

Chapter 1E. Complex Numbers.

Definition of Complex Number

A complex number is a number that can be written in the form $a + bi$, where a, b are real numbers and $i = \sqrt{-1}$. a is referred to as the **real part** and b is referred to as the **imaginary part**.

The standard form of the complex number is

$$a + bi.$$

ex) Write $\sqrt{24} - \sqrt{-64}$ in standard form.

$$= \sqrt{2^2 \cdot 2 \cdot 3} - \sqrt{-1} \cdot \sqrt{8^2}$$

$$= 2\sqrt{6} - 8\sqrt{-1}$$

$$= 2\sqrt{6} - 8i$$

The real part is $2\sqrt{6}$ and
the imaginary part is -8 .

Ex9) Find real numbers a and b such that

a) $(2a - 1) - 4i = -7 + \left(\frac{b}{3}\right)i$

real part.

$$(2a - 1) = -7$$
$$+1 \qquad \qquad +1$$

$$2a = -6$$

$$\therefore a = -3$$

imaginary part.

$$-4 = \frac{b}{3}$$

$$\boxed{\therefore b = -12}$$

b) $(a + b) + (a - b)i = 1$

real part

$$(a+b) = 1$$

imaginary part

$$(a-b) = 0$$

$$\Rightarrow \begin{cases} a+b = 1 \\ a-b = 0 \end{cases} \quad (+)$$

$$2a = 1$$

$$\therefore a = \frac{1}{2}, \quad b = \frac{1}{2}$$

Absolute value of a complex number

If $z = a + bi$ is a complex number,

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

The distance between two complex numbers z_1 and z_2 is

$$|z_1 - z_2|.$$

Ex10) Compute the absolute values of the following complex numbers.

$$\text{a) } |2 - 5i| = \sqrt{2^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$$

$$\text{b) } |2 + \sqrt{-9}| = |2 + 3i| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$\text{c) } |-5| = -(-5) = 5 \quad \text{or} \quad \sqrt{(-5)^2 + 0^2} = \sqrt{25} = 5$$

$$\text{d) } |-3i| = \sqrt{0^2 + (-3)^2} = \sqrt{9} = 3.$$

Properties of the absolute value of complex number

- For $z = a + bi$ a complex number, the absolute value of z represents the distance from the origin $(0, 0)$ to the point with coordinate (a, b) .
- Let z_1 and z_2 be any two complex numbers then

$$|z_1 z_2| = |z_1| |z_2|$$

and

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

- Let z_1 and z_2 be any two complex numbers then

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This inequality is called the **triangle inequality**.

Ex11) Show that the absolute value of the sum of $-3 + i$ and $2 + 2i$ is less than or equal to the sum of their absolute values.

$$\begin{aligned} |(-3+i) + (2+2i)| &= |(-3+2) + (1+2)i| \\ &= |-1+3i| = \sqrt{(-1)^2 + 3^2} \\ &= \sqrt{1+9} = \sqrt{10}. \end{aligned}$$

And.

$$\begin{aligned} |-3+i| + |2+2i| &= \sqrt{(-3)^2 + 1^2} + \sqrt{2^2 + 2^2} \\ &= \sqrt{9+1} + \sqrt{4+4} \\ &= \sqrt{10} + \sqrt{8} \end{aligned}$$

Since $\sqrt{10} < \sqrt{10} + \sqrt{8}$,

$$|(-3+i) + (2+2i)| < |-3+i| + |2+2i| .$$

Conjugate of a complex number

The conjugate of the complex number $z = a + bi$ is defined to be $a - bi$ denoted \bar{z} .

Ex12) Compute the conjugate of the following complex numbers.

a) $\overline{1 - 2i} = 1 + 2i$

b) $\overline{4} = 4$

c) $\overline{4i} = -4i$

d) $\overline{-2 + 4i} = -2 - 4i$

Properties of the conjugate of a complex number

$$1. z \cdot \bar{z} = |z|^2 \quad \because \bar{z} = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2 = |a+bi|^2 \\ = |z|^2$$

Let $z = a+bi$

2. If x is real number, then $\bar{x} = x$.

3. $\bar{\bar{z}} = z$ (the conjugate of the conjugate of z equals z .)

4. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (the conjugate of a sum is the sum of the conjugates)

$$\therefore \text{Let } z_1 = a+bi \Rightarrow \bar{z}_1 = a-bi$$

$$z_2 = c+di \Rightarrow \bar{z}_2 = c-di$$

$$\Rightarrow \overline{z_1 + z_2} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c) - (b+d)i \\ = (a-bi) + (c-di) = \bar{z}_1 + \bar{z}_2.$$

5. $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$ (The conjugate of a product is the product of the conjugates)

$$6. \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

(The conjugate of a quotient is the ~~conjugate~~ of the conjugates)
quotient

Ex13) Express the following in the form $a + bi$.

$$\begin{aligned}
 \text{a) } \frac{7+3i}{4i} &= \frac{7+3i}{4i} \cdot 1 = \frac{7+3i}{4i} \cdot \frac{4i}{4i} = \frac{(7+3i) \cdot 4i}{16i^2} \\
 &= \frac{28i + 12i^2}{16i^2} = \frac{28i - 12}{-16} \\
 &= \frac{+12i}{-16} + \frac{28i}{-16} = \frac{3}{4} - \frac{7}{4}i
 \end{aligned}$$

$$\therefore \text{real: } \frac{3}{4}, \quad \text{imaginary: } -\frac{7}{4}$$

$$\begin{aligned}
 \text{b) } \frac{3+5i}{1-2i} &= \frac{3+5i}{1-2i} \cdot 1 = \frac{3+5i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{3+6i+5i+10i^2}{1-(2i)^2} \\
 &= \frac{3+11i-10}{1-(-4)} = \frac{-7+11i}{5} \\
 &= -\frac{7}{5} + \frac{11}{5}i
 \end{aligned}$$

$$\text{real: } -\frac{7}{5}, \quad \text{imaginary: } \frac{11}{5}$$