ex) If $f(6) = 12$, f' is continuous, and

$\int_6^7 f'(x)\,dx = 19$, what is the value of $f(7)$?

$$\boxed{\text{F.T.C II.} \quad \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}$$

$$\int_6^7 f'(x)\,dx = 19$$

|| F.T.C 2

$$f(x)\Big|_6^7$$

$$\Rightarrow f(7) - \underset{12}{\underset{\shortparallel}{f(6)}} = 19$$

$$\Rightarrow f(7) - 12 = 19$$

$$\therefore f(7) = 31$$

Definition. **Average Value of $f(x)$ over $[a, b]$.**

If $f(x)$ is continuous on $[a, b]$, we define the **average value of $f(x)$ on $[a, b]$** to be

$$\boxed{\frac{1}{b-a}\int_a^b f(x)\, dx}$$

Ex20) Find the average value of the given functions on the given intervals.

a) $f(x) = 5$ on $[0, 2]$

$$\text{ave.} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\text{ave.} = \frac{1}{2-0}\int_0^2 5 \; dx$$

$$= 5$$

b) $f(x) = -2x^3 + x$ on $[-2, 1]$

$$\text{ave.} = \frac{1}{1-(-2)}\int_{-2}^{1} (-2x^3 + x)\,dx$$

$$= 2$$

Ex21) Given the supply function $p = 10(e^{0.02x} - 1)$, find the average price (in \$) over the interval $[20, 30]$.

$$\text{ave.} = \frac{1}{30-20} \cdot \int_{20}^{30} 10 \cdot (e^{0.02x} - 1)\,dx$$

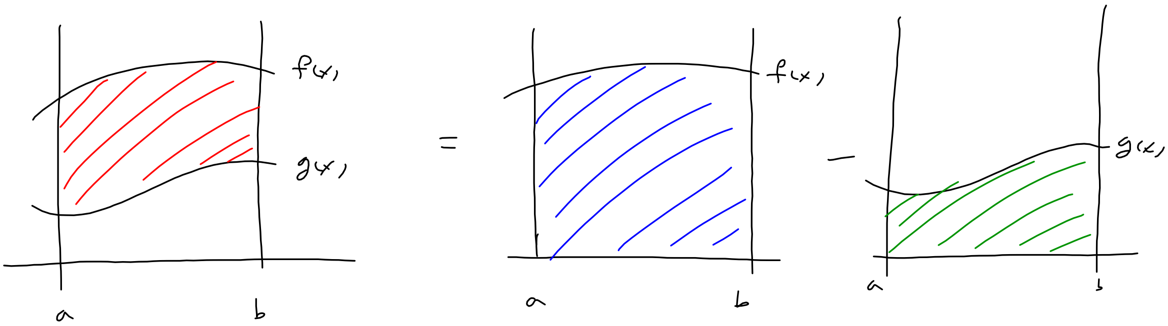
$$= \$6.51$$

Section 6.6 Area Between Two Curves

<u>Area Between Two Curves</u>

Let $y = f(x)$ and $y = g(x)$ be two continuous functions with $f(x) \geq g(x)$ on $[a, b]$. Then the area between the graphs of the two curves on $[a, b]$ is given by the definite integral

$$\int_a^b [f(x) - g(x)]\, dx$$

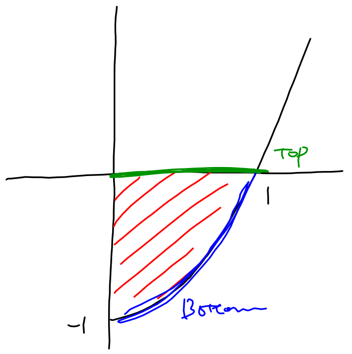


$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx$$

$$= \boxed{\int_a^b [\text{Top} - \text{Bottom}]\,dx}$$

Ex22) Find the area between $y = x^2 - 1$ and the x-axis on $[0, 1]$.

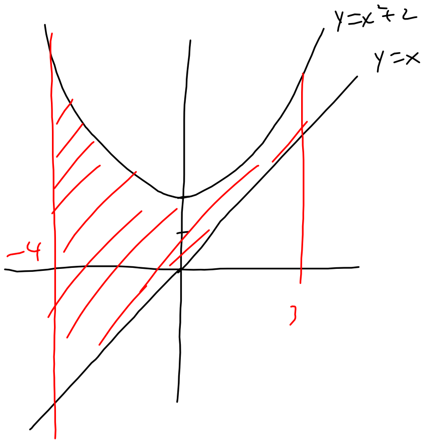


Top − Bottom

$$\text{Area} = \int_0^1 \left[0 - (x^2 - 1) \right] dx$$

$$= \frac{2}{3}$$

Ex23) Find the area between $y = x$ and $y = x^2 + 2$ on $[-4, 3]$.



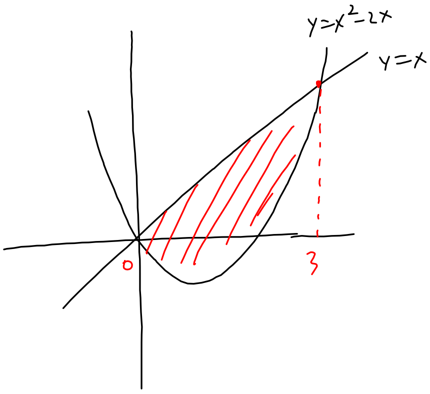
$$\text{Area} = \int_a^b [\text{Top} - \text{Bottom}]\, dx$$

$$= \int_{-4}^{3} \left[x^2 + 2 - (x) \right] dx$$

$$= \int_{-4}^{3} \left[x^2 - x + 2 \right] dx$$

$$= \frac{287}{6}$$

Ex24) Find the area bounded by the curves $y = x^2 - 2x$ and $y = x$.



Find Intersection

$: \ x^2 - 2x = x$

$\Rightarrow x^2 - 3x = 0$

$\Rightarrow x(x-3) = 0$

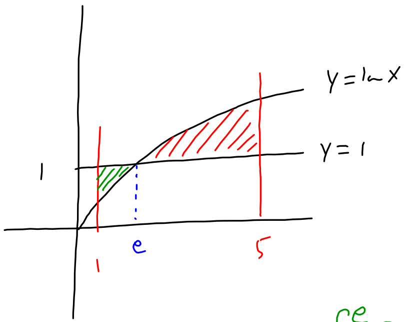
$\therefore x = 0, 3$

$$\text{Area} = \int_0^3 [\text{TOP} - \text{BOTTOM}]\, dx$$

$$= \int_0^3 [x - (x^2 - 2x)]\, dx$$

$$= \int_0^3 [-x^2 + 3x]\, dx = 4.5$$

Ex25) Find the area bounded by $y = \ln x$ and $y = 1$ on $[1, 5]$.



Find Intersection:

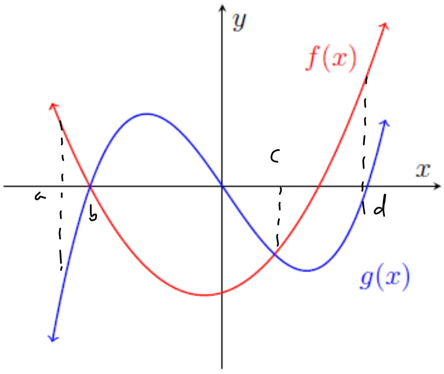
$\ln x = 1$

$\therefore x = e$

$$\text{Area} = \int_1^e [1 - \ln x]\, dx + \int_e^5 [\ln x - 1]\, dx$$

$$\approx 1.4838$$

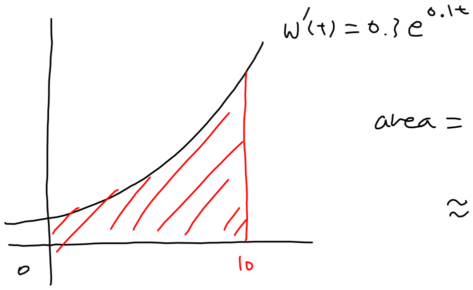
Ex26) Given the graph below write definite integrals to represent the total area bounded by $f(x)$ and $g(x)$ on $[a, d]$.



$$\text{Area} = \int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx$$

$$+ \int_c^d [f(x) - g(x)]\,dx$$

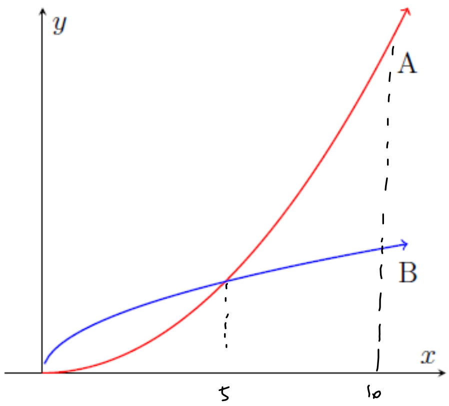
Ex27) A yeast culture is growing at a rate of $w'(t) = 0.3e^{0.1t}$ grams per hour. Find the area between the graph of $w'(t)$ and t-axis on the interval $[0, 10]$ and interpret the results.



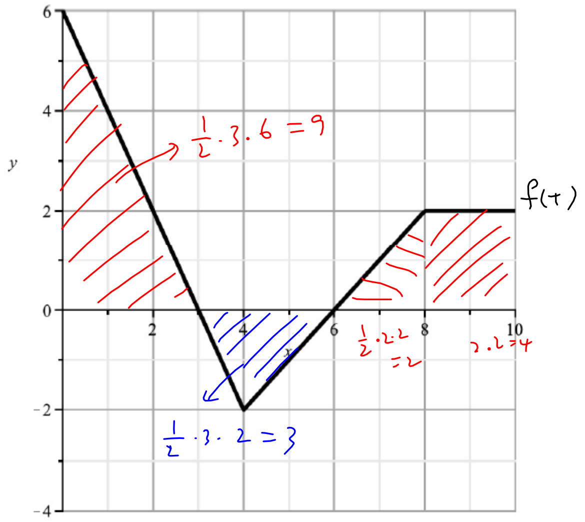
$$\text{area} = \int_0^{10} 0.3e^{0.1t}\,dt$$

$$\approx 5.1548 \text{ grams}$$

Ex28) The figure below shows the rate of growth of two trees. If the two trees are the same height at time $t = 0$, which tree is taller after 5 years? After 10 years?



Ex29) If $g(x) = \int_0^x f(t)dt$, where the graph of $f(t)$ is given below, where $0 \leq x \leq 10$, evaluate $g(0)$, $g(3)$, $g(6)$ and $g(10)$.



$g(0) = \int_0^0 f(t)\,dt = 0$

$g(3) = \int_0^3 f(t)\,dt = 9$

$g(6) = \int_0^6 f(t)\,dt = 9 - 3 = 6$

$g(10) = \int_0^{10} f(t)\,dt = 9 - 3 + 2 + 4$

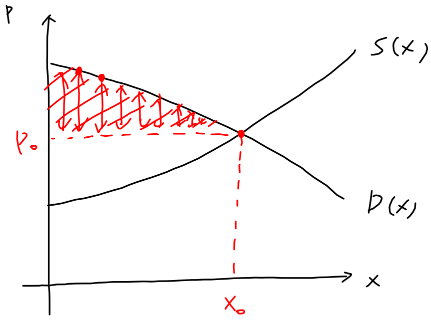
$= 12$

Section 6.7 Additional Applications of the Integral

Consumers' Surplus

Definition. If $p = D(x)$ is the demand equation, p_0 is the equilibrium price of the commodity, and x_0 is the equilibrium demand, then the **consumers' surplus** is given by

$$\int_0^{x_0} [D(x) - p_0]\, dx$$



$$\Rightarrow \text{area} = \int_0^{x_0} [\text{Top} - \text{Bottom}]\, dx$$

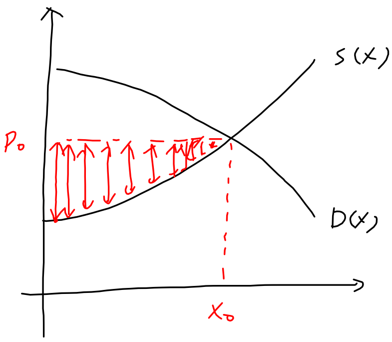
$$= \int_0^{x_0} [D(x) - p_0]\, dx = \text{C.S}$$

The **consumers' surplus** represents the total savings to consumers who are willing to pay more than p_0 for the product but are still able to buy the product for p_0.

Producers' Surplus

Definition. If $p = S(x)$ is the supply equation, p_0 is the equilibrium price of the commodity, and x_0 is the equilibrium demand, then the **producers' surplus** is given by

$$\int_0^{x_0} [p_0 - S(x)]\, dx$$



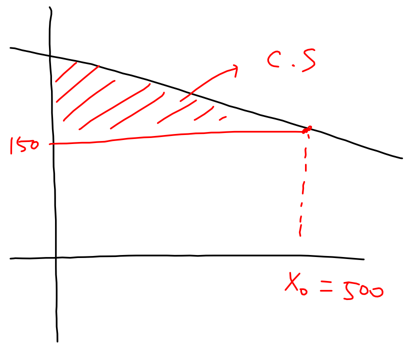
$$\text{Area} = \int_0^{x_0} [\text{Top} - \text{Bottom}]\, dx$$

$$= \int_0^{x_0} [P_0 - S(x)]\, dx = P.S$$

The **producers' surplus** represents the total gain to producers who are willing to supply units at a lower price than p_0 but are still able to supply units at p_0.

Ex30) Find the consumers' surplus at a price level of $150 for the price-demand equation $p = 400 - 0.5x$.

$P_0 =$



$D(x) = 400 - 0.5x$

$400 - 0.5x = 150$

$\Rightarrow -0.5x = -250$

$\therefore x = 500$

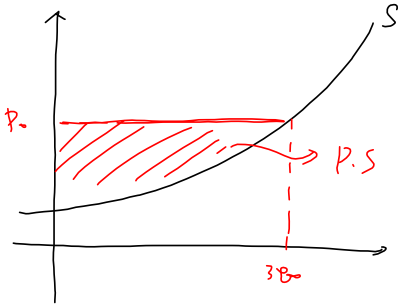
$$C.S = \int_0^{500} [D(x) - P_0]\, dx$$

$$= \int_0^{500} [400 - 0.5x - 150]\, dx$$

$$= \int_0^{500} [250 - 0.5x]\, dx$$

$$= \$62{,}500$$

Ex31) Find the producers' surplus when 380 $= x_0$ items are sold, if the supply equation is given by $p = e^{0.01x}$.



$S(x) = e^{0.01x}$

$\Rightarrow e^{0.01(380)} \approx 44.70 = P_0$

$$P.S = \int_0^{380} [P_0 - S(x)]\,dx$$

$$= \int_0^{380} [44.70 - e^{0.01x}]\,dx$$

$$\approx \$12{,}615.88$$

Ex32) If $p = D(x) = 80e^{-0.001x}$ and $p = S(x) = 30e^{0.001x}$, find the follwing:

a) Equilibrium Point (Intersection)

$D(x) = S(x)$

$\Rightarrow 80 \cdot e^{-0.001x} = 30 \cdot e^{0.001x}$

$\Rightarrow 80 \cdot \frac{1}{e^{0.001x}} = 30 \cdot e^{0.001x}$

$\Rightarrow 80 = 30 \cdot e^{0.002x}$

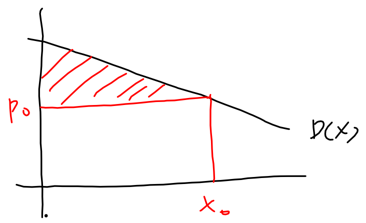
$\frac{8}{3} = e^{0.002x}$

$\Rightarrow \ln \frac{8}{3} = \ln e^{0.002x}$

$\Rightarrow \ln \frac{8}{3} = 0.002x$

$\therefore x = \frac{\ln \frac{8}{3}}{0.002} \approx 490.4146$

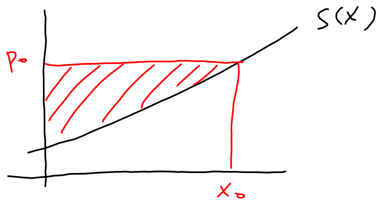
b) Consumers' surplus at the equilibrium price level. $\Rightarrow P_0 \approx 48.99$



$$C.S = \int_0^{490.4146} [80 \cdot e^{-0.001x} - 48.99]\,dx$$

$$\approx \$6984.89$$

c) Producers' surplus at the equilibrium price level.



$$P.S = \int_0^{490.4146} [48.99 - 30 \cdot e^{0.001x}]\,dx$$

$$\approx \$5035.52$$