$$P(t) = P_0 \cdot e^{rt}$$

## Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is directly proportional to the mass of the substance. This is analogous to population growth, except that the mass of radioactive material *decreases*. It can be shown that the mass $m(t)$ remaining at time $t$ is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r$ is the rate of decay expressed as a proportion of the mass and $m_0$ is the initial mass. Physicists express the rate of decay in terms of **half-life**, the time required for half the mass to decay.

### Radioactive Decay Model

If $m_0$ is the initial mass of a radioactive substance with half-life $h$, then the mass remaining at time $t$ is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$.

If $h$ : half-life, then a mass of 1 unit becomes $\frac{1}{2}$ unit when $t = h$

$$\Rightarrow \quad m(h) = \frac{1}{2} m_0$$

$$\boxed{\Rightarrow \quad \frac{1}{2} \cancel{m_0} = \cancel{m_0} \cdot e^{-rh}}$$

$$\Rightarrow \quad \frac{1}{2} = e^{-rh}$$

take ln

$$\Rightarrow \quad \ln \underbrace{\frac{1}{2}}_{=2^{-1}} = \ln e^{-rh}$$

$$\Rightarrow \quad -\ln 2 = -rh \cdot \underbrace{\ln e}_{=1}$$

$$\Rightarrow \quad \frac{-\ln 2}{-h} = \frac{-rh}{-h}$$

$$\therefore \quad r = \frac{-\ln 2}{-h} = \frac{\ln 2}{h}.$$

half-life

Ex18) The half life of uranium 235 is $7.1 \times 10^8$ years. If we start out with 1.5 kilograms of $^{235}U$ in 2014, how much uranium will be left after 10,000 years?

$m_0 = 1.5$

$$m(t) = m_0 \cdot e^{-rt}$$

Since half-life $= 7.1 \times 10^8$ years

$$\Rightarrow \frac{1}{2} \cancel{m_0} = \cancel{m_0} \cdot e^{-r(7.1\times10^8)}$$

$$\Rightarrow \frac{1}{2} = e^{-r\cdot(7.1\times10^8)}$$

take ln

$$\Rightarrow \ln\frac{1}{2} = \ln e^{-r(7.1\times10^8)}$$

$$= -r\cdot(7.1\times10^8)\cdot \underbrace{\ln e}_{=1}$$

$$\Rightarrow \ln\frac{1}{2} = \frac{-r\cdot(7.1\times10^8)}{-7.1\times10^8}$$

$$= \ln 2^{-1}$$

$$= \frac{-\ln 2}{-7.1\times10^8}$$

$$\therefore r = \frac{-\ln 2}{-7.1\times10^8} = \frac{\ln 2}{7.1\times10^8}$$

$$\boxed{\therefore m(t) = 1.5 \cdot e^{-\left(\frac{\ln 2}{7.1\times10^8}\right)\cdot t}}$$

After 10,000 years,

$$m(10000) = 1.5 \cdot e^{-\left(\frac{\ln 2}{7.1\times10^8}\right)\cdot(10,000)} \approx 1.499 \text{ kg}$$

$\therefore$ $^{235}U$ has decayed about 0.001 kg after 10,000 years

Ex19) Suppose a radioactive substance satisfies the exponential decay law $A(t) = A(0) \cdot 4^{-t}$, where $t$ is in centuries. What is the half-life of this substance?

Find $t$ which satisfy $A(t) = \frac{1}{2}A(0)$

$$\Rightarrow \frac{1}{2}\cancel{A(0)} = \cancel{A(0)} \cdot 4^{-t}$$

$$\Rightarrow \frac{1}{2} = 4^{-t}$$

take ln

$$\Rightarrow \ln \frac{1}{2} = \ln 4^{-t}$$

$$\Rightarrow \ln 2^{-1} = \ln 4^{-t}$$

$$\Rightarrow \frac{-\ln 2}{-\ln 4} = \frac{-t \cdot \ln 4}{-\ln 4}$$

$$\therefore t = \frac{-\ln 2}{-\ln 4} = \frac{\ln 2}{\ln 4} = \frac{\ln 2}{\ln 2^2} = \frac{\cancel{\ln 2}}{2 \cdot \cancel{\ln 2}} = \frac{1}{2}$$

Thus, the half-life of this substance equals 50 years.

### Newton's Law of Cooling

If $D_0$ is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature $T_s$, then the temperature of the object at time $t$ is modeled by the function

$$T(t) = T_s + D_0\, e^{-kt}$$

where $k$ is a positive constant that depends on the type of object.

$T_s = 70$

Ex20) A cup of coffee has a temperature of $200°F$ and is place in a room that has a temperature of $70°F$. After 10 minutes the temperature of the coffee is $150°F$.

a) Find the temperature of the coffee after 15 minutes.

b) When will the coffee have cooled to $100°F$.

$$T(t) = T_s + D_0 \cdot e^{-kt}$$

$T_s = 70, \quad D_0 = 200 - 70 = 130$

$$\Rightarrow T(t) = 70 + 130 \cdot e^{-kt}$$

after 10 min $\Rightarrow t = 10$

$\Rightarrow T(10) = 150$

$$T(10) = 70 + 130 \cdot e^{-k\cdot(10)} = 150$$

$-70 \qquad -70$

$$\Rightarrow \frac{130 \cdot e^{-10k}}{130} = \frac{80}{130}$$

$$\Rightarrow e^{-10k} = \frac{8}{13}$$

take ln

$$\Rightarrow \frac{-10k}{-10} = \frac{\ln \frac{8}{13}}{-10}$$

$$\therefore k = \frac{\ln \frac{8}{13}}{-10} \approx 0.04855$$

$$\boxed{\therefore T(t) = 70 + 130 \cdot e^{-0.04855 \cdot t}}$$

a) After 15 min. $\Rightarrow T(15)$

$$T(15) = 70 + 130 \cdot e^{-0.04855 \cdot (15)} \approx 133\,°F$$

b) $T(t) = 100$

$$\Rightarrow 70 + 130 \cdot e^{-0.04855 \cdot t} = 100$$

$-70 \qquad -70$

$$\Rightarrow \frac{130 \cdot e^{-0.04855 t}}{130} = \frac{30}{130}$$

$$\Rightarrow e^{-0.04855 \cdot t} = \frac{3}{13}$$

take ln

$$\Rightarrow \frac{-0.04855\, t}{-0.04855} = \frac{\ln \frac{3}{13}}{-0.04855}$$

$$\therefore t \approx 30.2 \text{ min.}$$

In Sum.

- exponential growth: $P(t) = P_0 \cdot e^{rt}$
- exponential decay: $m(t) = m_0 \cdot e^{-rt}$
- Newton's Law of Cooling: $T(t) = T_s + D_0 \cdot e^{-kt}$

# Chapter 7. Systems of Equations

## Chapter 7A. Systems of Linear Equations

A **system of equations** is a set of equations that involves the same variables. A **solution** of a system is an assignment of values for the variables that makes each equation in the system true.

To **solve** a system means to find all solutions of the system.

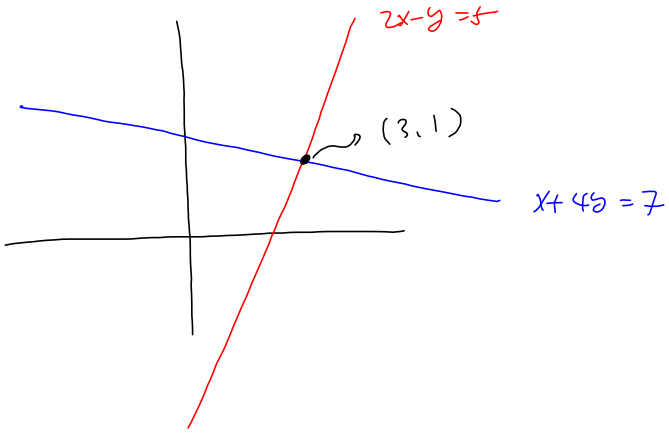
ex) $\begin{cases} 2x - y = 5 & - ① \\ x + 4y = 7 & - ② \end{cases}$

Check $x = 3$ and $y = 1$ is a solution of system.

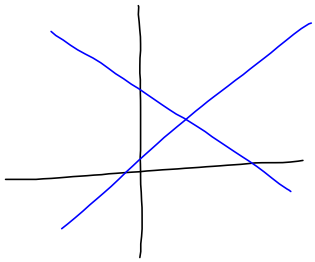
| Eq (1) | Eq (2) |
|---|---|
| $2x - y = 5$ | $x + 4y = 7$ |
| $2(3) - (1) = 5$ | $(3) + 4(1) = 7$ |
| $6 - 1 = 5$ ✓ | $3 + 4 = 7$ ✓ |

$\therefore$ $x = 3$ and $y = 1$ is a solution of system can also be written as the ordered pair $(3, 1)$

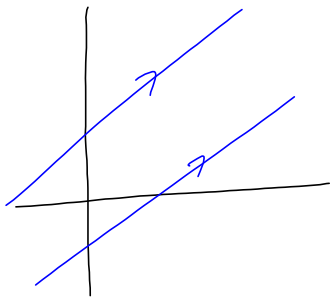
Graphically,




<u>Note</u> The solution of linear system is the point of intersection of the two lines

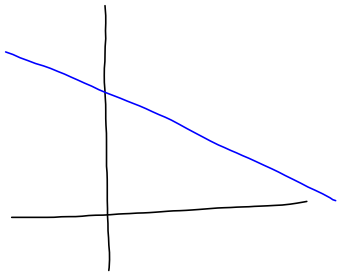


only one solution



line parallel

$\Rightarrow$ no solution



identical lines

$\Rightarrow$ infinite number of solutions

## Substitution Method

1. **Solve for One Variable.** Choose one equation and solve for one variable in terms of the other variable.
2. **Substitute.** Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.
3. **Back-Substitute.** Substitute the value you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

Ex21) Find all solutions of the system.

$$2x + y = 1 \quad — ①$$
$$3x + 4y = 14 \quad — ②$$

Step 1) Solve ① for $y$

$$2x + y = 1$$
$$-2x \quad -2x$$
$$\Rightarrow y = 1 - 2x \quad — (*)$$

Step 2) Substitute $(*)$ into ②

$$\Rightarrow 3x + 4(1-2x) = 14$$
$$\Rightarrow 3x + 4 - 8x = 14$$
$$\Rightarrow -5x = 10$$
$$\therefore x = -2$$

Step 3) Back-Substitute into $(*)$

$$y = 1 - 2(-2) = 1 + 4 = 5$$

$\therefore x = -2,\ y = 5$ or $(-2, 5)$ : solution

Ex22) Solve the system

$$x - 3y = 6$$
$$2x + 5y = 7$$

$$\left( \frac{51}{11}, -\frac{5}{11} \right)$$

Ex23) Solve the system

$$2x - 6y = 5 \quad — ①$$
$$-x + 3y = 4 \quad — ②$$

Step 1) Solve ② for x

$-x + 3y = 4$

$\Rightarrow x = 3y - 4 \quad — (*)$

Step 2) Substitute (*) into ①

$\Rightarrow 2(3y-4) - 6y = 5$

$\Rightarrow \cancel{6y} - 8 - \cancel{6y} = 5$

$\Rightarrow -8 = 5$ ✗ $\therefore$ no solution