MATH 142 Business Mathematics II

Summer, 2016, WEEK 3

JoungDong Kim

Week 3: 5.1, 5.2, 5.3, 5.4, 5.5, 5.6

Chapter 5 Curve Sketching and Optimization

Section 5.1 The First Derivative

Test for Increasing or Decreasing Functions

- If for all $x \in (a, b)$, $f'(x) > 0$, then $f(x)$ is increasing($\nearrow$) on (a, b)
- If for all $x \in (a, b)$, $f'(x) < 0$, then $f(x)$ is decreasing($\searrow$) on (a, b)

Definition. Critical Value: A value $x = c$ is a **critical value** for a function $f(x)$ if

1. c is in the domain of the function $f(x)$ and
2. $f'(c) = 0$ or $f'(x)$ does not exist.

Definition. Partition number: A **partition number** of $f'(x)$ is a value of x such that $f'(x) = 0$ or $f'(x)$ is undefined.

Ex1) Find the critical values and partition numbers for the following functions and then determine where the function is increasing/decreasing.

a) $f(x) = -x^3 + 12x - 5$

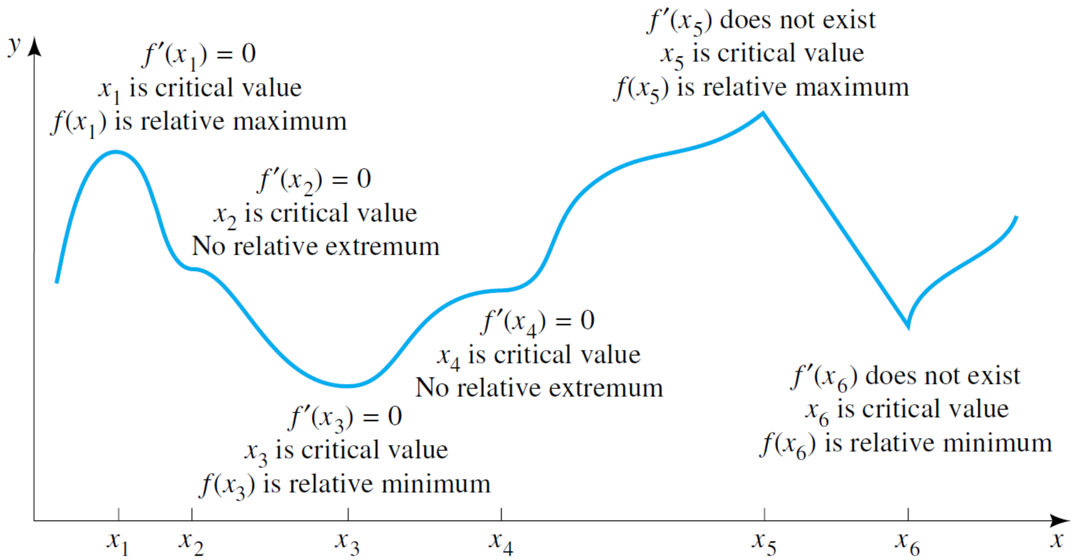
b) $g(x) = \dfrac{x+3}{5-x}$

c) $h(x) = x \ln x - x$

Definition. Relative Maximum and Relative Minimum:

- We say that the quantity $f(c)$ is a **relative (local) maximum** if $f(x) \leq f(c)$ for all x in some open interval (a, b) that contains c.
- We say that the quantity $f(c)$ is a **relative (local) minimum** if $f(x) \geq f(c)$ for all x in some open interval (a, b) that contains c.

Definition. Relative Extremum We say that $f(c)$ is a **relative (local) extremum** if $f(c)$ is a relative maximum or a relative minimum.



First Derivative Test

Suppose f is defined on (a, b) and c is a critical value in the interval (a, b).

1. If $f'(x) > 0$ for x near and to the left of c and $f'(x) < 0$ for x near and to the right of c, then we have $\nearrow\searrow$ and $f(c)$ is a relative maximum.

2. If $f'(x) < 0$ for x near and to the left of c and $f'(x) > 0$ for x near and to the right of c, then we have $\searrow\nearrow$ and $f(c)$ is a relative minimum.

3. If the sign of $f'(x)$ is the same on both sides of c, then $f(c)$ is not a relative extremum.

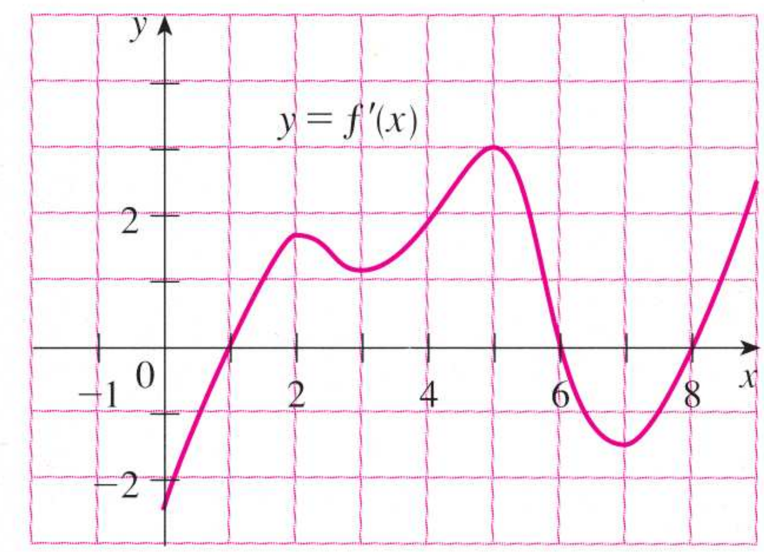
Ex2) Determine the intervals where the following functions are increasing and decreasing and find the local extrema.

a) $f(x) = x^4 + 2x^3 + 5$

b) $g(x) = \dfrac{2}{x^2 - 16}$

c) $h(x) = (x + 2)e^x$

Ex3) The graph of the derivative $f'(x)$ is shown below.



a) on what intervals is f increasing?

b) on what intervals is f decreasing?

c) the critical value(s).

d) At what values of x does $f(x)$ have a local maximum or minimum?

Section 5.2 The Second Derivative

Definition. Second Derivative: Given a function $y = f(x)$, the **second derivative**, denoted by $f''(x)$, is defined to be the derivative of the first derivative. Thus,

$$f''(x) = \frac{d}{dx}(f'(x))$$

Ex4) Find the first and second derivative of the following functions:

a) $f(x) = 2x^3 - 14x^2 + 3x - 16$

b) $y = 3x^2 \ln x$

Definition. Concave Up and Down:

1. We say that the graph of f is concave up ($\smile$) on (a, b) if $f'(x)$ is increasing on (a, b).
2. We say that the graph of f is concave down ($\frown$) on (a, b) if $f'(x)$ is decreasing on (a, b).

Test for Concavity

1. If $f''(x) > 0$ on (a, b), then the graph of f is concave up ($\smile$) on (a, b).
2. If $f''(x) < 0$ on (a, b), then the graph of f is concave down ($\frown$) on (a, b).

Definition. Inflection Point: A point $(c, f(c))$ on the graph of f is an **inflection point** and c is and **inflection value** if $f(c)$ is defined and the concavity of the graph of f changes at $(c, f(c))$.

Locating Inflection Points:

1. Determine the values of x where the second derivative is zero or where the second derivative is undefined.
2. Place these values on a number line and create a sign chart for the second derivative.
3. The point is an inflection point if the second derivative changes sign and if the x-value is in the domain of $f(x)$.

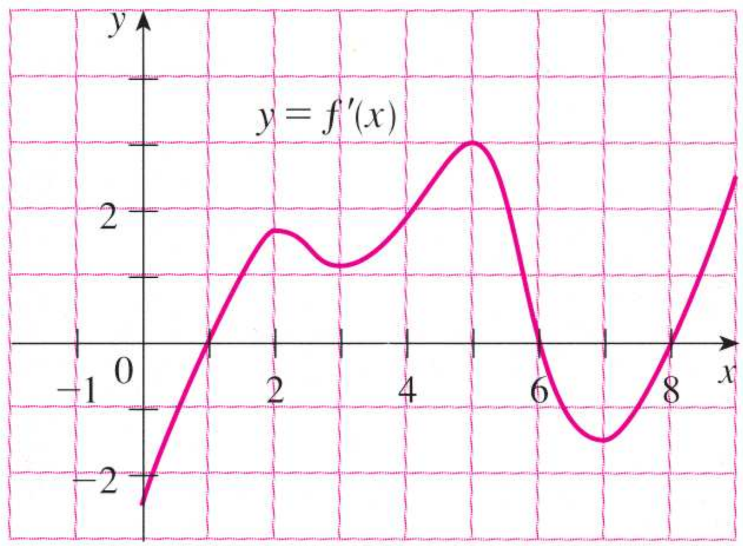
Ex5) Given $f(x) = -x^3+2x^2-3x+9$ determine the intervals where $f'(x)$ is increasing and decreasing.

Ex6) Determine the intervals where the functions below are concave up and concave down and locate any inflection points.

a) $f(x) = x^4 - 6x^2$

b) $f(x) = \ln(x^2 + 6x + 13)$

Ex7) The graph of the derivative f' of a function f is shown below



a) the interval(s) on which $f(x)$ is increasing.

b) the interval(s) on which $f(x)$ is decreasing.

c) At what values of x does f have a local maximum or minimum?

d) the interval(s) on which $f'(x)$ is increasing.

e) the interval(s) on which $f'(x)$ is decreasing.

f) the x-value of the inflection point(s) of $f(x)$.

g) Sketh a graph of f.

Second Derivative Test

Suppose that f is defined on (a, b), $f'(c) = 0$, and $c \in (a, b)$.

1. If $f''(c) > 0$, then $f(c)$ is a relative (local) minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative (local) maximum.

Ex8) Determine where the local extrema of $f(x) = -x^3 + 12x - 5$ occur (and classify) using the Second Derivative Test for Local Extrema.

Section 5.3 Limits at Infinity

Definition. Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Definition. Horizontal Asymptote: The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Limit of Power Function at Infinity

If p is a positive real number,

$$\lim_{x\to\pm\infty} \frac{1}{x^p} = 0.$$

Ex9) Find the limits:

a) $\lim_{x\to\infty} x$

b) $\lim_{x\to-\infty} x$

c) $\lim_{x\to\infty} (x - x^2)$

d) $\lim_{x\to-\infty} (x - x^3)$

e) $\lim_{x\to\infty} \frac{1}{x}$

f) $\lim_{x\to-\infty} \frac{1}{x}$

g) $\lim_{x\to\infty} \frac{1}{x^4}$

h) $\lim_{x\to\infty} \frac{7x + 12}{x^2 + 10x + 5}$

i) $\lim_{x\to\infty} \frac{7x^3 + 4x}{2x^3 - x^2 + 3}$

j) $\lim_{x\to-\infty} \frac{3x+4}{2x+1}$

k) $\lim_{x\to-\infty} \frac{x^2+7x+12}{x+5}$

l) $\lim_{t\to\infty} \frac{t^4-t^2+1}{t^5+t^3-t}$

m) $\lim_{x\to-\infty} \frac{x^4+2x+3}{x(x^2-1)}$

n) $\lim_{x\to\infty} \frac{\sqrt{1+4x^2}}{4+x}$

o) $\lim_{x\to-\infty} \frac{\sqrt{1+4x^2}}{4+x}$

p) $\lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{4x+1}$

Finding limits at infinity for a rational function, $f(x)$:

Look for the highest degree of x:

1. If it is in the denominator, then $\lim_{x\to\pm\infty} f(x) = 0$.
2. If it is in the numerator, then $\lim_{x\to\pm\infty} f(x) = \pm\infty$.
3. If the degree of the polynomial in the numerator and denominator is the same then $\lim_{x\to\pm\infty} f(x) =$ ratio of the leading coefficients.

Finding the Vertical Asymptote and Horizontal Asymptote.

1. Vertical asymptote: undefined point but if it could be cancelled, it is not vertical asymptote but hole.
2. Horizontal asymptote: use infinite limit $x \to \infty$ and $x \to -\infty$.

Ex10) Find all horizontal and vertical asymptotes of the functions below.

a) $f(x) = \dfrac{3x+2}{x-4}$

b) $g(x) = \dfrac{x^2+9}{x}$

c) $h(x) = \dfrac{x+3}{x^2+7x+12}$

Ex11) Find all horizontal and vertical asymptotes of $f(x) = \dfrac{6e^x}{1-4e^x}$.

Section 5.4 Additional Curve Sketching

Graphing Strategy:

1. Analyze $f(x)$
   (a) Find the domain of f.
   (b) Find the intercepts.
   (c) Find asymptotes.
2. Analyze $f'(x)$
   (a) Find the critical values.
   (b) Use sign chart for $f'(x)$ and determine intervals of increasing/decreasing.
   (c) Find all relative(local) extrema.
3. Analyze $f''(x)$
   (a) Use sign chart for $f''(x)$ and find intervals where the graph of the function is cancave up and concave down.
   (b) Find all inflection values.
4. Sketch the graph of f using all of the above information. Plot additional points as needed.

Ex12) Use the graphing strategy to sketch a graph of $f(x) = \dfrac{2+x}{3-x}$.

Ex13) Use the graphing strategy to sketch a graph of $f(x) = \dfrac{2x^2 + 11x + 14}{x^2 - 4}$.

Ex14) Use the graphing strategy to sketch a graph of $f(x) = (x - 2)e^x$.

Ex15) Use the given information to sketch the graph of f.

- Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
- Asymptotes: $x = -1$, $x = 1$, $y = 0$
- $f(-2) = 1$, $f(0) = 0$, $f(2) = 1$
- $f'(x) > 0$ on $(-\infty, -1) \cup (0, 1)$
- $f'(x) < 0$ on $(-1, 0) \cup (1, \infty)$
- $f''(x) > 0$ on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Section 5.5 Absolute Extrema

Definition. Maximum, Minimum. A function f has an **absolute maximum** at c if $f(c) \geq f(x)$ for all x in the domain of f. A function f has an **absoulte minimum** at c if $f(c) \leq f(x)$ for all x in the domain of f. In this case, we call $f(c)$ the maximum value or minimum value, repectively.

cf) A function f has an **local maximum** at c if $f(c) \geq f(x)$ when x is near c. A function f has an **local minimum** at c if $f(c) \leq f(x)$ when x is near c.

ex)

Extreme Value Theorem. A function f that is **continuous** on a **closed interval** $[a, b]$ has both an absolute maximum value and an absolute minimum value on that interval.

Finding Absolute Extrema on a Closed Interval

1. Check to make certain that f is continuous over $[a, b]$.
2. Find the critical values in the interval (a, b).
3. Evaluate f at the endpoints a and b and at the critical values found in step 2.
4. The largest value obtained from the previous step is the absolute maximum of $f(x)$ on $[a, b]$ and the smallest value obtained is the absolute minimum of $f(x)$ on $[a, b]$.

Ex16) Find the absolute extrema of the following functions on the given intervals.

a) $f(x) = 2x^3 - 3x^2 - 12x + 24$ on $[1, 4]$

b) $g(x) = x^3 + 3x^2 - 9x - 7$ on $[-2, 0]$

c) $h(x) = x^3 + 3x^2 - 9x - 7$ on $[-4, 0)$

Ex17) Find the absolute extrema of each function on the given interval.

a) $f(x) = 6x - x^2 + 4$ on $(0, \infty)$

b) $g(x) = 2x + \dfrac{8}{x}$ on $(0, 10)$

Section 5.6 Optimization and Modeling

Strategy for solving Optimization Problems

1. Introduce variables, look for relationships among these variables, and construct a mathematical model of the form.
2. Find the critical values of $f(x)$.
3. Use the procedures developed in Section 5.5 to find the absolute maximum (or minimum) value of $f(x)$ on the interval I and the value(s) of x where this occurs.
4. Use the solution to answer all questions asked in the problem.

Ex18) Find two positive numbers whose sum is 60 and whose product is a maximum.

Ex19) Find the dimensions of a rectangle of area 225 square centimeters that has the smallest perimeter. What is the perimeter?

Ex20) A university student center sells 1,600 cups of coffee per day at a price of $2.40. A market survey shows that for every $0.05 reduction in price 50 more cups of coffee will be sold. How much should the student center charge for a cup of coffee in order to maximize its revenue?

Ex21) A 300-room hotel in Las Vegas is filled to capacity every night at $80 a room. For each $1 increase in rent, 3 fewer rooms are rented. If each rented room costs $10 to service per day, how much should the management charge for each room to maximize gross profit? What is the maximum gross profit?

Ex22) An open-top box used to carry small toys is to be made from a 10 inch by 12 inch piece of cardboard by cutting identical squares from the corners and then folding up the flaps. Determine the dimensions of the box to maximize the volume.

Ex23) A homeowner has \$320 to spend on building a fence around a rectangular garden. Three sides of the fence will be constructed with wire fencing at a cost \$2 per foot. In order to provide a view block for a neighbor, the fourth side is to be constructed with wood fencing at a cost of \$6 per foot. Find the dimensions of the garden with largest area that can be enclosed with \$320 worth of fencing.