1. Write the difference quotient \( \frac{f(x) - f(a)}{x - a} \) and simplify it, so \( x - a \) can be cancelled, if possible.

Find \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \) by substituting \( x = a \) in the simplified D.Q.

Or use \( \frac{f(x + h) - f(x)}{h} \), simplify and cancel \( h \), find \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) by substituting \( h = 0 \) in the simplified D.Q.

\( a) \ f(x) = \sqrt{x} \)

\[
\frac{\sqrt{x + h} - \sqrt{x}}{h} = \frac{(\sqrt{x + h} - \sqrt{x})(\sqrt{x + h} + \sqrt{x})}{h(\sqrt{x + h} + \sqrt{x})} = \frac{h}{h(\sqrt{x + h} + \sqrt{x})} = \frac{1}{\sqrt{x + h} + \sqrt{x}}
\]

\[
= \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})} = \frac{h}{h(\sqrt{x + h} + \sqrt{x})} = \frac{1}{\sqrt{x + h} + \sqrt{x}}
\]

\[
\lim_{h \to 0} \frac{1}{\sqrt{x + h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = f'(x)
\]

Or \( \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}} \)

\[
\lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}} = f'(a) \text{ for any } a \neq 0.
\]

So \( \frac{1}{2\sqrt{x}} = f'(x) \)
1b) \[ \frac{\sqrt{7x^2+5} - \sqrt{7a^2+5}}{x-a} \] is the D. Q. We conjugate to get:

\[
\left( \frac{\sqrt{7x^2+5} - \sqrt{7a^2+5}}{x-a} \right) \left( \frac{\sqrt{7x^2+5} + \sqrt{7a^2+5}}{\sqrt{7x^2+5} + \sqrt{7a^2+5}} \right)
\]

\[
= \frac{7x^2+5 - (7a^2+5)}{(x-a)(\sqrt{7x^2+5} + \sqrt{7a^2+5})} = \frac{7(x^2-a^2)}{(x-a)(\sqrt{7x^2+5} + \sqrt{7a^2+5})}
\]

\[
= \frac{7(x+a)}{\sqrt{7x^2+5} + \sqrt{7a^2+5}}
\]

\[
\lim_{x \to a} \frac{7(x+a)}{\sqrt{7x^2+5} + \sqrt{7a^2+5}} = \frac{7(2a)}{2\sqrt{7a^2+5}} = \frac{7a}{\sqrt{7a^2+5}}
\]

\[ f'(x) = \frac{7x}{\sqrt{7x^2+5}} \]

Note the chain rule gives

\[ f'(x) = \frac{1}{2}(7x^2+5)^{-\frac{1}{2}} (14x) = \frac{7x}{\sqrt{7x^2+5}} \]
1c) \[ \frac{f(x) - f(a)}{x - a} = \frac{1}{x} - \frac{1}{a} = \frac{a - x}{xa} = \frac{x}{x - a} \]

\[ = \frac{a - x}{ax(x - a)} = \frac{-1}{ax} \]

\[ \lim_{x \to a} \frac{-1}{ax} = \frac{-1}{a^2} \quad f'(x) = \frac{-1}{x^2} \]

d) \[ \frac{1}{x^2 + 4} - \frac{1}{a^2 + 4} \]

\[ = \frac{a^2 + 4 - (x^2 + 4)}{(x - a)(x^2 + 4)(a^2 + 4)} \]

\[ = \frac{a^2 - x^2}{(x - a)(x^2 + 4)(a^2 + 4)} = \frac{(a - x)(a + x)}{(x - a)(x^2 + 4)(a^2 + 4)} \]

but \( a - x = -(x - a) \) so this is

\[ \frac{-2x}{(x^2 + 4)(a^2 + 4)} \]

\( x \to a \)

\[ f'(x) = \frac{-2x}{(x^2 + 4)^2} \]
2a) \( f(1) = 5+10+3 = 18 \) The Pt is \((1, 18)\)
\[ f'(x) = 10x + 10 \quad f'(1) = 20 \] The Slope is 20
Pt. Slope \( y - y_1 = m(x - x_1) \)
\[ y - 18 = 20(x - 1) = 20x - 20 \]
\[ y = 20x - 20 \]

b) \( f'(4) = \frac{16-16+32}{2} = 16 \) Pt \((4, 16)\)
\[ f'(x) = \frac{(2x-4)\sqrt{x} - (x^2-4x+32) \frac{1}{2\sqrt{x}}}{x} \]

Or Rewrite \( f(x) = x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 32x^{-\frac{1}{2}} \)
so \( f'(x) = \frac{3}{2} x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} = 16 x^{-\frac{3}{2}} \)

Either way, the slope is \( f'(4) = 0 \)
Pt. \((4, 16)\) \( m = 0 \)
The horizontal line \( y = 16 \) is the tangent line.

c) \( f(0) = 0 \)
\( f'(x) = e^x + xe^x \) \( f'(0) = 1 \) \( m = 1 \)
\[ y = x \]

\[ \text{Tangent line} \]

d) \( f'(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1 \) Pt \((\frac{\pi}{4}, 1)\)
\[ f'(x) = \sec^2 x \]
\( f'(\frac{\pi}{4}) = 2 = \text{Slope} \)
Pt. Slope \( y - 1 = 2(x - \frac{\pi}{4}) \)
\[ y = 2x - \frac{\pi}{2} + 1 \]
2. c) \( f(x) = \sec x \) at \( x = \frac{\pi}{4} \)
\[
f\left(\frac{\pi}{4}\right) = \sqrt{2}
\]
\[
f'(x) = \sec x \tan x
\]
\[
f'\left(\frac{\pi}{4}\right) = \sqrt{2} \cdot 1 = \sqrt{2} = m \quad y - \sqrt{2} = \sqrt{2}(x - \frac{\pi}{4})
\]
\[
y = \frac{\sqrt{2}x - \sqrt{2}\pi}{4} + \sqrt{2}
\]

2. f) \( f(2) = \ln(\frac{5}{4}) \)

\( f(x) \) can be rewritten using log rules to make \( f'(x) \) easier to compute.
\[
f(x) = \ln(x^2 + 1) - \ln(2x) \quad \text{using the chain rule:}
\]
\[
f'(x) = \frac{2x}{x^2 + 1} - \frac{2}{2x} = \frac{2x}{x^2 + 1} - \frac{1}{x}
\]
\[
f'(2) = \frac{4}{5} - \frac{1}{2} = \frac{3}{10} \quad y - \ln\frac{5}{4} = \frac{3}{10}(x - 2)
\]
\[
y = \frac{3}{10}x - \frac{3}{5} + \ln\frac{5}{4}
\]

\[
\lim_{x \to \frac{3}{2}} f(x) = 9 \quad \lim_{x \to 3} f'(x) = 9 = f(3)
\]
\[
f'(3) = e^{\frac{3}{2}} \quad f'(3) = 6 \quad \text{so since \( f \)'s are continuous at 3 and the slopes match, \( f \) is differentiable at \( x = 3 \).}
\]

Check \( x = 5 \)
\[
\lim_{x \to 5^-} f(x) = 21 \quad \lim_{x \to 5^+} f(x) = 21 \quad \sqrt{f \text{is continuous at } x = 5}
\]
\[
f'(5) = 6 \quad f'(5) = \frac{1}{14} = \frac{1}{14} \neq 6 \quad \text{so \( f \) has a corner at } x = 5.
\]
3. a) continued.

The derivative of \( \sqrt{9-x} + 19 \) does not exist at \( x=9 \) since \( \frac{1}{2\sqrt{9-x}} \) has a vertical asymptote at \( x=9 \). 

f has a vertical tangent at \( x=9 \). 

Final answer 

f is not differentiable at \( x=5 \) corner and at \( x=9 \) vertical tangent.

3b) \( 12x^{\frac{1}{3}} \) is not differentiable at \( x=0 \) since 

\[
\frac{d}{dx}(12x^{\frac{1}{3}}) = 4x^{-\frac{2}{3}} = \frac{4}{x^{\frac{2}{3}}}
\]

has a vertical asymptote at \( x=0 \), f has a vertical tangent at \( x=0 \).

Check \( x=8 \):

Is \( f \) continuous at \( x=8 \)?

\[
\lim_{x \to 8^-} f(x) = 24 \text{ but } f(8) = 5^- \text{ so No.}
\]

Since \( f \) is not continuous at 8, \( f \) is not differentiable at 8.

\( 24|x-9| \) has a corner at \( x=9 \) so 

f is not differentiable at \( x=9 \).

Answer f is not diff. at 8 — a discontinuity, at 0, f has a vertical tangent and at 9, a corner.
3c) Graph \( f(x) = (x^2 - 16)^{\frac{1}{5}} \) and observe the vertical tangents at \( x = -4 \) and \( x = 4 \).

Note: \( f'(x) = \frac{2x}{5(x^2 - 16)^{\frac{4}{5}}} \) has vertical asymptotes at \( x = -4 \) and \( x = 4 \).

4. a) \( s(t) \) has a minimum when \( v \) changes from \(-\) to \(+\).

\[
v(t) = s'(t) \quad \frac{1}{2} \quad - \quad + \quad 4\]

\[
s(t) \quad \frac{2}{5} \quad \frac{\text{max} \at \ t = 2}{\text{min} \at \ t = 5}\]

b) \( v(t) > 0 \) (eastward motion) and a max at \( t \approx 0.88 \) secs.

c) \( v(t) < 0 \) (westward motion) and \( v \) has a min (so \( v \) is most negative) at \( t \approx 3.79 \) secs.

d) \( s(t) \) has inflection pts at \( t \approx 0.88 \) and \( t \approx 3.79 \).

5. a) \( f(3) = 4(3) - 7 = 5 \quad f'(3) = 4 \)

b) \( f(5) = 15 \quad f'(5) = 0 \)

c) The tangent line has undefined slope so \( f(2) \) does not exist and \( f(2) \) cannot be determined from this tangent line.
6. \( g(1) = -4(1) + 6 = 2 \) and \( f(2) = 0 \) so \( f(g(1)) = 0 \).

\[
\frac{d}{dx}(f(g(x_1))) = f'(g(1))g'(x_1)
\]

from the chain rule

\[
x = 1 = -4(3) = -12
\]

\[
y = -12(x-1) + 0 = -12x + 12
\]

7. a) \( \frac{d}{dt}(1000 e^{0.06t}) = 1000(0.06)e^{0.06t} = 60e^{0.06t} \)

b) \[
\begin{align*}
\frac{d}{dx} \left( \ln\left( \frac{x^5}{\sqrt{x^4+1}} \right) \right) &= \frac{d}{dx} \left( 5 \ln x - \frac{1}{2} \ln(x^4+1) \right) \\
&= \frac{5}{x} - \frac{4x^3}{x^4+1} = \frac{5}{x} - \frac{2x^3}{x^4+1}
\end{align*}
\]

c) \[
\frac{d}{dx} \left( x^2 \cos^3 x \right) = 2x \cos^2 x + x^2 (2 \cos x)(-\sin x) = 2x \cos^2 x - 2x^2 \cos x \sin x
\]

d) \[
\frac{d}{dx} \left( e^x \sin^2 x \right) = e^x \sin^2 x + 2e^x \sin x \cos x
\]

e) \[
\frac{d}{dt} \left( e^{x^2-5x} \right) = (3x^2-5)e^{x^2-5x}
\]

f) \[
\frac{d}{dx} \left( \frac{e^{x^2}}{\tan x+4} \right) = \frac{2xe^{x^2}(\tan x+4) - (\sec^2 x)e^{x^2}}{(\tan x+4)^2}
\]
8. a) \( f(x) = \sqrt[4]{x} \) \( f(16) = 2 \)

Using the tangent line:

\[
f(16) = 2 \quad \text{and} \quad f'(16) = \frac{1}{4} \left( \frac{-3}{8} \right)x^{-\frac{3}{4}} |_{x=16} = \frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}
\]

\[
y = \frac{1}{32}(x-16) + 2
\]

Now substitute \( x = 15 \) into the tangent line.

\[
y(15) = \frac{1}{32}(15-16) + 2 = \frac{1}{32} + 2 = 2.03125
\]

Using differentials:

\[
df = f'(x) \, dx = \frac{1}{4} x^{-\frac{3}{4}} \, dx \bigg|_{x=16} = \frac{1}{32}(-1) = -\frac{1}{32}
\]

Then \( f(15) \approx f(16) + df = 2 - \frac{1}{32} = 2.03125 \)

b) \( f(x) = \sqrt[4]{x} \) same as above but now \( dx = 1 \)

\[
y = \frac{1}{32}(x-16) + 2 \quad \text{Substitute} \quad x = 17
\]

\[
y(17) = \frac{1}{32} + 2 = 2.03125
\]

or

\[
df = \frac{1}{4} x^{-\frac{3}{4}} \, dx = \frac{1}{32}(1) \quad f(17) \approx 2 + \frac{1}{32}
\]

\[
= 2.03125
\]
9. a) \( V = \frac{4}{3} \pi r^3 \)
\( dV = 4\pi r^2 \, dr \)
\( = \frac{4\pi}{3} r^2 \cdot 0.02 \)
\( = 2\pi \text{ cu. units} \)

b) \( \frac{dV}{\sqrt{\frac{4}{3} \pi r^3}} = \frac{3}{r} \, dr = \frac{3 \cdot 0.02}{5} = 0.06 \)

or \( \frac{6}{5} \% = 1.2\% \)

10. Local min \( f' = + \) This occurs only \( f \downarrow \uparrow \) at \( x = -6 \)

Local max \( f' = - \) only at \( x = 4 \)
\( f \downarrow \uparrow \)

Inflection pts \( f' \) changes direction \( x = -3, 0, 2 \)

\( f \) is increasing where \( f' > 0 \) so on \((-6, 0)\)
and on \((0, 4)\) so on \((-6, 4)\) since \( f \) is continuous at 0.

\( f \) is decreasing where \( f' < 0 \) so on \((-\infty, -6)\) and on \((4, \infty)\) (or to edge of picture)

\( f \) is concave up \( \cup \) where \( f' \) is increasing
so on \((-\infty, -3)\) and on \((0, 2)\)

\( f \) is concave down on \((-3, 0)\) and \((2, \infty)\).
9. \( V = \frac{4}{3} \pi r^3 \) \[ dll = 4\pi r^2 \, dr \]
\[ = 4\pi (5^2)(6.02) \]
\[ = 2\pi \text{ cu. units} \]

b) \[ \frac{dl}{\sqrt{\frac{4}{3} \pi r^2}} = \frac{3\, dr}{\sqrt{r}} = 3\left(\frac{0.02}{5}\right) = \frac{0.06}{5} \]

or \( \frac{6}{5} \% = 1.2\% \)

10. Local min \( f' \) - + This occurs only at \( x = -6 \)

Local max \( f' \) + - Only at \( x = 4 \)

Inflection pts. \( f' \) changes direction \( x = -3, 0, 2 \)

\( f \) is increasing where \( f' > 0 \) so on \((-6, 0)\) and on \((0, 4)\) so on \((-6, 4)\) since \( f \) is continuous at 0.

\( f \) is decreasing where \( f' < 0 \) so on \((-\infty, -6)\) and on \((4, \infty)\) lost to edge of picture

\( f \) is concave up \( \cup \) where \( f' \) is increasing so on \((-\infty, -3)\) and on \((0, 2)\)

\( f \) is concave down on \((-3, 0)\) and \((2, \infty)\).