Markov Chains

1

A Markov chain or process describes an experiment consisting of a finite number of stages.

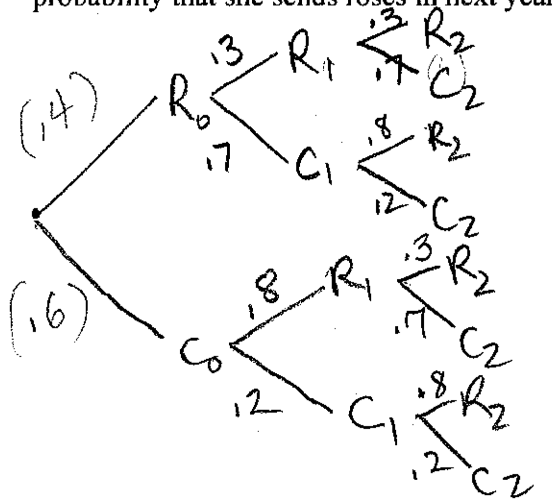
- The outcomes and associated probabilities at each stage depend only on the outcomes of the preceding stage.
- The outcome at any stage of the experiment in a Markov chain is called the state of the experiment.

Example

Every Mother's Day, a certain individual sends her mother either roses or carnations. If she sends roses one year, 30% of the time she will send roses again the next year. If she sends carnations one year, 80% of the time she will send roses the next year.

a) Is this a Markov chain?

b) If there is a 40% chance of giving roses this year, what is the probability that she sends roses in next year? In two years?



a) yes

b) $P(R_1) = (.4)(.3) + (.6)(.8) = .6$

Let $X_0 = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$ and $T = \begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix}$

$T X_0 = \begin{array}{c} \begin{bmatrix} .6 \\ .4 \end{bmatrix} \end{array}\begin{array}{l} R \\ C \end{array}$, $T^2 X_0 = \begin{array}{c} \begin{bmatrix} .5 \\ .5 \end{bmatrix} \end{array}\begin{array}{l} R \\ C \end{array}$

2

A transition matrix T is a matrix such that:

- The matrix is square
- All entries are nonnegative.
- The entries in each column sum to 1.
- The entries represent conditional probabilities

The initial state is stored as matrix X_0.

The matrix X_i represents the distribution after i stages.

$$X_n = T^n X_0$$

c) What is the probability that she will send roses in 10 years?

$X_{10} = \begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix}^{10} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = \begin{bmatrix} .5332 \\ .4668 \end{bmatrix}\begin{array}{l} R \\ C \end{array}$

d) Given she sent carnations this year, what is the probability that she will give carnations again 3 years from now?

$\Rightarrow X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$X_3 = T^3 X_0 = \begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .55 \\ .45 \end{bmatrix}$

③

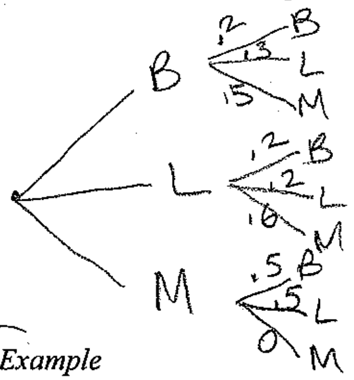
Example
A survey indicates that people in a certain area take their summer vacations either at the beach, at the lake, or in the mountains. The survey finds that among people who have gone to the beach, 20% go to the beach the next summer, 30% go to the lake, and 50% go to the mountains. Among those who have gone to the lake, 20% go to the beach the next summer, 20% go to the lake, and 60% go to the mountains. Finally amount those who have gone to the mountains. 50% go to the beach next summer and 50% go to the lake.

a) Find the transition matrix for this Markov process.

b) There was an oil spill last summer and no one went to the beach. Instead, half went to the mountains and half to the lake. What is the probability that a people will go to each of these locations two summers after the spill?

(on back)

c) What will be the long term distribution of vacation locations?



$$T = \begin{array}{c|ccc} & B & L & M \\ B & .2 & .2 & .5 \\ L & .3 & .2 & .5 \\ M & .5 & .6 & 0 \end{array}$$

Example
What is the long term distribution for flowers on Mother's Day?

Long term: $TX = X_L \Rightarrow \begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} .3x+.8y \\ .7x+.2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{array}{l} .3x+.8y=x \\ .7x+.2y=y \end{array} \Rightarrow \begin{array}{l} -.7x+.8y=0 \\ .7x-.8y=0 \end{array}$

$\rightarrow$ parametric. But $x+y=1$ (probability)

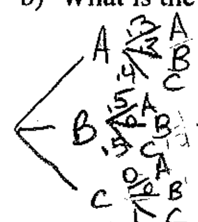
$\begin{array}{l} -.7x+.8y=0 \\ .7x-.8y=0 \\ x+y=0 \end{array} \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 8/15 \\ 0 & 1 & 7/15 \\ 0 & 0 & 0 \end{array}\right]$ $x = 8/15 = P$ roses, $y = 7/15 = P$ carn

④

Example
A company offers three different cars to its executives each year. Those who have a brand A car ask for a brand A car again 30% of the time, they ask for a brand B car 30% of the time and a brand C car 40% of the time. Those who are driving a brand B car ask for a brand A car 50% of the time and a brand C car 50% of the time. Those who are driving a brand C car ask for a brand C car all of the time.

a) Find the transition matrix for this Markov process.

b) What is the long term distribution of cars? All C



$$\begin{array}{c|ccc} & A & B & C \\ A & .3 & .5 & 0 \\ B & .3 & 0 & 0 \\ C & .4 & .5 & 1 \end{array}$$ absorbing state

A transition matrix T is a regular Markov chain if the sequence $T, T^2, T^3, \ldots$ approaches a steady state matrix with all positive entries.

Example
Classify the following matrices as regular, absorbing, or not a transition matrix.

a) $\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \longrightarrow$ Regular

b) $\begin{bmatrix} 0.7 & 1 \\ 0.3 & 0 \end{bmatrix} \longrightarrow$ Regular

c) $\begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix} \longrightarrow$ Absorbing

b) $X_0 = \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}$

$$T^2 X_0 = \begin{bmatrix} .2 & .2 & .5 \\ .3 & .2 & .5 \\ .5 & .6 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} .29 \\ .325 \\ .385 \end{bmatrix} \end{array} \begin{array}{c} B \\ L \\ M \end{array}$$

$T X_L = X_L$

$$\begin{bmatrix} .2 & .2 & .5 \\ .3 & .2 & .5 \\ .5 & .6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} -x \quad .2x + .2y + .5z &= x \quad -x \\ -y \quad .3x + .2y + .5z &= y \quad -y \\ -z \quad .5x + .6y + 0z &= z \quad -z \end{aligned}$$

$$\left.\begin{aligned} -.8x + .2y + .5z &= 0 \\ .3x - .8y + .5z &= 0 \\ .5x + .6y - z &= 0 \end{aligned}\right\} \text{parametric}$$

$$x + y + z = 0$$

$$\left[\begin{array}{ccc|c} -.8 & .2 & .5 & 0 \\ .3 & -.8 & .5 & 0 \\ .5 & .6 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 50/163 \\ 0 & 1 & 0 & 55/163 \\ 0 & 0 & 1 & 58/163 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

About 30.67% to beach, 33.74% to lake and 35.58% to mountains.