Section 5.4 Additional Curve Sketching

Graphing Strategy:

1. Analyze $f(x)$
   (a) Find the domain of f.
   (b) Find the intercepts.
   (c) Find asymptotes.
2. Analyze $f'(x)$
   (a) Find the critical values.
   (b) Use sign chart for $f'(x)$ and determine intervals of increasing/decreasing.
   (c) Find all relative(local) extrema.
3. Analyze $f''(x)$
   (a) Use sign chart for $f''(x)$ and find intervals where the graph of the function is cancave up and concave down.
   (b) Find all inflection values.
4. Sketch the graph of f using all of the above information. Plot additional points as needed.

Ex12) Use the graphing strategy to sketch a graph of $f(x) = \dfrac{2+x}{3-x}$.

Step 1) $f(x) = \dfrac{2+x}{3-x}$

· Domain: $(-\infty, 3) \cup (3, \infty)$

· x-intcp $(y=0)$: $\dfrac{2+x}{3-x} = 0$ (numerator $= 0$)

$\Rightarrow 2+x = 0$

$\therefore x = -2$ $\quad\therefore$ x-intcp: $(-2, 0)$

y-intcp $(x=0)$: $y = \dfrac{2+0}{3-0} = \dfrac{2}{3}$ $\quad\therefore$ y-intcp: $(0, \frac{2}{3})$

· V.A: $x = 3$

· H.A: $\displaystyle\lim_{x\to\infty} \frac{2+x}{3-x} \overset{H.O}{=} \lim_{x\to\infty} \frac{\cancel{x}}{-\cancel{x}} = -1$ $\qquad\therefore\ y=-1$: H.A

$\displaystyle\lim_{x\to-\infty} \frac{2+x}{3-x} \overset{H.O}{=} \lim_{x\to-\infty} \frac{\cancel{x}}{-\cancel{x}} = -1$

Step 2) $f'(x)$

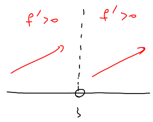
$f(x) = \dfrac{2+x}{3-x} \Rightarrow f'(x) = \dfrac{(1)\cdot(3-x) - (2+x)\cdot(-1)}{(3-x)^2}$ $\quad = 3-x+2+x = 5$

$= \boxed{\dfrac{5}{(3-x)^2}}$

$\therefore$ no C.V

P.N : $x = 3$

Sign chart for f' $\qquad\therefore$ inc. $(-\infty, 3) \cup (3, \infty)$



Step 3) $f''(x)$

$f'(x) = \dfrac{5}{(3-x)^2} = 5(3-x)^{-2}$

$f''(x) = -10(3-x)^{-3}\cdot(-1)$

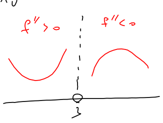
$= 10(3-x)^{-3} = \dfrac{10}{(3-x)^3}$

no C.V

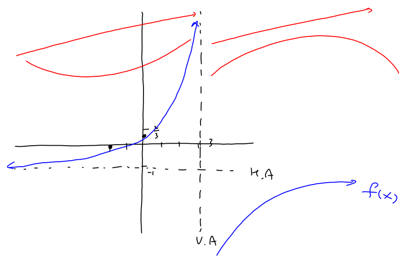
P.N : $x = 3$

Sign chart for f'' $\qquad\therefore$ C.U: $(-\infty, 3)$

C.D: $(3, \infty)$



Step 4)



Ex13) Use the graphing strategy to sketch a graph of $f(x) = \dfrac{2x^2+11x+14}{x^2-4}$.

$A=11$, $M=28$: $4, 7$

$= 2x^2+7x+4x+14 = x(2x+7)+2(2x+7) = (2x+7)(x+2)$

Step 1) $f(x) = \dfrac{(2x+7)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} = \dfrac{2x+7}{x-2}$ $\quad = \dfrac{(2x+7)(x+2)}{(x+2)(x-2)}$

· Domain: $\mathbb{R}$ except $x=-2, 2$

$(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

· X-Intcp $(y=0)$: $\dfrac{2x+7}{x-2}=0$

$\Rightarrow 2x+7=0$

$\therefore x=-\frac{7}{2}$ $\quad \therefore$ X-Intcp: $\left(-\frac{7}{2}, 0\right)$

· Y-Intcp $(x=0)$: $y=\dfrac{2(0)+7}{(0)-2}=\dfrac{7}{-2}$ $\quad \therefore$ Y-Intcp : $\left(0, -\frac{7}{2}\right)$

· V.A : $x=2$

hole : $\left(-2, \lim\limits_{x\to-2} f(x)\right) = \lim\limits_{x\to-2} \dfrac{2x+7}{x-2} = \dfrac{2(-2)+7}{(-2)-2} = \dfrac{3}{-4}$

$= \left(-2, -\frac{3}{4}\right)$

· H.A : $\lim\limits_{x\to\infty} \dfrac{2x^2+11x+14}{x^2-4} \overset{H.O}{=} \lim\limits_{x\to\infty} \dfrac{2\cancel{x^2}}{\cancel{x^2}} = 2$

$\lim\limits_{x\to-\infty}$ 〃 〃 〃 $=2$

$\therefore$ HA : $y=2$

Step 2) $f'(x)$

$f(x) = \dfrac{2x+7}{x-2}$

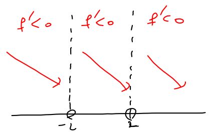
$f'(x) = \dfrac{(2)\cdot(x-2)-(2x+7)\cdot(1)}{(x-2)^2} = \boxed{\dfrac{-11}{(x-2)^2}}$

$= 2\cancel{x}-4 \; -2\cancel{x}-7$

$\therefore$ no C.V

P.N : $x=2, -2$

Sign chart for f'



$\therefore$ dec. $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

Step 3) $f''(x)$

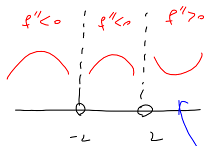
$f'(x) = \dfrac{-11}{(x-2)^2} = -11(x-2)^{-2}$

$f''(x) = 22(x-2)^{-3}\cdot(1) = \boxed{\dfrac{22}{(x-2)^3}}$

$\therefore$ no C.V

P.N : $x=-2, 2$

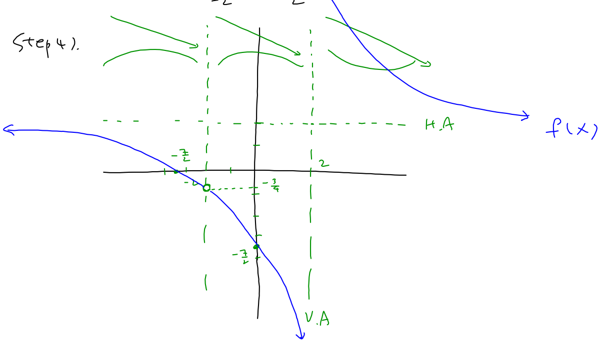
Sign chart for f''



$\therefore$ C.D : $(-\infty,-2)\cup(-2,2)$

C.U : $(2,\infty)$

Step 4).



Ex14) Use the graphing strategy to sketch a graph of $f(x) = (x-2)e^x$.

Step 1) $f(x)$

- Domain : $(-\infty, \infty)$
- x-intcp ($y=0$) : $(x-2)\cdot \underbrace{e^x}_{\neq 0} = 0$

  $\therefore x = 2$ $\therefore$ x-intcp: $(2, 0)$

  y-intcp ($x=0$): $y = (0-2)\cdot \underbrace{e^0}_{=1} = -2$ $\therefore$ y-intcp: $(0, -2)$

- V.A: no.
- H.A: $\lim_{x\to\infty} (x-2)e^x = \infty$

  $\lim_{x\to-\infty} (x-2)e^x \overset{H.O}{=} \lim_{x\to-\infty} e^x = e^{-\infty} = 0$

  $\therefore$ H.A: $y=0$ on negative side

Step 2) $f'(x)$

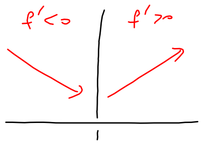
$f(x) = (x-2)\cdot e^x$

$f'(x) = (1)\cdot e^x + (x-2)\cdot e^x$

$= e^x + x\cdot e^x - 2\cdot e^x = x\cdot e^x - e^x = \boxed{(x-1)\cdot e^x} = 0$

$\therefore x = 1$: C.V & P.N

sign chart for f'



$\therefore$ dec. $(-\infty, 1)$

inc. $(1, \infty)$

local min $(1, f(1)) = (1, -e)$

Step 3) $f''(x)$

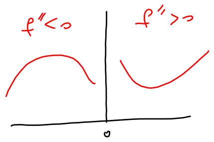
$f'(x) = (x-1)\cdot e^x$

$f''(x) = (1)\cdot e^x + (x-1)\cdot e^x$

$= \cancel{e^x} + x\cdot e^x - \cancel{e^x} = \boxed{x\cdot e^x} = 0$

$\therefore x = 0$: C.V & P.N

sign chart for f''

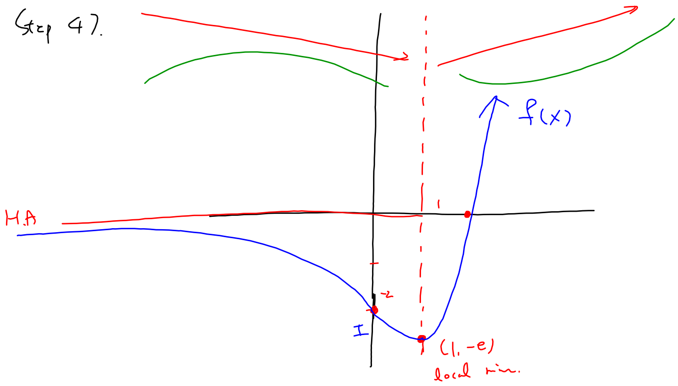


$\therefore$ C.D: $(-\infty, 0)$

C.U: $(0, \infty)$

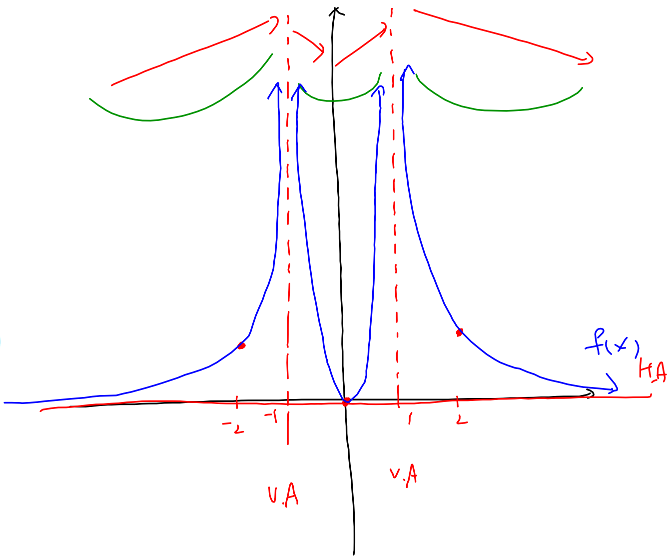
Inf: $(0, f(0)) = (0, -2)$

Step 4)



Ex15) Use the given information to sketch the graph of f.

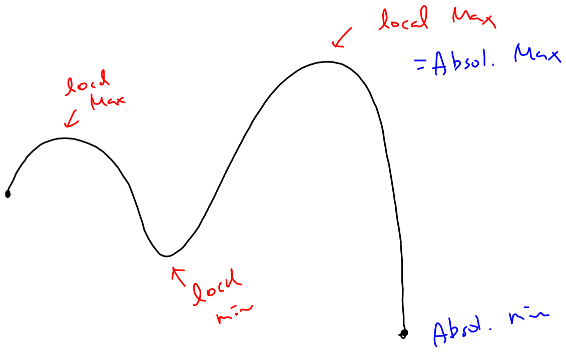
- Domain: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$
- Asymptotes: $x=-1$, $x=1$, $y=0$
- $f(-2)=1$, $f(0)=0$, $f(2)=1$
- $f'(x)>0$ on $(-\infty,-1)\cup(0,1)$
- $f'(x)<0$ on $(-1,0)\cup(1,\infty)$
- $f''(x)>0$ on $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

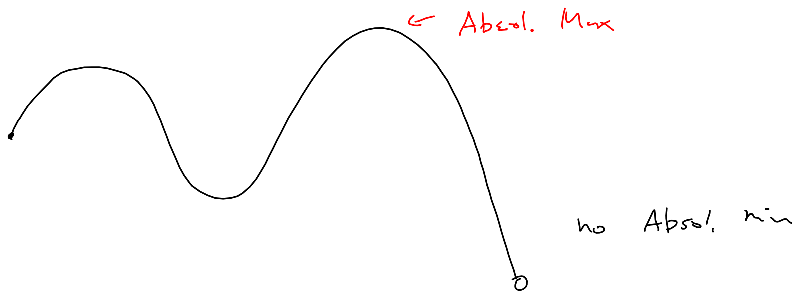


Section 5.5 Absolute Extrema

Definition. Maximum, Minimum. A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f. A function f has an **absoulte minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f. In this case, we call $f(c)$ the maximum value or minimum value, repectively.

cf) A function f has an **local maximum** at c if $f(c) \geq f(x)$ when x is near c. A function f has an **local minimum** at c if $f(c) \leq f(x)$ when x is near c.





Extreme Value Theorem. A function f that is **continuous** on a **closed interval** $[a, b]$ has both an absolute maximum value and an absolute minimum value on that interval.

Finding Absolute Extrema on a Closed Interval

1. Check to make certain that f is continuous over $[a, b]$.
2. Find the critical values in the interval (a, b).
3. Evaluate f at the endpoints a and b and at the critical values found in step 2.
4. The largest value obtained from the previous step is the absolute maximum of $f(x)$ on $[a, b]$ and the smallest value obtained is the absolute minimum of $f(x)$ on $[a, b]$.

★ Candidates : C.Vs & end points

Ex16) Find the absolute extrema of the following functions on the given intervals.

a) $f(x) = 2x^3 - 3x^2 - 12x + 24$ on $[1, 4]$

$f'(x) = 6x^2 - 6x - 12 = 0$

$\Rightarrow 6(x^2 - x - 2) = 0$

$\Rightarrow 6(x+1)(x-2) = 0$

$\therefore x = -1, 2$: C.Vs

$\therefore$ Candidates : $x = 1, 2, 4$

$f(1) = 11$

$f(2) = 4$: Absol. min

$f(4) = 56$: Absol. Max

b) $g(x) = x^3 + 3x^2 - 9x - 7$ on $[-2, 0]$

$g'(x) = 3x^2 + 6x - 9 = 0$

$\Rightarrow 3(x^2 + 2x - 3) = 0$

$\Rightarrow 3(x-1)(x+3) = 0$

$\therefore x = 1, -3$: C.Vs

Candidates : $x = -2, 0$

$g(-2) = 15$: Absol. Max

$g(0) = -7$: Absol. min

c) $h(x) = x^3 + 3x^2 - 9x - 7$ on $[-4, 0)$

$h'(x) = 3x^2 + 6x - 9 = 0$

$\Rightarrow 3(x^2 + 2x - 3) = 0$

$\Rightarrow 3(x-1)(x+3) = 0$

$\therefore x = 1, -3$

Second Derivative Test

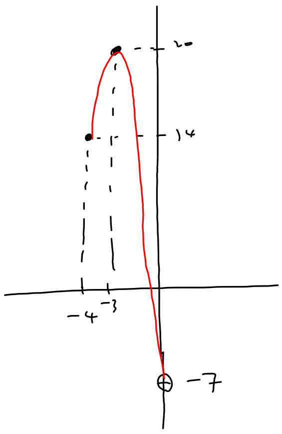
$h''(x) = 6x + 6$

$\Rightarrow h''(-3) = 6(-3) + 6 < 0$: $\frown$: local Max

$(-3, h(-3)) = (-3, 20)$

$h(-4) = 14,$

$h(0) = -7$



$\therefore$ Absol. Max $= 20$ at $x = -3$

no Absol. min

Ex17) Find the absolute extrema of each function on the given interval.

a) $f(x) = 6x - x^2 + 4$ on $(0, \infty)$

$f'(x) = 6 - 2x = 0$

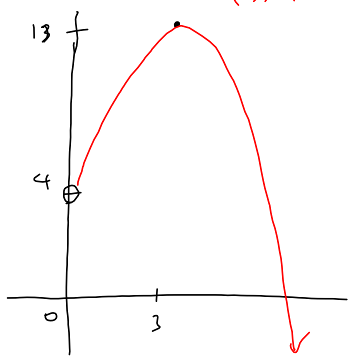
$\therefore x = 3$: C.V

$f''(x) = -2$

$\Rightarrow f''(3) = -2 < 0$: $\cap$: local Max

: $(3, f(3)) = (3, 13)$

$f(0) = 4$



$\therefore$ Absol. Max $= 13$ at $x = 3$

no Absol. min

b) $g(x) = 2x + \dfrac{8}{x}$ on $(0, 10)$

$= 2x + 8 \cdot x^{-1}$

$g'(x) = 2 - 8x^{-2} = 2 - \dfrac{8}{x^2} = 0$

$\Rightarrow \cancel{2}^{1} = \dfrac{\cancel{8}^{4}}{x^2}$

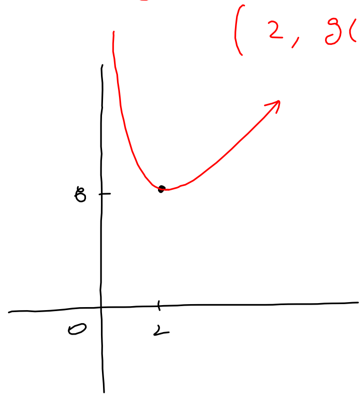
$\Rightarrow \sqrt{x^2} = \sqrt{4}$

$\therefore x = 2, -2$

$g''(x) = 16x^{-3} = \dfrac{16}{x^3}$

$g''(2) = \dfrac{16}{(2)^3} > 0$: $\cup$: local min

$(2, g(2)) = (2, 8)$



$\therefore$ Absol. min $= 8$ at $x = 2$

no Absol. Max