.

Knowing solution expression $x = x_p + x_0$ (k-plane) where $Ax_p = b, Ax_0 = 0$.

Linearly Independent(LI): $c_1v_1 + ... + c_nv_n = 0$ has ONLY zero solution $c_1 = ... = c_n = 0$. Otherwise linearly dependent(LD). n=2, $u = \alpha v$. Always ask: is there a nonzero solution $(c_1, ..., c_n)$?

Knowing how to get the reduced form R of A by 3ERO. Columns with leading 1's are LI and a column without a leading 1 can be expressed as LI of columns with leading 1's to the left. Knowing how to check LI and get coefficients of LC.

Knowing how to get $A^{-1}_{n \times n}$ by 3ERO $[A|I] \Longrightarrow [I|A^{-1}]$. For n=2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Determinants: n=2: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. n=3: $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & | & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & | & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & | & a_{31} & a_{32} \end{vmatrix}$.

Cofactor expansion: $|A| = \sum_{j=1}^n (-1)^{i+j} a_{ij}|A_{ij}| = \sum_{i=1}^n (-1)^{i+j} a_{ij}|A_{ij}|$ along ith row or jth column. Choose a row (column) with most zeros. Knowing how to compute $|A|$ by 3ERO.

(1) $|A^T| = |A|$, (2) $|A$=triangular$| = a_{11}a_{22} \cdots a_{nn}$, (3) $|A$ has a row (column) of zeros$| = 0$, (4) $|A$ has 2 rows (columns) proportional$| = 0$, (5) $|\alpha(i)$ of $A| = \alpha|A|$, (6) $|(i) \leftrightarrow (j)$ of $A| = -|A|$, (7) $|\alpha(i) + \longrightarrow (j)$ of $A| = |A|$, (8) $|A| \ne 0$ iff A is invertible, (9) $|AB| = |A||B|$, (10) $|A^{-1}| = \frac{1}{|A|}$.

$f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, then $f(x) = Ax$ where $A_{m \times n}$ is the matrix (representation) of f with $A = [f(e_1)f(e_2)...f(e_n)]$. Range of f =all LC of columns of A

f is one-to-one if $x_1 \ne x_2$ implies $f(x_1) \ne f(x_2)$. $f(x) = Ax$ is 1-to-1 iff the columns of A are LI.