Ex10) Suppose that $f(2) = -1$, $g(2) = 3$, $f'(2) = -4$, and $g'(2) = 6$. Find $h'(2)$ for each of the following:

a) $h(x) = 2f(x) - 3g(x)$

$$h'(x) = 2f'(x) - 3g'(x)$$

$$\Rightarrow h'(2) = 2\underbrace{f'(2)}_{=-4} - 3\underbrace{g'(2)}_{=6}$$

$$= 2(-4) - 3(6) = -8 - 18 = -26$$

b) $h(x) = f(x)g(x)$

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$\Rightarrow h'(2) = f'(2)g(2) + f(2)g'(2)$$

$$= (-4)(3) + (-1)(6) = -12 - 6 = -18$$

c) $h(x) = \dfrac{f(x)}{g(x)}$

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

$$\Rightarrow h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{g(2)^2} = \frac{(-4)(3) - (-1)(6)}{(3)^2} = \frac{-12 + 6}{9}$$

$$= \frac{-6}{9} = -\frac{2}{3}$$

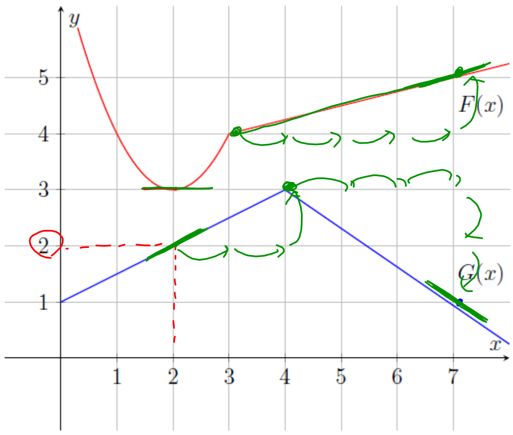
d) $h(x) = \dfrac{f(x)}{1 + g(x)}$

$$h'(x) = \frac{f'(x) \cdot (1 + g(x)) - f(x) \cdot (0 + g'(x))}{(1 + g(x))^2}$$

$$\Rightarrow h'(2) = \frac{f'(2)(1 + g(2)) - f(2)(g'(2))}{(1 + g(2))^2} = \frac{(-4)(1+3) - (-1)(6)}{(1+3)^2}$$

$$= \frac{-16 + 6}{16} = \frac{-10}{16} = \frac{-5}{8}$$

Ex11) Let $P(x) = F(x)G(x)$ and $Q(x) = \dfrac{F(x)}{G(x)}$, where F and G are the functions whose graphs are shown below.



$F(7) = 5$

$F'(7) = \frac{1}{4}$

$G(7) = 1$

$G'(7) = -\frac{2}{3}$

a) Find $P'(2)$

$$P'(x) = F'(x)\cdot G(x) + F(x)\cdot G'(x)$$

$$\Rightarrow P'(2) = \underset{0}{F'(2)}\cdot \underset{2}{G(2)} + \underset{3}{F(2)}\cdot \underset{\frac{1}{2}}{G'(2)} = 0 + \frac{3}{2} = \frac{3}{2}$$

b) Find $Q'(7)$

$$Q'(x) = \frac{F'(x)\cdot G(x) - F(x)\cdot G'(x)}{G(x)^2}$$

$$\Rightarrow Q'(7) = \frac{F'(7)\cdot G(7) - F(7)\cdot G'(7)}{G(7)^2} = \frac{(\frac{1}{4})(1) - (5)(-\frac{2}{3})}{(1)^2}$$

$$= \frac{\frac{1}{4} + \frac{10}{3}}{1} = \frac{3}{12} + \frac{40}{12}$$

$$= \frac{43}{12}$$

Section 4.3, 4.4 The Chain Rule

The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F(x) = f(g(x))$ is differentiable at x and is given by

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$F(x) = (x^2+1)^3$$

$$F'(x) = 3(x^2+1)^2 \cdot (2x+0)$$

General Derivative Rules

- If $y = [f(x)]^n$ then

$$y' = \underbrace{n[f(x)]^{n-1}}_{J'} \cdot \underbrace{f'(x)}_{T'}$$

- If $y = e^{f(x)}$ then

$$y' = \underbrace{e^{f(x)}}_{J'} \cdot \underbrace{f'(x)}_{T'}$$

- If $y = \ln(f(x))$ then

$$y' = \frac{1}{\underbrace{f(x)}_{J'}} \cdot \underbrace{f'(x)}_{T'}$$

- If $y = b^{f(x)}$ then

$$y' = \underbrace{b^{f(x)} \cdot \ln b}_{J'} \cdot \underbrace{f'(x)}_{T'}$$

- If $y = \log_b f(x)$ then

$$y' = \frac{1}{\underbrace{f(x) \cdot \ln b}_{J'}} \cdot \underbrace{f'(x)}_{T'}$$

Ex12) Differentiate the following:

a) $f(x) = (4x^2 + 7x)^5$ $T = 4x^2+7x$

$\frac{df}{dx} = f'(x) = 5(4x^2+7x)^4 \cdot (8x+7)$

b) $g(x) = 6(x^{\frac{1}{2}} - 3x)^4$ $T = x^{\frac{1}{2}} - 3x \Rightarrow T' = \frac{1}{2}x^{\frac{1}{2}-1} - 3 = \frac{1}{2}x^{-\frac{1}{2}} - 3$

$\frac{dg}{dx} = g'(x) = 24(x^{\frac{1}{2}} - 3x)^3 \cdot (\frac{1}{2}x^{-\frac{1}{2}} - 3)$

c) $y = \frac{3}{(t^2 + 3t + 4)^4} = 3(t^2+3t+4)^{-4}$ $T = t^2+3t+4 \Rightarrow T' = 2t+3$

$\frac{dy}{dt} = y' = -12(t^2+3t+4)^{-5} \cdot (2t+3)$

d) $F(x) = e^{x^2}$ $\qquad T = x^2 \Rightarrow T' = 2x$

$$F'(x) = e^{x^2} \cdot 2x$$

e) $H(x) = \underbrace{3x^5}_{F=}\underbrace{e^{x^4}}_{G=}$

$F = 3x^5$

$F' = 15x^4$

$G = e^{x^4}$ $\qquad T = x^4 \quad T' = 4x^3$

$G' = e^{x^4} \cdot 4x^3$

$= 4 \cdot x^3 \cdot e^{x^4}$

$$H'(x) = F' \cdot G + F \cdot G'$$

$$= (15x^4) \cdot e^{x^4} + 3x^5 \cdot 4 \cdot x^3 \cdot e^{x^4}$$

$$= 15x^4 \cdot e^{x^4} + 12x^8 \cdot e^{x^4}$$

f) $h(x) = \dfrac{3}{\sqrt[4]{(3x+2)^5}}$

$$= 3\,(3x+2)^{-\frac{5}{4}}$$

$T = 3x+2 \Rightarrow T' = 3$

$$h'(x) = 3\left(-\frac{5}{4}\right)(3x+2)^{-\frac{5}{4}-1} \cdot 3$$

$$= -\frac{45}{4}(3x+2)^{-\frac{9}{4}}$$

g) $f(x) = \underbrace{(4x^2+5)^6}_{F=}\underbrace{(\sqrt[3]{(3x^4-5x+7)^4})}_{G=}$

$F = (4x^2+5)^6$

$F' = 6(4x^2+5)^5 \cdot (8x)$

$= 48x(4x^2+5)^5$

$G = (3x^4-5x+7)^{\frac{4}{3}}$

$G' = \frac{4}{3}(3x^4-5x+7)^{\frac{4}{3}-1} \cdot (12x^3-5)$

$= \frac{4}{3}(3x^4-5x+7)^{\frac{1}{3}}(12x^3-5)$

$f'(x) = F' \cdot G + F \cdot G'$

$= 48x(4x^2+5)^5 \cdot (3x^4-5x+7)^{\frac{4}{3}} + (4x^2+5)^6 \cdot \frac{4}{3}(3x^4-5x+7)^{\frac{1}{3}} \cdot (12x^3-5)$

h) $y = \log_8(\sqrt{1+x^2} + 10x)$

$T = (1+x^2)^{\frac{1}{2}} + 10x$

$T' = \frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}} \cdot \cancel{2}x + 10$

$= x \cdot (1+x^2)^{-\frac{1}{2}} + 10$

$\frac{dy}{dx} = y' = \frac{1}{(\sqrt{1+x^2} + 10x) \cdot \ln 8} \cdot \left(x \cdot (1+x^2)^{-\frac{1}{2}} + 10\right)$

$= \frac{x(1+x^2)^{-\frac{1}{2}} + 10}{(\sqrt{1+x^2} + 10x) \cdot \ln 8}$

i) $y = \log_6\left(\frac{x-1}{x+2}\right)$

$T = \frac{x-1}{x+2} \Rightarrow T' = \frac{(1)\cdot(x+2) - (x-1)\cdot(1)}{(x+2)^2} = \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2}$

$y' = \frac{1}{\left(\frac{x-1}{x+2}\right) \cdot \ln 6} \cdot \frac{3}{(x+2)^2} = \frac{\cancel{x+2}}{(x-1) \cdot \ln 6} \cdot \frac{3}{(x+2)^{\cancel{2}}} = \frac{3}{(x-1)(x+2) \cdot \ln 6}$

Or Use properties of log. : $y = \log_6\left(\frac{x-1}{x+2}\right) = \log_6(x-1) - \log_6(x+2)$

$\Rightarrow y' = \frac{1}{(x-1) \cdot \ln 6} \cdot (1) - \frac{1}{(x+2)\ln 6}(1)$

$= \frac{1}{(x-1)\ln 6} - \frac{1}{(x+2)\ln 6}$

$= \frac{3}{(x-1)(x+2)\ln 6}$

j) $y = \frac{5\ln((x^2+x)^5)}{x^3}$

Find $f'(x)$ for $f(x) = \ln\left(\dfrac{x^4 \cdot (x^2+2)^3}{(2x+4)^2}\right)$

Use properties of log.

$$f(x) = \ln x^4 + \ln (x^2+2)^3 - \ln (2x+4)^2$$

$$= 4 \ln x + 3 \ln (x^2+2) - 2 \ln (2x+4)$$

$$f'(x) = 4 \cdot \frac{1}{x} + 3 \cdot \frac{1}{x^2+2} \cdot 2x - 2 \cdot \frac{1}{2x+4} \cdot 2$$

$$= \frac{4}{x} + \frac{6x}{x^2+2} - \frac{4}{2x+4}$$

Ex13) Find the value(s) of x where the tangent line is horizontal for $f(x) = \dfrac{x^2}{(2-3x)^3}$

$f'(x) = 0$

$$f(x) = \frac{\overset{F=}{x^2}}{\underset{G}{(2-3x)^3}}$$

$F = x^2$ $\qquad$ $G = (2-3x)^3$

$F' = 2x$ $\qquad$ $G' = 3(2-3x)^2 \cdot (-3)$

$= -9(2-3x)^2$

$$f'(x) = \frac{F' \cdot G - F \cdot G'}{G^2}$$

$$= \frac{(2x)\cdot(2-3x)^3 - x^2 \cdot (-9(2-3x)^2)}{((2-3x)^3)^2} = 0$$

(numerator = zero)

$$\Rightarrow (2x)(2-3x)^3 - x^2(-9(2-3x)^2) = 0$$

$$\Rightarrow 2x(2-3x)^3 + 9x^2(2-3x)^2 = 0$$

$$\Rightarrow x(2-3x)^2\left(2(2-3x) + 9x\right) = 0$$

$= 4 - 6x + 9x$

$= 4 + 3x$

$$\Rightarrow x\,\cancel{(2-3x)^2}\,(4+3x) = 0$$

$x = 0$

undefined

$4 + 3x = 0$

$\Rightarrow 3x = -4$

$x = -\frac{4}{3}$

$\therefore x = 0, -\frac{4}{3}$

Ex14) Suppose $w(x) = u(v(x))$ and $u(0) = 1$, $v(0) = 2$, $u'(0) = 3$, $u'(2) = 4$, $v'(0) = 5$, and $v'(2) = 6$. Find $w'(0)$.

$$w'(x) = u'(v(x)) \cdot v'(x)$$

$$\Rightarrow w'(0) = u'(\underbrace{v(0)}_{=2}) \cdot \underbrace{v'(0)}_{=5}$$

$$= \underbrace{u'(2)}_{=4} \cdot 5$$

$$= 4 \cdot 5 = 20$$

Ex15) Let $y = \ln u$ and $u = 5x^4 + x^6$. Find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{5x^4+x^6} \cdot (20x^3 + 6x^5) = \frac{20x^3+6x^5}{5x^4+x^6}$$

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{5x^4+x^6}$$

$$\frac{du}{dx} = 20x^3 + 6x^5$$

Or. $y = \ln u = \ln(5x^4 + x^6)$

$$\frac{dy}{dx} = y' = \frac{1}{5x^4+x^6} \cdot 20x^3 + 6x^5$$

$$= \frac{20x^3+6x^5}{5x^4+x^6}$$

Ex16) Keith invests \$5,000 into a savings account offering interest at an annual rate of 2.4% compounded continuously. How fast is the balance growing after 8 years?

$A'(8) = ?$

$A(t) = P \cdot e^{rt}$

$$\Rightarrow A(t) = 5000 \cdot e^{0.024t}$$

$$A'(t) = 5000 \cdot e^{0.024t} \cdot (0.024)$$

$$\Rightarrow A'(8) = 5000 \cdot e^{0.024(8)} \cdot (0.024) \approx \$145.40 \text{ /year.}$$