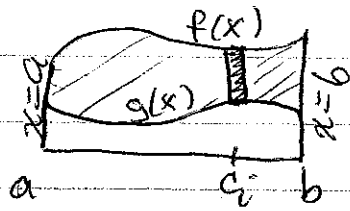
Section 6.1 Area between functions

I

A region is bounded by $y=f(x)$, $y=g(x)$, $x=a$ and $x=b$ and $f(x) \geq g(x)$ on $[a, b]$.



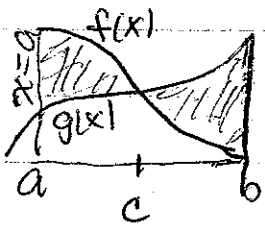
The height of any rectangle in a Riemann sum is $f(c_i) - g(c_i)$ where c_i is the chosen evaluation point.

So the area of the region is

$$A = \int_a^b f(x) - g(x)\ dx$$

II

A region is bounded by $y=f(x)$, $y=g(x)$, $x=a$ and $x=b$. $f(x)$ and $g(x)$ cross in $[a, b]$.



$$A = \int_a^c f(x) - g(x)\ dx + \int_c^b g(x) - f(x)\ dx$$

$$= \left| \int_a^c f(x) - g(x)\ dx \right| + \left| \int_c^b f(x) - g(x)\ dx \right|$$

III

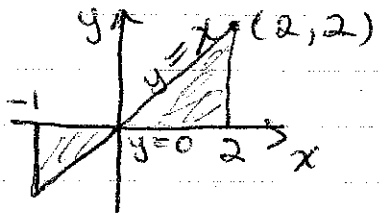
A region is bounded by $y = f(x)$, $y = g(x)$ and no interval is given.

Find any intersection pts. eg, $x_1 < x_2 < x_3$ where $f(x) = g(x)$.

$$A = \left| \int_{x_1}^{x_2} f(x) - g(x)\, dx \right| + \left| \int_{x_2}^{x_3} f(x) - g(x)\, dx \right|$$

Examples 6.1 Area

1) Find the area of the region bounded by $y=x$, $y=0$, $x=-1$ and $x=2$.

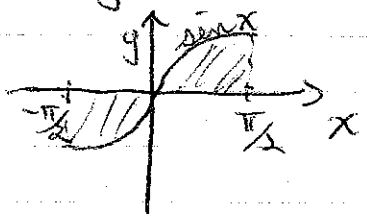


The Δ on the left has area $\frac{1}{2}$.
The Δ on the right has area 2.
Total area $= 2 + \frac{1}{2} = \frac{5}{2}$.

Notice: Total area $= 2 + \frac{1}{2}$ whereas

$$\int_{-1}^{2} x\,dx = 2 - \frac{1}{2}$$

2) Find the area of the region bounded by $y = \sin x$ $x=0$, $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.



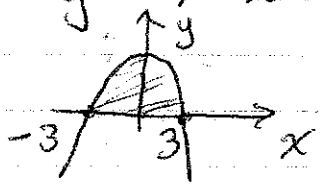
By symmetry, the total area is

$$A = 2\int_0^{\pi/2} \sin x\,dx$$

$$= -2\cos x\Big|_0^{\pi/2} = 0-(-2)$$

$$= 2$$

3) Find the area of the region bounded by $y = 9 - x^2$ and $y = 0$. $y = 9 - x^2 = 0$ if $x =$ or $x =$

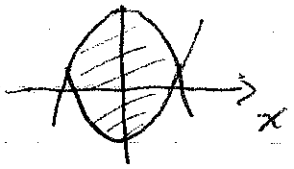


By symmetry

$$\text{Area} = 2\int_0^3 9 - x^2\,dx = 2\left[9x - \frac{1}{3}x^3\Big|_0^3\right]$$

$$= 2\left[27 - \frac{27}{3} - 0\right] = \frac{108}{3} = 36$$

4) Find the area between $y = 26 - x^2$ and $y = x^2 - 24$.



Find the intersection pts:

$x^2 - 24 = 26 - x^2$

$2x^2 = 50 \qquad x^2 = 25 \qquad x = 5$ and $x = -5$

By symmetry, $A = 2\int_0^5 26 - x^2 - (x^2 - 24)\,dx$

$= 2\int_0^5 50 - 2x^2\,dx = 2\left[50x - \frac{2}{3}x^3 \Big|_0^5\right]$

$= 2\left[250 - \frac{250}{3} - 0\right] = \frac{1000}{3}$

5) Find the area bounded by $y = 26 - x^2$, $y = x^2 - 24$, $x = 0$ and $x = 6$.

$A = \left|\int_0^5 (26 - x^2 - (x^2 - 24))\,dx\right| +$

$+ \left|\int_5^6 (26 - x^2) - (x^2 - 24)\,dx\right|$

$= \left|\int_0^5 50 - 2x^2\,dx\right| + \left|\int_5^6 50 - 2x^2\,dx\right|$

$= \frac{500}{3} + \left|-\frac{32}{3}\right| = \frac{532}{3}$