Chapter 2B. Solving Inequalities

$a < b$ a is less than b

$a \leq b$ a is less than or equal to b

$a > b$ a is greater than b

$a \geq b$ a is greater than or equal to b

ex) $x = 4$ $x \geq 4$

Notation

1. (a, b) represents the set of all real numbers x such that $a < x < b$. (All numbers between a and b)
2. $[a, b]$ represents the set of all real numbers x such that $a \leq x \leq b$. (including two endpoints of interval)
3. The set $(-\infty, \infty)$ is used to represent all real numbers.

In summary open parentheses mean exclude the endpoint and square brackets mean include the endpoint.

Ex1) Which of the following numbers $(-7, -5, -3, -1, 0)$ are in the interval $[-5, -1)$,

Linear Inequalities

To solve a linear inequality, isolate the variable in the same fashion as solving a linear equation. However, there is one major difference: Whenever you multiply or divide an inequality by a negative number, you must switch the inequality symbol (or direction).

ex) $5 < 7$ $\quad (-1) \quad (-1)$

$-5 > -7$

Rules for Inequalities (A,B and C: real numbers)

1. $A \leq B \iff A + C \leq B + C$
2. $A \leq B \iff A - C \leq B - C$
3. If $C > 0$, then $A \leq B \iff CA \leq CB$
4. If $C < 0$, then $A \leq B \iff CA \geq CB$
5. If $A > 0$, and $B > 0$ then $A \leq B \iff \frac{1}{A} \geq \frac{1}{B}$ ex) $2 < 10 \Rightarrow \frac{1}{2} > \frac{1}{10}$
6. If $A \leq B$ and $C \leq D$, then $A + C \leq B + D$

Ex2) Solve the inequality $3x < 9x + 4$

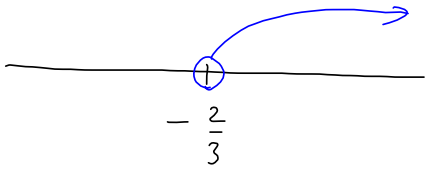
$-9x \quad -9x$

$\Rightarrow \quad -6x < 4$

$\frac{}{-6} \quad \frac{}{-6}$

$\Rightarrow \quad x > -\frac{4^2}{6_3} = -\frac{2}{3}$

$\therefore \quad x > -\frac{2}{3}$



or $\left(-\frac{2}{3}, +\infty\right)$

Ex3) Solve for y: $\frac{(y-1)}{3} - \frac{3y}{5} > -1$ $\cdot 15$ multiply LCD = 15

$\Rightarrow \quad 5(y-1) - (3y)\cdot 3 > -15$

$\Rightarrow \quad 5y - 5 - 9y > -15$

$\Rightarrow \quad -4y - 5 > -15$

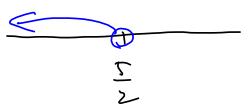
$+5 \quad +5$

$\Rightarrow \quad -4y > -10$

$\frac{}{-4} \quad \frac{}{-4}$

$\Rightarrow \quad y < \frac{+10^5}{+4_2} = \frac{5}{2}$

$\therefore \quad y < \frac{5}{2}$ or



or $\left(-\infty, \frac{5}{2}\right)$

Pair of Simultaneous Inequalities

$a < x < b$ means the number x is between a and b. More formally, we say that

$$a < x \text{ and } x < b.$$

Ex4) Solve the inequalities $4 \leq 3x - 2 < 13$.

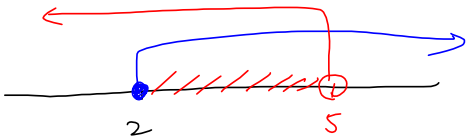
Method 1) Directly solve, because x is isolated.

$$4 \leq 3x - 2 < 13$$
$$+2 \quad +2 \quad +2$$
$$6 \leq 3x < 15$$
$$\frac{6}{3} \leq \frac{3x}{3} < \frac{15}{3}$$
$$\therefore 2 \leq x < 5 \quad \text{or} \quad [2, 5)$$

Method 2) The combined inequality can be rewritten as

$$3x - 2 \geq 4 \quad \text{and} \quad 3x - 2 < 13$$
$$+2 \quad +2 \qquad +2 \quad +2$$
$$\frac{3x}{3} \geq \frac{6}{3} \qquad \frac{3x}{3} < \frac{15}{3}$$
$$\therefore x \geq 2 \quad \text{and} \quad x < 5$$

$\cap$: intersection



$$\therefore 2 \leq x < 5 \quad \text{or} \quad [2, 5).$$

Ex5) Solve for x: $x + 4 < 2x - 7 < 3 - 3x$.

$x+4<2x-7$

$-2x \quad -2x$

$-4 \quad -4$

$-x < -11$

$-1 \quad -1$

$x > 11$

and

$2x - 7 < 3 - 3x$

$+3x \quad +3x$

$+7 \quad +7$

$5x < 10$

$5 \quad 5$

$x < 2$

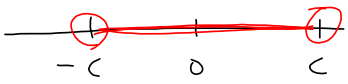
∴ No solution

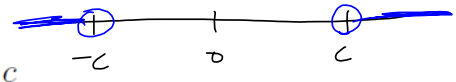
Absolute Value Inequalities

We use the following properties to solve inequalities that involve absolute value.

Properties of Absolute Value Inequalities

1. $|x| < c \iff -c < x < c$
2. $|x| \leq c \iff -c \leq x \leq c$
3. $|x| > c \iff x < -c \text{ or } x > c$
4. $|x| \geq c \iff x \leq -c \text{ or } x \geq c$





Ex6) Solve the inequality $|x - 5| < 2$

Method 2) Use Sign Chart

Find $|x-5| = 2$

$x - 5 = 2$
$+5 \quad +5$
$\therefore x = 7$

$x - 5 = -2$
$+5 \quad +5$
$x = 3$

Method 1) use properties

$-2 < x - 5 < 2$
$+5 \quad +5 \quad +5$

$3 < x < 7$ //

Sign chart for $|x-5| < 2$

take $x = -100$: $|-100-5| < 2$, $|-105| < 2$, $105 \not< 2$

take $x = 5$: $|5-5| < 2$, $0 < 2$ ✓

take $x = 100$: $|100-5| < 2$, $|95| < 2$, $95 \not< 2$

F 3 T 7 F

$\therefore 3 < x < 7$ or $(3, 7)$

Ex7) Solve the inequality $|3x+2|-6 \geq -2$

$+6 \quad +6$

$\Rightarrow |3x+2| \geq 4$

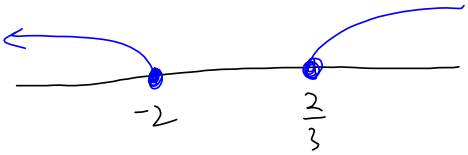
Method 1) Use properties

$3x+2 \leq -4$ or $3x+2 \geq 4$

$-2 \quad -2 \qquad -2 \quad -2$

$\frac{3x}{3} \leq \frac{-6}{3} \qquad \frac{3x}{3} \geq \frac{2}{3}$

$\therefore x \leq -2$ or $x \geq \frac{2}{3}$



$\therefore x \leq -2$ or $x \geq \frac{2}{3}, \Leftrightarrow (-\infty, -2] \cup [\frac{2}{3}, +\infty)$

Method 2) Use sign chart.

Find $|3x+2| = 4$

$3x+2 = 4 \qquad 3x+2 = -4$

$3x = 2 \qquad 3x = -6$

$x = \frac{2}{3} \qquad x = -2$

Sign chart for $|3x+2| \geq 4$

take $x=-100$: $|-300+2| \geq 4$, $|-298| \geq 4$, $298 \geq 4$

take $x=0$: $|0+2| \geq 4$, $|2| \not\geq 4$

take $x=100$: $|300+2| \geq 4$, $|302| \geq 4$

T -2 F $\frac{2}{3}$ T

$\therefore x \leq -2$ or $x \geq \frac{2}{3}$

$\Leftrightarrow (-\infty, -2] \cup [\frac{2}{3}, +\infty)$

Ex8) Solve the inequality $|3x-4|+6 \leq 3$

$-6 \quad -6$

$\Rightarrow |3x-4| \leq -3$

$\therefore$ no solution.

Nonlinear Inequalities

If an inequality is not linear,

1. **Move all terms to one side.** If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign. If the nonzero side of the inequality involves quotients, bring them to a common denominator.
2. **Factor.** Factor the nonzero side of the inequality.
3. **Find the intervals.** Determine the values for which each factor is zero. These numbers will divide the real line into intervals. List the intervals determined by these numbers.
4. **Solve.** To determine which, test a number in the interval by substituting it into the inequality. If the chosen number satisfies the inequality, then so does every other number in the interval, and if the number does not satisfy the inequality, then no other number from that interval will satisfy the inequality.

Sign chart.

Ex9) Solver the inequality $x^2 - 5x + 6 \leq 0$

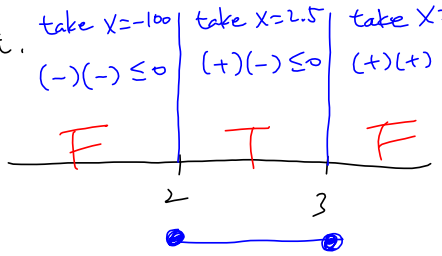
$A=+$
$h=6$: -2 -3

Factoring : $(x-2)(x-3) \leq 0$

$x = 2, 3$

Sign chart.

take $x=-100$	take $x=2.5$	take $x=100$
$(-)(-) \leq 0$	$(+)(-) \leq 0$	$(+)(+) \leq 0$
F	T	F

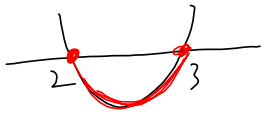


$\therefore$ $2 \leq x \leq 3$ or $[2, 3]$

Method 2) by properties

$2 \leq x \leq 3$

Method 3) by graph



$2 \leq x \leq 3$

Ex10) Solve for x: $x^2 - 4x > 12$

$-12 \quad -12$

$x^2 - 4x - 12 > 0$

$A = -4$
$M = -12$: $2, -6$

$\Rightarrow (x+2)(x-6) > 0$

$x = -2 \quad | \quad x = 6$

Sign Chart

take $x = -100$: $(-)(-) > 0$

take $x = 0$: $(+)(-) > 0$

take $x = 100$: $(+)(+) > 0$

T F T

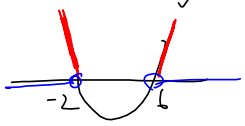
$-2 \qquad 6$

$(-\infty, -2) \cup (6, +\infty)$

Method 2) Use properties

$x < -2$ or $x > 6$

Method 3) by Graph.



$x < -2$, or $x > 6$

Ex11) Solve for x: $x^4 + 3x^3 - 4x^2 \geq 0$