Math 131 In Class Exam 2 Review Solutions

1. The limit definition of the derivative function is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For $f(x) = e^x + x^2$, this is

$$f'(x) = \lim_{h \to 0} \frac{e^{(x+h)} + (x+h)^2 - (e^x + x^2)}{h}$$

2. Tangent line: $y = f'(a)(x-a) + f(a)$

a) $f'(x) = 3x^{-1/4} - 2$

$f'(16) = 3(16^{-1/4}) - 2 = \frac{3}{2} - 2 = -\frac{1}{2}$

$f(16) = 4(16^{3/4}) - 2(16) + 9$

$= 4(8) - 32 + 9 = 9$

$y = -\frac{1}{2}(x - 16) + 9$ is the tangent line.

b) $f'(x) = e^x + \frac{1}{x}$ $\quad f'(1) = e + 1$

$f(1) = e + \ln 1 = e$

$y = (e+1)(x-1) + e$ is the tangent line

3. Look for V.A.'s of f' which make V.T.'s in f, corners, and any discontinuities.

a) $f(x) = x^{2/5}$ $f'(x) = \frac{2}{5}x^{-3/5} = \frac{2}{5}\cdot\frac{1}{x^{3/5}}$

f' has a V.A. @ $x=0$ so f has a V.T.
that is f' DNE at $x=0$.
Note: f is continuous for all x.

b) $f(x) = (x^2-9)^{2/5}$ f is continuous for all x

$$f'(x) = \frac{2}{5}(x^2-9)^{-3/5}(2x) = \frac{4x}{5(x^2-9)^{3/5}}$$

At $x=-3$ and $x=3$, f' DNE (V.A.)
f has V.T.

c) $|x|$ has a corner at $x=0$
so f is not differentiable at $\boxed{x=0}$

Check continuity at 1. ✓ $\lim_{x\to 1}|x| = 1 = \lim_{x\to 1}x^2 = f(1)$

x^2 is differentiable everywhere and $6(x-3)+12$ is also.
Is f continuous at $x=3$? $9 = 3^2 = \lim_{x\to 3^-} f(x)$

but $\lim_{x\to 3^+} f(x) = 6(3-3)+12 = 12$ Not Cont.

So f is not differentiable at $\boxed{x=3}$

Question:
If we make continuous by changing the right piece to $6(x-3)+9$, then is f differentiable? Slope of x^2 at 3 is $2x|_{x=3} = 6$ and the slope of $6(x-3)+9$ is 6 so yes.

4. $h(1) = f(g(1))$ Since the tangent to g at 1 is $y = -x + 4$, $g(1) = -1 + 4$ and $g'(1) = -1$.

$g(1) = y(1) = -1 + 4 = 3$ $\boxed{h(1) = f(3) = 3^2 - 2 = 7}$

$g'(1) = -1$ $h'(1) = f'(g(1))\,g'(1)$ Chain Rule

$= f'(3)\,g'(1).$

$f'(u) = 2u$ so $f'(3) = 6$

$\boxed{h'(1) = 6(-1) = -6}$

$y = -6(x-1) + 7$ is the tangent to h at $x = 1$.

5. Pro ct Rule $(f \cdot g)' = f'g + fg'$

Quotient Rule $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

a) $\frac{d}{dx}\left[\frac{x}{e^{2x}+1}\right] = \frac{e^{2x}+1 - x(e^{2x} \cdot 2)}{(e^{2x}+1)^2}$

$= \frac{e^{2x}+1-2xe^{2x}}{(e^{2x}+1)^2}$

b) $f(x) = \frac{x^2}{x} + \frac{4\cancel{x}}{\cancel{x}} - \frac{7}{x} = x + 4 - 7x^{-1}$

$f'(x) = 1 + 7x^{-2} = 1 + \frac{7}{x^2}$

5c) Rewrite $f(x)$ as

$3\ln(x^2+4) + \frac{1}{4}\ln x - \ln(e^x+1)$

$$f'(x) = 3\cdot\frac{2x}{x^2+4} + \frac{1}{4}\cdot\frac{1}{x} - \frac{e^x}{e^x+1}$$

$$= \frac{6x}{x^2+4} + \frac{1}{4x} - \frac{e^x}{e^x+1}.$$

d) $\frac{d}{dx}\left[\cos(\pi x)\right] = -\pi\sin(\pi x)$

e) $\frac{d}{dx}\left[\sin^2 x\right] = \frac{d}{dx}\left[(\sin x)^2\right]$

$= 2\sin x\cos x$

f) $\frac{d}{dx}\left[\tan(x^2)\right] = 2x\sec^2(x^2)$

g) $\frac{d}{dx}\left[\sec(3x)\right] = 3\sec(3x)\tan(3x)$

6. a) $f(x) = (x+2)^2(x-3)^3$

$$f'(x) = 2(x+2)(x-3)^3 + 3(x+2)^2(x-3)^2$$

$$= (x+2)(x-3)^2[2(x-3) + 3(x+2)]$$

$$= (x+2)(x-3)^2[\,2x-6 + 3x+6\,]$$

$$= (x+2)(x-3)^2\, 5x$$

f' $+ + +$ | $- - -$ | $+ + +$ | $+ + + +$

$-2 \qquad 0 \qquad 3$

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local max at $x=-2$ local min at $x=0$

$$f'(x) = (x+2)(x-3)^2\, 5x = (5x^2+10x)(x-3)^2$$

$$f''(x) = (10x+10)(x-3)^2 + (5x^2+10x)\cdot 2(x-3)$$

$$= (x-3)\big((10x+10)(x-3) + (5x^2+10x)\cdot 2\big)$$

$$= (x-3)(10x^2 \cancel{-20x} - 30 + 10x^2 \cancel{+20x})$$

$$= (x-3)(20x^2 - 30)$$

$$= (x-3)\cdot 10(2x^2-3)$$

$$= 10(x-3)(\sqrt{2}\,x - \sqrt{3})(\sqrt{2}\,x + \sqrt{3})$$

inflection pts. at $x=3$ $x = \frac{\sqrt{3}}{\sqrt{2}}$ and $x = -\frac{\sqrt{3}}{\sqrt{2}}$

f'' $-$ | $+$ | $-$ | $+$

6 b) $f(x) = xe^x$

$f'(x) = e^x + xe^x = (1+x)e^x$

$e^x > 0$ for all x so the sign chart for f' is the same as the sign chart for $x+1$.

sign f' $\quad - - - \, 0 \, + + +$

$\qquad\qquad\quad -1$

$f \qquad \searrow \qquad \nearrow$

local min at $x = -1$

$f''(x) = e^x + (1+x)e^x = (2+x)e^x$

sign f'' $\quad - - - \, | \, + + +$ $\qquad$ inflection pt. at $x = -2$

$\qquad\qquad\quad -2$

$f \qquad \frown \qquad \smile$

7. $v(t)$ is <u>positive</u> and $a(t)$ is <u>positive</u>.

... $v(t)$ has a local max, $a(t)$ is <u>0</u> and f has an <u>inflection pt.</u>. Note: $a(t)$ must change sign for v to have a local max.

If $g' < 0$ and decreasing then g is decreasing and concave down.

8. a) $f'(t) = \frac{1}{(\arcsin t)} \cdot \frac{1}{\sqrt{1-t^2}}$

b) $f'(t) = \frac{1}{1+(e^t)^2} \cdot e^t = \frac{e^t}{1+e^{2t}}$