Limits of Polynomial and Rational Functions

1. $\lim_{x\to a} f(x) = f(a)$, f any polynomial function
2. $\lim_{x\to a} r(x) = r(a)$, r any rational function with nonzero denominator at $x = a$

Ex34) Find each limit.

a) $\lim_{x\to 3} x^3 - 4x^2 + 2 = (3)^3 - 4(3)^2 + 2 = -7$

b) $\lim_{x\to 2} \frac{x^2-1}{x+2} = \frac{(2)^2-1}{(2)+2} = \frac{3}{4}$

c) If $f(x) = \begin{cases} x^2+3 & \text{if } x \le 3 \\ x-4 & \text{if } x > 3 \end{cases}$, find

i) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2+3 = (0)^2+3 = 3$

ii) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2+3 = (3)^2+3 = 12$

$x<3$

iii) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x-4 = (3)-4 = -1$

$x>3$

iv) $\lim_{x\to 3} f(x) = \text{DNE}$

d) $\lim_{x\to -2} \frac{x^2-4}{x+2} = \frac{(-2)^2-4}{(-2)+2} = \frac{0}{0}$: indeterminate form

$= \lim_{x\to -2} \frac{\cancel{(x+2)}(x-2)}{\cancel{(x+2)}} = \lim_{x\to -2} (x-2) = (-2)-2 = -4$

How to find limits algebraically $\lim_{x\to a} f(x)$

Try plugging a into the function:

1. If you get a real number, that is your answer (unless you are dealing with a piecewise function)
2. If you get $\frac{0}{0}$ (indeterminate form), algebraically manipulate (usually factor), cancel, and plug in a again.
3. If you get $\frac{\text{none zero number}}{0}$ then the limit does not exist.

$$\frac{\#}{0} = \begin{cases} +\infty \\ -\infty \end{cases}$$

Ex35) Evaluate each limit:

a) $\lim_{x\to 6} |x+3| = |6+3| = |9| = 9$

b) $\lim_{x\to 2} \frac{2x^2 - x - 6}{x-2} = \frac{0}{0}$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(2x+3)}{\cancel{(x-2)}} = \lim_{x\to 2} (2x+3) = 2(2)+3$$
$$=7$$

$2x^2 - x - 6$

$A = -1$
$M = -12$: $-4,\ 3$

$= 2x^2 - 4x + 3x - 6$

$= 2x(x-2) + 3(x-2)$

$= (x-2)(2x+3)$

c) $\lim_{x\to 5} \frac{|x-5|}{x-5} = \frac{0}{0}$

Left: $\lim_{x\to 5^-} \frac{|x-5|}{x-5} = \lim_{x\to 5^-} \frac{-\cancel{(x-5)}}{\cancel{x-5}} = \lim_{x\to 5^-} -1 = -1$

$x<5$

$\neq$

Right: $\lim_{x\to 5^+} \frac{|x-5|}{x-5} = \lim_{x\to 5^+} \frac{\cancel{x-5}}{\cancel{x-5}} = \lim_{x\to 5^+} 1 = 1$

$x>5$

$\therefore \lim_{x\to 5} \frac{|x-5|}{x-5} = DNE$

d) $\lim_{x\to 4^-} \sqrt{16-x^2}$

e) $\lim_{x\to 4^+} \sqrt{16-x^2}$

Ex36) Evaluate $\lim_{x \to -4} \frac{\sqrt{x^2+9}-5}{x+4}$.

$= \frac{\sqrt{(-4)^2+9}-5}{(-4)+4} = \frac{0}{0}$

Multiply Conjugate

$$= \lim_{x \to -4} \frac{(\sqrt{x^2+9}-5)}{(x+4)} \frac{(\sqrt{x^2+9}+5)}{(\sqrt{x^2+9}+5)}$$

$$= \lim_{x \to -4} \frac{(x^2+9)-5^2}{(x+4)(\sqrt{x^2+9}+5)} = \lim_{x \to -4} \frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x \to -4} \frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(\sqrt{x^2+9}+5)} = \lim_{x \to -4} \frac{x-4}{\sqrt{x^2+9}+5}$$

$$= \frac{(-4)-4}{\sqrt{(-4)^2+9}+5} = \frac{-8}{10} = -\frac{4}{5}$$

Ex37) If $f(x) = 3x^2 + 4x - 7$ find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. $= f'(x)$

$$f(x+h) = 3\underbrace{(x+h)^2}_{= x^2+2xh+h^2} + 4(x+h) - 7$$

$$= 3(x^2+2xh+h^2) + 4(x+h) - 7$$

$$= 3x^2 + 6xh + 3h^2 + 4x + 4h - 7$$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{3x^2+6xh+3h^2+4x+4h-7-(3x^2+4x-7)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2}+6xh+3h^2+\cancel{4x}+4h\cancel{-7}\cancel{-3x^2}\cancel{-4x}\cancel{+7}}{h}$$

$$= \lim_{h \to 0} \frac{6xh+3h^2+4h}{h}$$

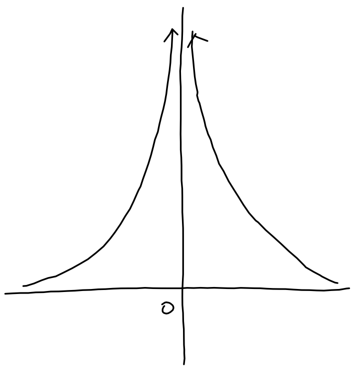
$$= \lim_{h \to 0} \frac{\cancel{h}(6x+3h+4)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x+3h+4$$

$$= 6x+3(0)+4 = 6x+4$$

Ex38) Evaluate and examine the behavior of $f(x) = \dfrac{7-x}{x^2}$ at $x = 0$.

$x<0$	$f(x)$	$x>0$	$f(x)$
-0.01	70100	0.01	69900
-0.001	700100	0.001	6999000
-0.0001	70001000	0.0001	69999000
$\vdots$	$\vdots$	$\vdots$	$\vdots$



V.A

$$\lim_{x\to 0^-} f(x) = +\infty = \lim_{x\to 0^+} f(x) = +\infty$$

$$\lim_{x\to 0} f(x) = +\infty$$

Definition. The vertical line $x = a$ is a **vertical asymptote** for the graph of $y = f(x)$ if

$$\lim_{x\to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \pm\infty$$

Locating Vertical Asymptotes of Rational Functions

If $f(x) = \dfrac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

V.A : undefined point(s) of rational function

BUT if it is cancelled, it is NOT V.A, But "hole"

Ex39) Find vertical asymptote(s) of $f(x) = \dfrac{x-3}{x^2-4x+3}$.

$$= \frac{x-3}{(x-3)(x-1)}$$

V.A: $x = 1$

Hole: $x = 3$: $\left(3, \lim_{x\to 3} f(x)\right)$

$$= \lim_{x\to 3} \frac{\cancel{x-3}}{\cancel{(x-3)}(x-1)} = \lim_{x\to 3} \frac{1}{x-1} = \frac{1}{3-1} = \frac{1}{2}$$

$$= \left(3, \tfrac{1}{2}\right)$$

Ex40) Find each limit.

a) $\lim_{x\to -3} \frac{x^2}{x+3}$ $= \frac{9}{0} = \begin{cases} +\infty \\ -\infty \end{cases}$

Left : $\lim_{x\to -3^-} \frac{x^2}{x+3} \begin{matrix} + \\ - \end{matrix} = -\infty$

-3

$x < -3$

$\neq$ $\therefore \lim_{x\to -3} \frac{x^2}{x+3} = DNE$

Right : $\lim_{x\to -3^+} \frac{x^2}{x+3} \begin{matrix} + \\ + \end{matrix} = +\infty$

-3

$x > -3$

b) $\lim_{x\to -2} \frac{x^2+x-2}{x+2}$ $= \lim_{x\to -2} \frac{(x-1)\cancel{(x+2)}}{\cancel{(x+2)}} = \lim_{x\to -2} (x-1) = -2-1 = -3$

$\therefore$ hole : $\left(-2, \lim_{x\to -2} \frac{x^2+x-2}{x+2}\right)$

$= (-2, -3)$

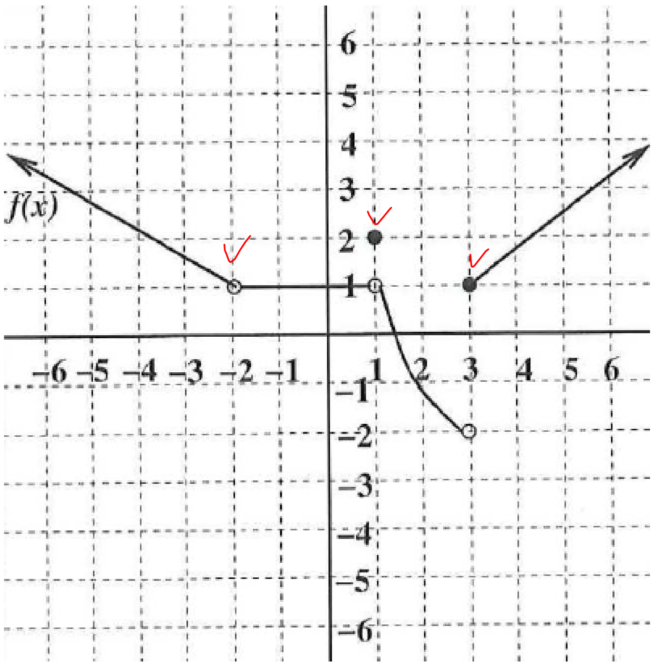
Continuity

A function f is **continuous** at the point $x = a$ if all of the following are true:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists. $\left(\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \right)$
3. $\lim_{x \to a} f(x) = f(a)$

If one or more of the conditions are not met, we say f is discontinuous at $x = a$.

Ex41) For what values of x is the function discontinous? Explain.



Definition. A function is **continous on an interval** if it is continuous at each point on the interval.

Ex42) Find the intervals on which the following functions are continuous.

a) $f(x) = 3x^9 + 4x^5 - 7$ $\quad (-\infty, \infty)$

b) $g(x) = 2^{3x} + 4$ $\quad (-\infty, \infty)$

c) $h(x) = \log_8(x-3) - 2$ $\quad (3, \infty)$

>0

$x - 3 > 0$

$x > 3$

d) $k(x) = \dfrac{x+2}{x^2 - 3x - 10} = \dfrac{x+2}{(x+2)(x-5)}$

$A = -3$

$M = -10: \ 2, -5$

$-2 \quad 5$

$\therefore (-\infty, -2) \cup (-2, 5) \cup (5, \infty)$

e) $m(x) = \begin{cases} \dfrac{x-5}{x+2} & \text{if } x \le -3 \\ 2x^2 & \text{if } -3 < x \le 0 \\ \dfrac{x}{x-4} & \text{if } x > 0 \end{cases}$

~~$x \neq -2$~~

$x = 4$: discontinuous

At $x = -3$

① $m(-3) = \dfrac{(-3)-5}{(-3)+2} = \dfrac{-8}{-1} = 8$

② Left: $\displaystyle\lim_{x \to -3^-} m(x) = \lim_{x \to -3^-} \frac{x-5}{x+2} = \frac{(-3)-5}{(-3)+2} = \frac{-8}{-1} = 8$

$x < -3$

Right: $\displaystyle\lim_{x \to -3^+} m(x) = \lim_{x \to -3^+} 2x^2 = 2(-3)^2 = 18$

$x > -3$

$\therefore \displaystyle\lim_{x \to -3} m(x) =$ DNE

$\therefore m(x)$ is discontinuous at $x = -3$

At $x = 0$

① $m(0) = 2(0)^2 = 0$

② Left: $\displaystyle\lim_{x \to 0^-} m(x) = \lim_{x \to 0^-} 2x^2 = 2(0)^2 = 0$

$x < 0$

Right: $\displaystyle\lim_{x \to 0^+} m(x) = \lim_{x \to 0^+} \frac{x}{x-4} = \frac{0}{0-4} = 0$

$x > 0$

$\therefore \displaystyle\lim_{x \to 0} m(x) = 0$

③ $m(0) = 0 = \displaystyle\lim_{x \to 0} m(x)$

$\therefore m(x)$ is continuous at $x = 0$

$-3 \quad 4$

$\therefore m(x)$ is continuous on $(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

Ex43) Find the value(s) of k that make $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} 2x - 5 & \text{if } x \leq 1 \\ x^2 + k & \text{if } x > 1 \end{cases}$$

At $x=1$

① $f(1) = 2(1) - 5 = -3$

② Left : $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x - 5 = 2(1) - 5 = -3$

$x < 1$

Right : $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 + k = (1)^2 + k = 1 + k$

$x > 1$

$1 + (-4) = -3$

$\Rightarrow -3 = 1 + k$

$\therefore k = -4$

$\therefore \lim_{x \to 1} f(x) = -3$ when $k = -4$.

③ $f(1) = -3 = \lim_{x \to 1} f(x)$

$\therefore$ when $k = -4$, $f(x)$ is continuous everywhere.