1. An alphabet of 46 symbols is used for transmitting messages in a network. A packet consists of 16 distinct symbols. How many distinct packets are there?

2. A *staircase path* consists of a sequence of movements in the plane, where each movement is one of $R : (i, j) \to (i + 1, j)$ or $U : (i, j) \to (i, j + 1)$. How many different staircase paths begin at $(0, 0)$ and end at $(4, 5)$?

3. How many eight-bit bytes have (a) exactly three 1's? (b) at most three 1's?

4. Consider strings of length 5 made up of the digits 0 through 9. How many have weight (sum of digits) equal to four? (The string "01111" is an example.)

5. Regard two n-digit strings as equivalent if the digits are the same. For example, 0004 is equivalent to 4000. How many distinct unequivalent digital strings of length four are there?

6. How many nonnegative integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10?$$

7. In how many ways can one place ten pennies in six containers?

8. How many ways can you toss a penny 12 times, getting 6 heads and 6 tails, with no consecutive tails? How many of these ways also have no consecutive heads?

9. Construct a truth table for $\neg(p \vee \neg q) \to \neg p$.

10. One of four children ate a cookie that was supposed to be for Santa. The children made the following statements:

Kelly:	Chuck ate the cookie.
Dawn:	I did not eat the cookie.
Chuck:	Tyler ate the cookie.
Tyler:	Chuck lied when she said I ate the cookie.

    Assume that three of the statements are true and one is false. Who ate the cookie?

11. Find all positive integers n such that

$$\sum_{i=1}^{2n} i = \sum_{i=1}^{n} i^2.$$

12. Give a recursive definition of the intersection of a family of n sets $\{A_1, \ldots, A_n\}$.

13. Let $A = \{1, \{1\}, \{\{1\}\}, \{2\}\}$. (one point each) Which of the following statements is true? (One that is "meaningless" is not true!)
    1) $1 \in A$
    2) $1 \subseteq A$
    3) $\{1\} \in A$
    4) $\{1\} \subseteq A$
    5) $\{\{1\}\} \in A$
    6) $2 \in A$

7) $2 \subseteq A$
8) $\{2\} \in A$
9) $\{2\} \subseteq A$

14. Let $A = \{1, \{1\}, \{2\}\}$ and $B = \{2, \{2\}, \{1\}\}$.
    Find
    1) $A \cup B$
    2) $A \, \Delta \, B$
    3) $A - B$

15. Prove or disprove
    1) $\forall A \, \forall B \, \exists C, \; A \cup C = B \cup C \Rightarrow B = A$
    2) $\forall A \, \forall B \, \exists C, \; A \cap C = B \cap C \Rightarrow B = A$
    3) $\forall A \, \forall B \, \exists C, \; A \subseteq C \wedge B \subseteq C \Rightarrow B = A$

16. How many different arrangements of the letters in word $\mathcal{MISSISSIPPI}$ have no consecutive $\mathcal{S}$'s?

17. How many terms are in the expansion of $(x + y + z)^4$?

18. What is the sum of the coefficients in the expansion of $(x + y + z)^4$?

19. Use mathematical induction to prove the formula

$$\sum_{i=1}^{n} (2i - 1) = n^2.$$

20. Given that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

derive the formula in the first problem without using mathematical induction.

Let $A = \{1, 2, 3\}$ and $B = \{0, 2, 4, 6\}$.

21. Let $\mathcal{R}_1 \subseteq A \times B$ be defined by $\mathcal{R}_1 = \{(a, b) \mid a \leq b\}$. What is $|\mathcal{R}_1|$?
22. Let $\mathcal{R}_2 \subseteq A \times B$ be defined by $\mathcal{R}_2 = \{(a, b) \mid \exists x \in \mathbf{Z}, ax = b\}$. What is $|\mathcal{R}_2|$?

Three true/false

23. $\forall n \in \mathbf{Z}, \; \lfloor n \rfloor = \lceil n \rceil$.
24. $\forall x \in \mathbf{R}, \; \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$.
25. $\forall x \in \mathbf{R}, \; \lfloor 2x \rfloor \leq 2 \lfloor x \rfloor$.
26. Tom and Harry play a game in which Tom flips a coin two or three times. Tom wins when he has two heads in a row. The game ends each time when it is impossible that Tom can win. Draw a tree diagram of the game. What is the probability that Tom wins?
27. Beth and Sue play a game in which Beth flips a coin three times. Beth wins when she has two heads in a row. Draw a tree diagram of the game. What is the probability that Beth wins?

28. Does the binary operation f on $\mathbf{Z}^+$ defined by $f(x, y) = x^y$ have an identity?

29. What is wrong with thinking of f in the previous problem as a binary operation on $\mathbf{Z}$?

30. If you pick ten integers from the set $\{1, 2, \ldots, 98, 99\}$ prove that at least two, say x and y, satisfy the inequality $0 < |\sqrt{x} - \sqrt{y}| < 1$.

31. Let $f, g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = ax + b$ and $g(x) = cx + d$. for what condition on the (constant) coefficients a, b, c, d is $f \circ g = g \circ f$?

32. Let

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_n &= a_{n-1} + a_{n-2}, \quad n \geq 3. \end{aligned}$$

Show that

$$a_n \leq \left(\frac{7}{4}\right)^n \quad \text{for all} \quad n \geq 1.$$

In these problems, f and g are functions $f, g : \mathbf{Z}^+ \to \mathbf{R}$.

33. Give an example of functions f and g such that $f \in O(g)$ and $g \notin O(f)$.

34. Let $h : \mathbf{Z}^+ \to \mathbf{R}$ be defined by $h(n) = 2n^3$. Give examples of functions f and g such that $f \in O(h)$ and $h \in O(g)$, but $h \notin O(f)$ and $g \notin O(h)$,

True/false; $\mathcal{R}$ etc. is a relation on a finite set A with $|A| = n > 1$.

35. If $\mathcal{R}$ is reflexive then $|\mathcal{R}| \geq n$.

36. If $|\mathcal{R}| \geq n$ then $\mathcal{R}$ is reflexive.

For these problems, suppose $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

37. If $\mathcal{R}_1$ is reflexive then $\mathcal{R}_2$ is reflexive.

38. If $\mathcal{R}_1$ is symmetric then $\mathcal{R}_2$ is symmetric.

39. If $\mathcal{R}_1$ is antisymmetric then $\mathcal{R}_2$ is antisymmetric.

40. If $\mathcal{R}_1$ is transitive then $\mathcal{R}_2$ is transitive.

41. What is wrong with the following argument?

Theorem. *Let A be a set and let $\mathcal{R}$ be a symmetric transitive relation on A. Then $\mathcal{R}$ is reflexive.*

Proof: Let $(x, y) \in \mathcal{R}$. Then $(y, x) \in \mathcal{R}$ because $\mathcal{R}$ is symmetric, and so, since $\mathcal{R}$ is transitive, $(x, x) \in \mathcal{R}$. Therefore, $\mathcal{R}$ is reflexive.

42. Let $A = \{u, v, w, x, y\}$ and consider the (0,1)-matrix M below. Write the relation R which has that matrix and draw the directed graph of R.

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

43. Let $A = \{1, 2, 3, 4, 5\}$ and let

$$\mathcal{R} = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$$

a For each $x \in A$, find $[x]$.

b Find the (0,1)-matrix of $\mathcal{R}$.

c Draw the directed graph associated with $\mathcal{R}$.

44. Solve the recurrence relation

$$a_{n+1} = 3a_n, \qquad a_0 = 7.$$

45. Solve the linear recurrence relation with variable coefficients

$$a_{n+1} = na_n, \qquad n \geq 0.$$

Hint: think!

46. Solve the linear homogeneous recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = 0, \quad n \geq 0, \qquad a_0 = 1, \quad a_1 = 0.$$

47. A parking lot is layed out with spaces for small cars and motorcycles. Each motorcycle takes one space and each car takes two spaces. Find a recurrence relation for the number of ways to park cars and motorcycles in a row of n spaces. (All cars are identical in appearance, as are the motorcycles, and we use up all n spaces.)

Consider the linear inhomogeneous recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = n, \quad n \geq 0, \qquad a_0 = 1, \quad a_1 = 0.$$

I want you to outline the steps you would take to solve the relation using the method of generating functions.

48. What do you multiply $a_{n+2} - 5a_{n+1} + 6a_n = n$ by?

49. How do you handle $\sum_{n=0}^{\infty} nx^{n+2}$?

50. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, how do you solve for $f(x)$ in the equation?

51. What technique can you use to find a_n from your solution?

52. Write the power series for the function $f(x) = \frac{1}{1-ax}$.