Math 142 J. Lewis

Exam 2 Review (out of class) Solutions

Product Rule and Chain Rule

1 a) $f'(x) = 3(4x+1)^2 \cdot 4e^{5x} + (4x+1)^3 e^{5x} \cdot 5$

$$= (4x+1)^2 e^{5x} \left[12 + 5(4x+1) \right]$$

$$= (4x+1)^2 e^{5x} (20x+17)$$

b) Derivative of Exponential Rule $\frac{d}{dx}\left[b^{u(x)} \right] = b^{u(x)} u'(x) \ln b$

$$f'(x) = 7^{\sqrt{3x^2+8}} \left(\frac{1}{2}(3x^2+8)^{-1/2} (6x) \right) \ln 7$$

$$= \frac{3x}{\sqrt{3x^2+8}} 7^{\sqrt{3x^2+8}} \ln 7$$

c) Derivative of Log Rule $\frac{d}{dx}\left[\log_b u(x) \right] = \frac{u'(x)}{u(x) \ln b}$

Combined with $\log_b M^t = t \log_b M$

$$f(x) = \frac{\log_4 x^{1/2}}{x^6+1} = \frac{\frac{1}{2}\log_4 x}{x^6+1} \quad \text{so}$$

$$f'(x) = \frac{\frac{1}{2} \cdot \frac{1}{x} \frac{(x^6+1)}{\ln 4} - 6x^5 \cdot \frac{1}{2} \log_4 x}{(x^6+1)^2}$$

Use log rules

d) $f(x) = 2\ln(4x^3-6) + 3\ln(2x+1) - \frac{1}{2}\ln(5x+3)$

$$f'(x) = 2 \cdot \frac{12x^2}{4x^3-6} + 3 \cdot \frac{2}{2x+1} - \frac{1}{2} \cdot \frac{5}{(5x+3)}$$

$$= \frac{24x^2}{4x^3-6} + \frac{6}{2x+1} - \frac{5}{10x+6}$$

2. a) $P'(x) = -.6x + 182$

b) $P'(370) = -.6(370) + 182$
$= -40$

A decrease of about \$40/unit

3. $\frac{C(x)}{x} = .5x + 190 + \frac{4548}{x} = AC$

To find the minimum solve $\left(\frac{C(x)}{x}\right)' = 0$.

$$MAC = \left(\frac{C(x)}{x}\right)' = .5 - \frac{4548}{x^2}$$

$0 = .5 - \frac{4548}{x^2}$ if $x = \sqrt{\frac{4548}{.5}}$

and $x > 0$

$\boxed{x \cong 95.373}$

$MC = C'(x) = 1.x + 190$ $\quad AC(95.37) \cong 285.37$

$MC(95.37) \cong 285.37$

4. a) $A(t) = Pe^{.056t}$ b) $A'(t) = .056Pe^{.056t}$

c) $$\frac{A'(t)}{A(t)} = \frac{.056\cancel{Pe^{.056t}}}{\cancel{Pe^{.056t}}} = .056$$

5a is Too long for a test question.

5. a) $x = f(p) = \sqrt{3600 - .25p^2}$

$$E(p) = -\frac{p\,f'(p)}{f(p)}$$

$$f'(p) = \tfrac{1}{2}(3600 - .25p^2)^{-\frac{1}{2}}(-.5p)$$

$$= \frac{-.25p}{\sqrt{3600 - .25p^2}}$$

$$-p\frac{f'(p)}{f(p)} = -p\,\frac{-.25p}{(3600 - .25p^2)}$$

$$E(p) = \frac{.25p^2}{(3600 - .25p^2)}$$

Demand is inelastic where $E(p) < 1$.

$\dfrac{.25p^2}{(3600 - .25p^2)} < 1$ if $.25p^2 < 3600 - .25p^2$

$.5p^2 < 3600$

$0 < p < \dfrac{60}{\sqrt{.5}} \approx 84.853$

5 b) $f(p) = \sqrt{3600 - .25p^2}$

so $R(p) = p\sqrt{3600 - .25p^2}$

$R'(p) = \sqrt{3600 - .25p^2} + p\frac{1}{2}(3600 - .25p^2)^{-\frac{1}{2}}(-.5p)$

$R'(p) = 0$ if $\sqrt{3600 - .25p^2} = \dfrac{.25p^2}{\sqrt{3600 - .25p^2}}$

so $3600 - .25p^2 = .25p^2$

$3600 = .5p^2$

$\boxed{\dfrac{60}{\sqrt{.5}} = p}$ same answer

6. $E(p) = .01p^2$ $E(p) = 1$

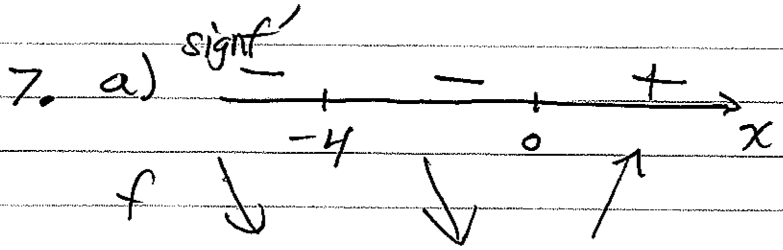
if $.01p^2 = 1$ $p^2 = \frac{1}{.01} = 100$ $p = 10$

Revenue is max @ $p = 10$.

b) Demand is inelastic and Revenue is increasing on $0 < p < 10$.

c) $E(8) \times \frac{1}{8} \times 100\% = \frac{.01(64)}{8} \times 100\% = 8\%$

7. a)



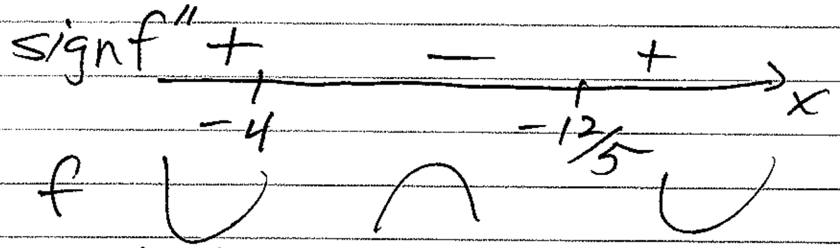
f is decreasing on $(-\infty, 0)$
increasing on $(0, \infty)$

f has a relative min at $x = 0$.

b) $f''(x) = 3x^2(x+4)^2 + x^3 \cdot 2(x+4)$

$= x^2(x+4)(3(x+4) + 2x)$

$= x^2(x+4)(5x+12)$

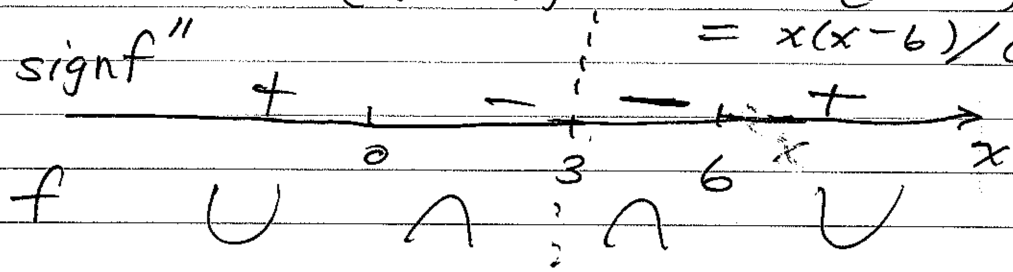


inflection points @ $x = -4$ and $x = \frac{-12}{5}$

Concave up on $(-\infty, -4)$ and $(-12/5, \infty)$

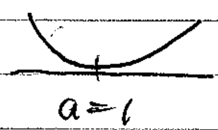
8. $f''(x) = \dfrac{2x(x-3) - x^2}{(x-3)^2} = \dfrac{x^2 - 6x}{(x-3)^2}$

$= x(x-6)/(x-3)^2$

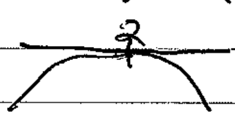


inflection points at $x=0$ and $x=6$

9. At $a=1$, f has a relative min



At $a=2$ f has a relative max



At $a=3$ No relative extremum.
2nd Deriv. test does
not apply. $f'(3) \neq 0$.

At $a=4$ No conclusion.

$f(x) = x^3 - 3x + 10$ on $[0, 2]$

10 a) $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

The c.v.'s are $-1, 1$.

In $[0, 2]$, the only c.v. is $x = 1$.

x	$f(x)$
0	10
1	8 ← absolute min.
2	$2^3 - 3(2) + 10 = 12$ ← absolute max.

In the calculator, enter $Y1 = x^3 - 3x + 10$

Set the window $X_{MIN} = 0$, $X_{MAX} = 2$ } the given interval

guess Y_{MIN}, Y_{MAX}

Observe and use calc maximum,
calc minimum.

b) $f(x) = x^3 - 9x^2 + 15x$ on $[-1, 2]$

b) $f'(x) = 3x^2 - 18x + 15$

$= 3(x^2 - 6x + 5)$

$= 3(x-5)(x-1)$

c.v.'s are $x = 1$ and $x = 5$.

The only c.v. in $[-1, 2]$ is $x = 1$.

x	$f(x)$
-1	$-1 - 9 - 15 = -25$ ← abs. min
1	$1 - 9 + 15 = 7$ ← abs. max
2	$8 - 36 + 30 = 2$

10 c) $f(x) = x^3 - 3bx^2$ on $[0, 4b]$ $b > 0$

$f'(x) = 3x^2 - 6bx = 3x(x - 2b)$

c.v.'s are $x = 0$ and $x = 2b$

x	$f(x)$
	$= 0$
0	0
$2b$	$(2b)^3 - 3b(2b)^2 = 8b^3 - 12b^3 = -4b^3$
$4b$	$(4b)^3 - 3b(4b)^2 = 64b^3 - 48b^3 = 16b^3$

Abs. min $= -4b^3$ at $x = 2b$

Abs. max $= 16b^3$ at $x = 4b$

d) $f(x) = x^3 - 3a^2x$ on $[-2a, 3a]$ $a > 0$

$f'(x) = 3x^2 - 3a^2 = 3(x - a)(x + a)$

c.v.'s are $x = -a$ and $x = a$

x	$f(x)$
$-2a$	$(-2a)^3 - 3a^2(-2a) = -2a^3$
$-a$	$(-a)^3 - 3a^2(-a) = -a^3 + 3a^3 = 2a^3$
a	$a^3 - 3a^2(a) = -2a^3$
$3a$	$27a^3 - 3a^2(3a) = 18a^3$

Abs min is $-2a^3$ at $x = -2a$ and $x = a$

Abs max is $18a^3$ at $x = 3a$

11. Find the demand price, $p(x)$.

$(0, 40)$ $m = \frac{-3}{100} = -.03$ $p = -.03x + 40$

$R(x) = x \cdot p(x) = -.03x^2 + 40x$

$C(x) = 2000 + 12x$

a) $P(x) = -.03x^2 + 28x - 2000$

b) $P'(200)$: $P'(x) = -.06x + 28$

$P'(200) = \$16$ per unit

c) $AP = \frac{P(x)}{x} = -.03x + 28 - 2000x^{-1}$

$MAP = \left(\frac{P(x)}{x}\right)' = -.03 + 2000x^{-2}$

12. a) $f'(x) = \dfrac{4x^3(e^{5x}+1) - x^4 \cdot 5e^{5x}}{(e^{5x}+1)^2}$

b) $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} \cdot 8^{\sqrt{x}} \ln 8$

c) $f'(x) = (2x-3)e^{x^2-3x+5}$

d) Rewrite $f(x)$ using log rules to simplify finding f'.

$f(x) = 4\ln(e^x+7) + \ln(x^2+6x) - \frac{1}{3}\ln(x+2)$

$f'(x) = \dfrac{4e^x}{e^x+7} + \dfrac{2x+6}{x^2+6x} - \dfrac{1}{3} \cdot \dfrac{1}{(x+2)}$

13. $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ is the chain rule.

$$\frac{df}{dx} = (7u^6) \cdot \frac{5x}{x^3+1}$$

$$\left.\frac{df}{dx}\right|_{x=3} = 7(u(3))^6 \cdot \frac{5(3)}{3^3+1} = 7 \cdot \frac{15}{28} = \frac{15}{4}$$

$$= \boxed{3.75}$$

14. a) Solve $E(p) = 1$. $\frac{3p^2}{2500 - p^2} = 1$

$3p^2 = 2500 - p^2$ $\quad 4p^2 = 2500$

$p^2 = 625$ $\quad \boxed{p = 25}$

(since -25 is not sensible for a price.)

b) on $(0, 25)$

c) Since 20 and 21 are in the inelastic range, revenue will increase.

d) $E(20) = \frac{3(20)^2}{2500 - (20)^2} = \frac{4}{7}$

% change in price $= \frac{1}{20} \times 100\% = 5\%$

$E(20) \times$ % change in price $= \frac{4}{7} \times 5\% = \frac{20}{7}\%$

$\approx 2.857\%$ demand decreases

15. $f(p) = 2000pe^{-p^2}$

a) Relative R.O.C. $= \dfrac{f'(p)}{f(p)}$

$$f'(p) = 2000e^{-p^2} + 2000pe^{-p^2}(-2p)$$

$$= 2000e^{-p^2}(1-2p^2)$$

$$\frac{f'(p)}{f(p)} = \frac{\cancel{2000}\,\cancel{e^{-p^2}}(1-2p^2)}{\cancel{2000}\,p\,\cancel{e^{-p^2}}} = \frac{1-2p^2}{p}$$

b) $E(p) = -p\dfrac{f'(p)}{f(p)} = -(1-2p^2) = 2p^2-1$

16. a) $x=-3$ and $x=4$

b) sign f': $-$ on $(-3, 4)$, $+$ after 4; points -3, 4 on the x-axis

f: $\searrow$ $\searrow$ $\nearrow$

relative (local) min at $x=4$

c) $f''(x) = 2(x+3)(x-4)^3 + (x+3)^2 \cdot 3(x-4)^2$

$$= (x+3)(x-4)^2[\underbrace{2(x-4)}_{2x-8} + \underbrace{3(x+3)}_{3x+9}]$$

$$= (x+3)(x-4)^2[5x+1]$$

$f''=0$ at -3, 4, $-1/5$

sign f'': $+$ before -3, $-$ on $(-3, -1/5)$, $+$ on $(-1/5, 4)$, $+$ after 4

IP $x=-3$ IP $x=-1/5$

$$f'(x) = \frac{e^{-x}}{(x+2)^2} \quad \text{given}$$

17.

a) $$f''(x) = \frac{-e^{-x}(x+2)^2 - e^{-x} \cdot 2(x+2)}{(x+2)^4}$$

$$= e^{-x}(x+2)\left[-(x+2) - 2\right] / (x+2)^4$$

$$= \frac{-e^{-x}\left[x+2+2\right]}{(x+2)^3}$$

$$= \frac{-e^{-x}(x+4)}{(x+2)^3}$$

b) sign f'': $-$ $\quad$ $+$ $\quad$ $-$ $\quad$ $e^{-x} > 0$

x: $\quad -4 \quad -2 \quad$ for all x

f: $\cap$ $\quad$ $\cup$ $\quad$ $\cap$

I.P. at $x = -4$

c) f is concave up on $(-4, -2)$

18. a) $f'(x) = (2x-5)\,8^{(x^2-5x+12)}\ln 8$

b) Rewrite f(x) using log rules to simplify finding f'.

$f(x) = \ln(x^2+6x) + \frac{1}{2}\ln(x-4) - 3\ln(e^x+1)$

$f'(x) = \dfrac{2x+6}{x^2+6x} + \dfrac{1}{2(x-4)} - \dfrac{3e^x}{e^x+1}$

19. a) Solve $E(p) = 1$

$\dfrac{p^2}{675-2p^2} = 1$ $\quad p^2 = 675 - 2p^2$

$3p^2 = 675$

$p^2 = 225$ $\quad \boxed{p = 15}$

($p = -15$ makes no sense)

b) Inelastic for $E(p) < 1$,

p in $(0, 15)$

c) Since \$12 and \$13 are in the inelastic range, revenue will increase. Demand always decreases (we assume) when price increases.

d) % Change in p $= \frac{1}{12} \times 100\%$

$= 8\frac{1}{3}\%$

$E(12) = \dfrac{144}{675-288} \approx .3721$

$E(12) \times 8\frac{1}{3}\% \approx \boxed{3.10\%}$

20\. a) $x = -2$ and $x = 6$

b) f': $-$ | -2 | $-$ | 6 | $+$ $\rightarrow x$

f: $\searrow$ -2 $\searrow$ 6 $\nearrow$

relative min at $x = 6$

c) $f''(x) = 2(x+2)(x-6)^3 + (x+2)^2 \cdot 3(x-6)^2$

$= (x+2)(x-6)^2 [\overset{2x-12}{2(x-6)} + \overset{3x+6}{3(x+2)}]$

$= (x+2)(x-6)^2(5x-6)$

$-2 \quad 6 \quad 6/5$

sign f'': $+$ | -2 | $-$ | $6/5$ | $+$ | 6 | $+$ $\rightarrow x$

f: $\cup$ -2 IP $\cap$ $6/5$ IP $\cup$ 6 $\cup$

I.P. at $x = -2$

$x = 6/5$

21\. a) $f''(x) = \dfrac{2x(x^2+75)^2 - x^2 \cdot 2(x^2+75) \cdot 2x}{(x^2+75)^4}$

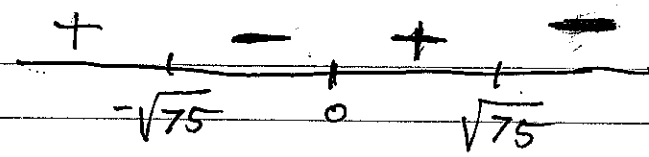
$= \dfrac{2x(x^2+75)[x^2+75 - 2x^2]}{(x^2+75)^4}$

$= \dfrac{2x(75 - x^2)}{(x^2+75)^3}$

Note: $x^2 + 75 \geq 75$ for all x,

sign f''

b)



I.P. at $x = -\sqrt{75}$, $x = 0$ and $x = \sqrt{75}$

c) Concave up on $(-\infty, -\sqrt{75})$
and on $(0, \sqrt{75})$.

22. q = quantity at x weeks from now.
p = price at x weeks from now.

Revenue $= q \cdot p = (200 - 10x)(1 + .25x)$

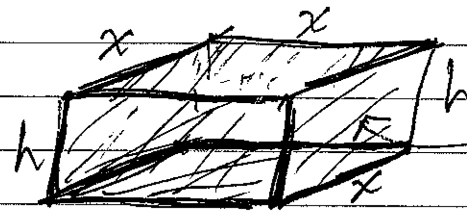
$R'(x) = -10(1 + .25x) + (200 - 10x)(.25)$

$= -10 - 2.5x + 50 - 2.5x$
$= 40 - 5x$

$R' = 0$ at $x = \frac{40}{5} = 8$ weeks.

2nd derivative test shows it's a max pt.
$R''(x) = -5 < 0$ R is concave down at $x = 8$. (at any x here)

relative max

23. 

area of any side $= xh$
area of top $= x^2$

Volume $= x^2 h$ (or $l \cdot w \cdot h$ but $l = w = x$) since top & bottom are square.

$1350 = x^2 h$ given

so $h = \dfrac{1350}{x^2}$

Cost of top and bottom $= (2x^2)\ \$2/\text{sf}$.
Cost of 4 sides $= (4\,xh)\ \$5/\text{sf}$

Total cost $= 4x^2 + 20xh$
Substitute $h = \dfrac{1350}{x^2}$

$$C(x) = 4x^2 + 20x\,\frac{(1350)}{x^2}$$
$$= 4x^2 + \frac{27000}{x}$$

Minimize C: Solve $C'(x) = 0$

$$C'(x) = 8x - \frac{2700}{x^2}$$

$$0 = 8x - \frac{2700}{x^2} \qquad 8x = \frac{27000}{x^2}$$

$$x^3 = \frac{27000}{8}$$

$$x = \sqrt[3]{\frac{27000}{8}} = 15$$

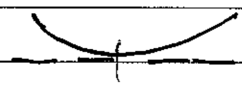
$$h = \frac{1350}{15^2} = 6$$

Show C has a min at C.V. $x=15$.
$x=15$ and $h=6$

$C'(x)$ was $8x - \frac{27000}{x^2}$

2nd Deriv. test:

$$C''(x) = 8 + \frac{2(27000)}{x^3} > 0 \text{ if } x=15$$



$x=15$ relative min.