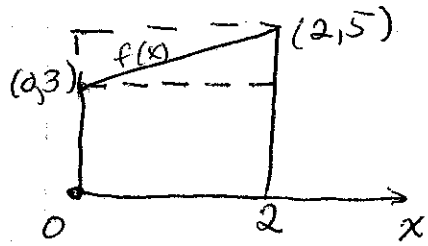
5.1

The Area under a non-negative function over $[a, b]$.

Introduction: We will approximate the area of the trapezoid shown using rectangles.



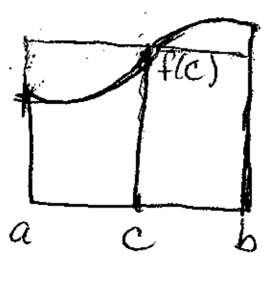
The lower rectangle has area base × ht = 6

The upper rectangle has area 10.

The trapezoid area then is between 6 and 10.

The lower rectangle has height 3 which is the value on the function $f(x)$ at the left end. $f(0) = 3$. We call this a <u>left-hand rectangle</u>.
The upper rectangle has height $5 = f(2)$ and is a <u>right-hand rectangle</u>.

The true area of the trapezoid is actually $8 = \frac{10+6}{2}$ the average of the left hand and right hand rectangle areas.



In general we can choose any c in $[a, b]$, form the rectangle with base $= b - a$ and height $f(c)$ and estimate the area as

$$\text{Area} \approx f(c)(b-a)$$

Calculus Concepts & Contexts,
James Stewart

From Figures 8 and 9 it appears that, as n increases, both L_n and R_n become better and better approximations to the area of S. Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

TEC In Visual 5.1 you can create pictures like those in Figures 8 and 9 for other values of n.

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \tfrac{1}{3}$$

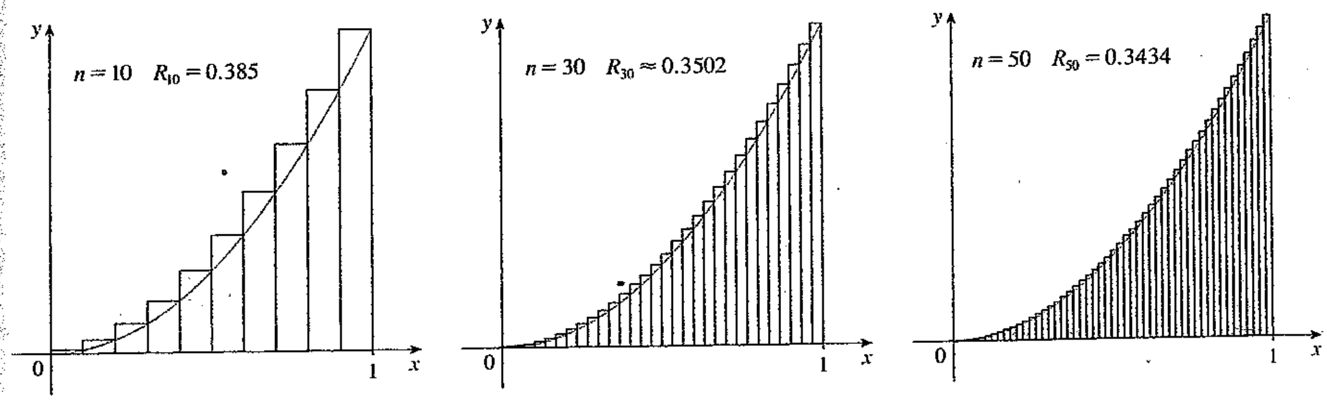


FIGURE 8

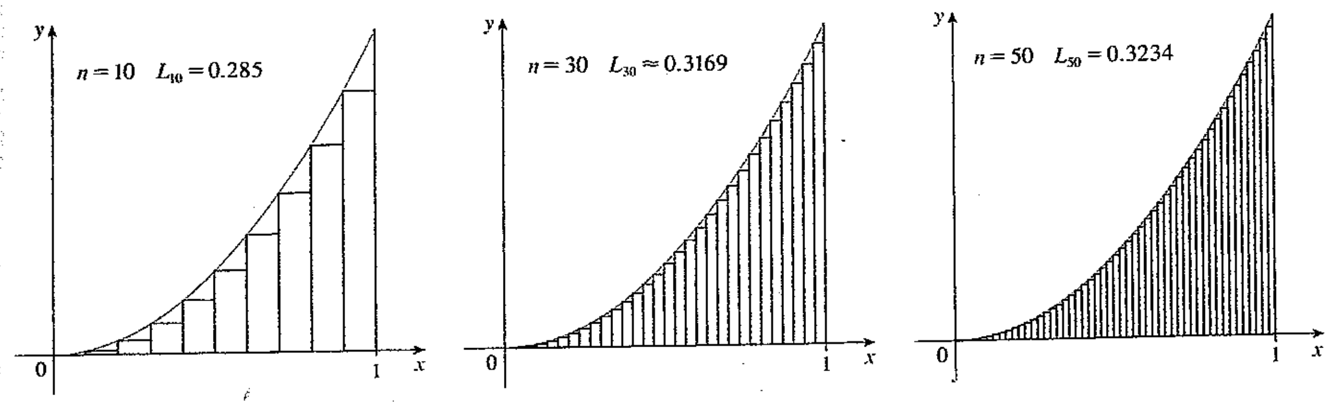


FIGURE 9 The area is the number that is smaller than all upper sums and larger than all lower sums

Let's apply the idea of Examples 1 and 2 to the more general region S of Figure 1. We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 10.

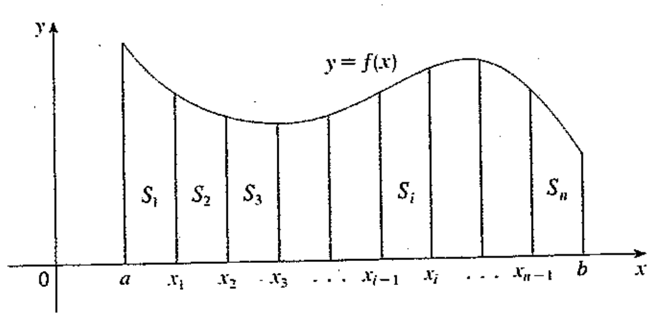


FIGURE 10

We get a still better estimate by partitioning $[a, b]$ into many small intervals and forming many thin rectangles and adding their areas.

$a = x_0 < x_1 < x_2 < x_3 < b = x_4$ — Notation for a partition with 4 subintervals.

Example: Partition the interval $[1, 5]$ into 8 equal subintervals. Then each small interval has length $\frac{5-1}{8} = \frac{1}{2}$. Partition pts.

1 2 3 4 5

start at 1 and occur every ½ unit up to 5.

$1 = x_0 < 1.5 < 2 < 2.5 < 3 < 3.5 < 4 < 4.5 < 5 = x_8$

Suppose $f(x) = x^3$. In each small interval we must choose an x-value to plug into $f(x) = x^3$ to form the height of that rectangle. If we choose the left endpoint every time we form the Left-Hand-Sum = LHS.
(Sum of rectangle areas formed at left endpts)
Similarly, choosing the right endpoint every time we form the Right-Hand-Sum = RHS.

To finish this exampe:

Add up $\frac{1}{2} \times f($
sum of base×ht

$$LHS = \frac{1}{2}\left[f(1) + f(1.5) + \dots + f(4.5)\right]$$

$$= \frac{1}{2}\left[1^3 + 1.5^3 + \dots + (4.5)^3\right] = 126.5$$

$$RHS = \frac{1}{2}\left[1.5^3 + 2^3 + \dots + 5^3\right] = 188.5$$

A better estimate of the area under $f(x) = x^3$ over $[1, 5]$ can be found by averaging LHS and RHS.

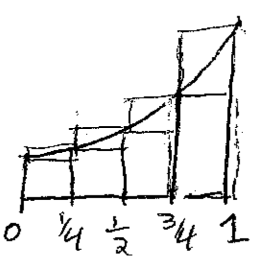
$$\text{Area} \approx \frac{LHS + RHS}{2} = \frac{126.5 + 188.5}{2} = 157.5$$

The true area, which you will be able to find after a later section, is 156.

Example: Approximate the area under $f(x) = 16^x$ over the interval $[0, 1]$, using 4 equal subintervals, find LHS, RHS and their average.

The subintervals have lenght $\frac{1-0}{4} = 1/4$

$0 < 1/4 < 1/2 < 3/4 < 1$ are the partition numbers.



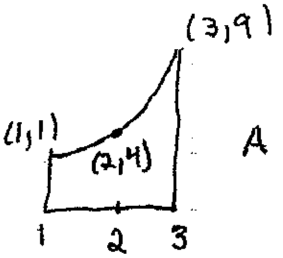
The left endpts. are $0, 1/4, 1/2, 3/4$.
The right endpts. $1/4, 1/2, 3/4, 1$

$$LHS = 1/4\left[16^0 + 16^{1/4} + 16^{1/2} + 16^{3/4}\right] = 1/4\left[1 + 2 + 4 + 8\right]$$

$$LHS = 15/4$$

$$RHS = 1/4\left[16^{1/4} + 16^{1/2} + 16^{3/4} + 16^1\right] = 15/2$$

$$\frac{LHS + RHS}{} = \frac{45}{} = 5.625 \qquad \text{Actual Area} = 5.41..$$

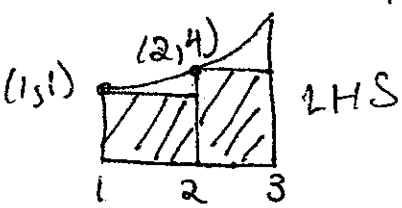
Example: 

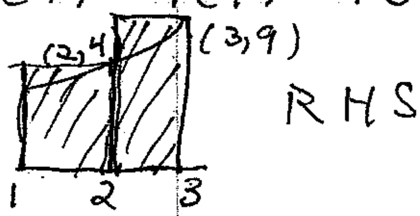
Approximate the area under the graph of $f(x)=x^2$ using 2 equal subintervals.

Find LHS and RHS and their average.

$f(x)$	1	4	9
x	1	2	3

$$LHS = 1(1) + 1(4) = 5$$
$$RHS = 1(4) + 1(9) = 13$$





The average of LHS and RHS is $\frac{5+13}{2} = 9$

The actual area, which we will learn how to find later, is $8\frac{2}{3}$, provided FYI.

Note: The base of each rectangle when the subinter are of equal size is $\frac{b-a}{n}$ where n subintervals are used.

Application: Suppose the function graphed is the nonnegative velocity of an object moving in a straight line.

For any time interval $[t_1, t_2]$,

$$\underbrace{s(t_2) - s(t_1)}_{\text{distance traveled}} \approx v(t)(t_2 - t_1) \text{ if } t_1 \le t \le t_2$$

$v(t)(t_2 - t_1)$ is the area of a rectangle using the function value at one point in the interval as the height.

This means we can approximate the distance traveled between $t=a$ and $t=b$ using rectangles to approximate the area under $v(t)$ over $[a, b]$.

"$s(b) - s(a) = \lim_{\text{mesh} \to 0} (\text{rectangle areas summed})$" if the limit exists.

Example:

$v(t) = -18t + 60 \quad 0 < t < 2$.

Find $s(2) - s(0)$.

Since $v(t)$ is a line, its graph over $[0, 2]$ forms a trapezoid. We can use geometry to find the true area.

$$s(2) - s(0) = \text{base} \times \text{avg. ht.}$$

$$= 2\left(\frac{60 + 24}{2}\right) = 84$$

The object traveled 84 units in the 1st 2 time units