2.5 Multiplication of Matrices

Example

A flower shop sells 96 roses, 250 carnations and 130 daisies in a week. The roses sell for \$2 each, the carnations for \$1 each and the daisies for \$0.50 each. Find the revenue of the shop during the week. using matrices.

Answer

$R = (96)(2) + (250)(1) + (130)(.5) = \$507$

Express the number of flowers in a 1×3 matrix:

$$A = \begin{array}{cc} & \begin{array}{ccc} R & C & D \end{array} \\ \# & \begin{bmatrix} 96 & 250 & 130 \end{bmatrix} \end{array}$$

Next express the price as a 3×1 matrix:

$$B = \begin{array}{cc} & \$ \\ \begin{array}{c} R \\ C \\ D \end{array} & \begin{bmatrix} 2 \\ 1 \\ .5 \end{bmatrix} \end{array}$$

$$A\cdot B = [96\times 2 + 250\times 1 + 130\times .5]$$
$$= [507] \rightarrow \$507$$

In general, if A is $1\times n$ and B is $n\times 1$, the product AB is a 1×1 matrix:

Labels must match

$(1\times n)\cdot(n\times 1) = 1\times 1$

$$A\cdot B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}\cdot\begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{n1} \end{bmatrix} = \begin{bmatrix} a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + \ldots + a_{1n}\cdot b_{n1} \end{bmatrix}$$

$(m\times n)\cdot(n\times p) = m\times p$

If A is an $m\times n$ matrix and B is a $n\times p$ matrix, then the product matrix $A\cdot B = C$ is an $m\times p$ matrix.

$$A\cdot B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

row 1 of A, col 2 of B

row 1 of A, col p of B

$$= \begin{bmatrix} a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + \ldots + a_{1n}\cdot b_{n1} & (ab)_{12} & \cdots & (ab)_{1p} \\ (ab)_{21} & (ab)_{22} & \cdots & (ab)_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ (ab)_{m1} & (ab)_{m2} & \cdots & (ab)_{mp} \end{bmatrix}$$

row 2 of A, col 1 of B

Matrix multiplication is not commutative. In general, $AB \neq BA$

Example

Find the products AB and BA where

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} (1)(-1)+(0)(0) & (1)(2)+(0)(-3) \\ (-2)(-1)+(0)(3) & (-2)(2)+(3)(-3) \end{bmatrix}$$

$(2\times 2) \cdot (2\times 2)$

user row 1 of A, col 2 of B

2×2

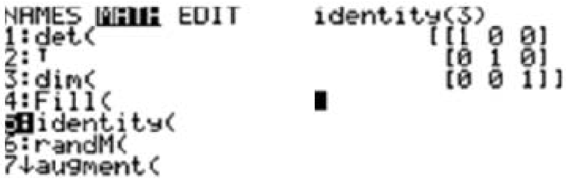
$$BA = \begin{bmatrix} -5 & 6 \\ 6 & -9 \end{bmatrix}$$

One special matrix is called the identity matrix, I.
It is a square matrix with 1's on the diagonal and zeros elsewhere,

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

I? I 1 l

I_2 is a 2×2 identity matrix and I_n is an $n \times n$ identity matrix.



The identity matrix has the following property

$IA = A = AI$

Matrix multiplication and linear equations:

Example

Write the following system of linear equations as a matrix equation

$$\begin{aligned} 2x - 3y &= 6 \\ -x + 2y &= 4 \end{aligned} \Longleftrightarrow \begin{aligned} 2x - 3y &= 6 \\ -x + 2y &= 4 \end{aligned}$$

Answer

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 2x + (-3)y \\ -1x + 2y \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$A \cdot X = B$

(2×2) (2×1) (2×1)

$ax = b \Rightarrow \frac{\cancel{a}x}{\cancel{a}} = \frac{b}{a} \Rightarrow x = \frac{b}{a}$

2.6 Inverse of a Square Matrix

$\cancel{\frac{1}{A}}$

For any non-zero real number r, the reciprocal (or inverse) is $\frac{1}{r}$ or r^{-1}

Multiplicative identity:

$2 \cdot \frac{1}{2} = 1$ $\quad 2\overline{)1.0}$ $= .5$

For matrices, the inverse is A^{-1} and it is defined by

$A^{-1}A = I = AA^{-1}$

A matrix with no inverse is called singular.

If needed, find the inverse with the x^{-1} function on the calculator.

The one use of matrix inverses is to solve matrix equations.

Solve the matrix equation $AX = B$ for X $\quad \cancel{AXA^{-1}} \rightarrow$ a mess

$\underbrace{A^{-1}A}X = A^{-1}B$

$IX = A^{-1}B$

$\boxed{X = A^{-1}B}$

EXAM 1 next week

Solve the matrix equation $D = X - AX$ for X.

$D = IX - AX$
$= (I - A)X \quad [\neq X(I - A)]$

$d = 1x - ax$

$\frac{d}{1-a} = \frac{\cancel{(1-a)}x}{\cancel{1-a}}$

$\frac{d}{1-a} = 1x$

$(I-A)^{-1}D = \underbrace{(I-A)^{-1}(I-A)}_{I}X$

$(I-A)^{-1}D = IX = \boxed{X = (I-A)^{-1}D}$

Example - Cost Analysis - The Mundo Candy Company makes three types of chocolate candy: cheery cherry (cc), mucho mocha (mm) and almond delight (ad).

The candy is produced in San Diego (SD), Mexico City (MC) and Managua (Ma) using two main ingredients, sugar (S) and chocolate (C).

Each kilogram of cheery cherry requires 0.5 kg of sugar and 0.2 kg of chocolate.
Each kilogram of mucho mocha requires 0.4 kg of sugar and 0.3 kg of chocolate.
Each kilogram of almond delight requires 0.3 kg of sugar and 0.3 kg of chocolate.

(a) Put this information in a 2x3 matrix.

$$A = \begin{array}{c} \\ S \\ C \end{array} \begin{array}{c} \begin{array}{ccc} cc & mm & ad \end{array} \\ \begin{bmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{bmatrix} \end{array} = 2\times3$$

B_1 that is 2×3 or B_2 thats 3×2

(b) The cost of 1 kg of sugar is \$3 in San Diego, \$2 in Mexico City and \$1 in Managua. The cost of 1 kg of chocolate is \$3 in San Diego, \$3 in Mexico City and \$4 in Managua.

Put this information into a matrix $\boxed{\text{in such a way}}$ that when it is multiplied by the matrix in part (a) it will tell us the cost of producing each kind of candy in each city. result are 3×3

1. $B_1 \cdot A = (2\times3)\cdot(2\times3)$ NOT POSSIBLE
2. $A \cdot B_1 = (2\times3)\cdot(2\times3)$ " "
3. $A \cdot B_2 = (2\times3)(3\times2) = 2\times2$ MEANINGLESS
4. $B_2 \cdot A = (3\times2)\cdot(2\times3) = 3\times3$ probably

$$\boxed{\begin{array}{c} \\ \text{Cost in SD} \\ \text{" " MC} \\ \text{" " Ma} \end{array} \begin{array}{c} \begin{array}{cc} S & C \end{array} \\ \begin{pmatrix} 3 & 3 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \end{array}} \cdot \begin{array}{c} \\ S \\ C \end{array} \begin{array}{c} \begin{array}{ccc} cc & mm & ad \end{array} \\ \begin{pmatrix} .5 & .4 & .3 \\ .2 & .3 & .3 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ SD \\ MC \\ Ma \end{array} \begin{array}{c} \begin{array}{ccc} cc & mm & ad \end{array} \\ \begin{pmatrix} \underbrace{3 \times .5}_{S\ cost} + \underbrace{(3\times.2)}_{C\ cost} & 2.1 & 1.8 \\ 1.6 & 1.7 & 1.5 \\ 1.3 & 1.6 & 1.5 \end{pmatrix} \end{array}$$

Introduction to Leontief Input-Output Models

The Old School Village has an economy made up of farmers and weavers. The farmers produce all of the food for the villagers. To sustain all their hard work, for each $1 worth of *food* produced by the farmers, the farmers eat $0.25 worth of *food*.

If the farmers produce $200.00 worth of food, how much food will the farmers eat? $200 \times .25 = \$50$

If the farmers produce $\$f$ worth of food, how much food will the farmers eat? $.25f$

While making cloth the weavers also get hungry and need to eat. For each $1 worth of *cloth* produced by the weavers, the weavers eat $0.40 worth of *food*.

If the weavers produce $300.00 worth of cloth, how much food will the weavers eat? $300 \times .4 = \$120$

If the weavers produce $\$c$ worth of cloth, how much food will the weavers eat? $.4c$

If the farmers produce $200 worth of food and the weavers produce $300 worth of cloth, how much food will be eaten by all the villagers?

$$\underbrace{\$50}_{\text{food for farmers}} + \underbrace{\$120}_{\text{food weavers}} = \$170$$

Is there any food left over? If so, how much?

Yes, $200 - 170 = \$30$ of food

If the farmers produce $\$f$ worth of food and the weavers produce $\$c$ worth of cloth, how much total food will be eaten by all the villagers?

$.25f + .4c =$ food eaten by villagers

The weavers produce all of the cloth used in clothing the villagers. For each \$1 worth of *cloth* produced by the weavers, the weavers use \$0.15 of the *cloth* to clothe themselves.

If the weavers produce \$300.00 worth of cloth, how much cloth will the weavers use? $300 * .15 = \$45$

If the weavers produce $\$c$ worth of cloth, how much cloth will the weavers use? $.15c$

The farmers' clothes become worn and torn as they farm the fields. For each \$1 worth of *food* produced by the farmers, the farmers require \$0.20 worth of *cloth* to clothe themselves.

If the farmers produce \$200.00 worth of food, how much cloth will the farmers use? $200 * .2 = \$40$

If the farmers produce $\$f$ worth of food, how much cloth will the farmers use? $.2f$

If the farmers produce \$200 worth of food and the weavers produce \$300 worth of cloth, how much cloth will be used by all the villagers?

$$\underbrace{\$40}_{\text{cloth for farmer}} + \underbrace{\$45}_{\text{cloth for weavers}} = \$85$$

Is there any cloth left over? If so, how much?

yes! $\$300 - \$85 = \$215$

If the farmers produce $\$f$ worth of food and the weavers produce $\$c$ worth of cloth, how much total cloth will be used by all the villagers?

cloth used internally $= .2f + .15c$

Other villages develop and the king in the nearby city requires the Old School Village to produce food and cloth for all the other villages.

If the farmers produce $\$f$ worth of food and the weavers produce $\$c$ worth of cloth, how much food and cloth will the Old School villagers use for themselves?

internal food eaten = $.25f + .4c$
internal cloth used = $.2f + .15c$

Total Supply (Production) = Total Demand (Consumption)

Let f = total food produced $= .25f + .4c + 892$
c = total cloth produced $= .2f + .15c + 446$

Suppose the king tells the Old School Village that the other villages re~~quire $892 worth of food and~~ $446 worth of cloth.

$$\text{FOOD: } f = \overbrace{(0.25f + .4\,c)}^{\text{Internal Demand}} + \overbrace{892}^{\text{External}}$$

$$\text{CLOTH: } c = (.2\,f + 0.15c) + 446$$

$$\begin{bmatrix} f \\ c \end{bmatrix} = \boxed{\begin{bmatrix} 0.25f + .4\,c \\ .2\,f + 0.15c \end{bmatrix}} + \begin{bmatrix} 892 \\ 446 \end{bmatrix} \rightarrow$$

$$X = AX + D$$

$$\underbrace{\begin{bmatrix} f \\ c \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0.25 & .4 \\ .2 & 0.15 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} f \\ c \end{bmatrix}}_{X} + \underbrace{\begin{bmatrix} 892 \\ 446 \end{bmatrix}}_{D}$$

X = the production matrix
D = the external demand matrix
A = the input-output matrix

Note: Matrix A can be constructed without starting from equations using these conventions:

- The *first column* of A contains information about how much food and cloth is used when *making one $1 worth of food*, and
- the *second column* of A contains information about how much food and cloth is used when *making $1 worth of cloth*.

Solve the matrix equation for X.

$$A = \begin{array}{c} \\ f \\ c \end{array}\begin{array}{c} \begin{array}{cc} f & c \end{array} \\ \begin{pmatrix} .25 & .4 \\ .2 & .15 \end{pmatrix} \end{array}$$

$$X = AX + D$$

$$X - \underline{AX} = D$$

$$IX - \underline{AX} = D$$

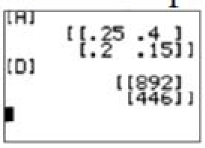
$$(I-A)^{-1} \quad (\underline{I} - \underline{A})X = D \; (I-A)^{-1}$$

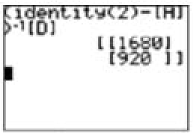
$$X = (\underline{I} - \underline{A})^{\boxed{-1}} D$$

The total amount of goods that must be produced to satisfy both internal and external demands in an economy with input-output matrix A and external demand matrix D, is given by the formula:

$$X = (I - A)^{-1} D$$

Solve for the production needs of the Old School Village.





$$X = \begin{bmatrix} f \\ c \end{bmatrix} = \begin{bmatrix} 1680 \\ 920 \end{bmatrix}$$

Thus, the Old School Village needs to produce $ 1680 worth of food and $ 920 worth of cloth to satisfy both internal and external demands.

2. How much of the food and cloth produced was used internally by the Old School villagers

$ 1680-892 of food and $ 920-446 of cloth

$788 $474