Math
142 In Class Exam 2 Review Solutions

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1. $p = 30e^{-.02x}$

$R(x) = x \cdot p = 30x\,e^{-.02x}$

Marginal Revenue $= R'(x)$ — Use the product rule.

$R'(x) = 30e^{-.02x} + 30x(-.02)e^{-.02x}$

$= 30e^{-.02x}[1 - .02x]$

$R'(x) = 0$ if $1 - .02x = 0$

$.02x = 1$

critical value $x = \frac{1}{.02} = 50$

sign R': $+$ on $(0, 50)$, $-$ after 50

R: $\nearrow$ $\searrow$

Shows R has a max at $x = 50$

2. Find $P(x)$, the profit function.

$R(x) = x \cdot p = -0.5x^2 + 30x$

$P(x) = R(x) - C(x) = -0.5x^2 + 30x - (12x + 40)$

$P(x) = -0.5x^2 + 18x - 40$

a) Marginal Profit $= P'(x) = -x + 18$

$P(16) - P(15) \approx P'(15) = 3$

$P(16) \approx 3 + P(15) = 117.5$

2 b) Find Avg Profit function:

$$\frac{P(x)}{x} = -0.5x + 18 - 40x^{-1}$$

Marginal Avg Profit $= \left(\frac{P(x)}{x}\right)'$

To find max of Avg. Profit, find Marginal Avg Profit and Solve for x so Marginal Avg Profit $= 0$.

$$\left(\frac{P(x)}{x}\right)' = -0.5 + 40x^{-2}$$

Solve $\quad -0.5 + 40x^{-2} = 0$

$$\frac{40}{x^2} = \frac{1}{2} \qquad 80 = x^2$$

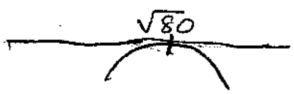
$$\sqrt{80} = x$$

(since $x > 0$)

How do we know this is a max?

Use 2nd derivative test:

$$\left(\frac{P(x)}{x}\right)'' = -80x^{-3} < 0 \text{ when } x = \sqrt{80}$$



horiz tangent
concave down

3. $f(x) = (x^2 - x - 6)^{1/3}$ $\qquad$ Domain $f = (-\infty, \infty)$

$$f'(x) = \frac{1}{3}(x^2 - x - 6)^{-2/3}(2x - 1)$$

$$= \frac{2x - 1}{3(x^2 - x - 6)^{2/3}}$$

a) Horizontal Tangent: Solve $f'(x) = 0$:

$2x - 1 = 0 \qquad x = \frac{1}{2}$

b) Vertical Tangent: Set den $= 0$

$x^2 - x - 6 = 0 \qquad (x - 3)(x + 2) = 0 \qquad x = 3, -2$

4. a) $$\left[\frac{e^{5x}}{2x^3+7}\right]' = \frac{5e^{5x}(2x^3+7)-e^{5x}\cdot 6x^2}{(2x^3+7)^2}$$

b) $$\left(3^{x^4-6x^2+7}\right)' = 3^{x^4-6x^2+7}(\ln 3)(4x^3-12x)$$

c) Rewrite $f(x)$ using logarithm rules:

$$f(x) = \log\left[\frac{\sqrt{x}}{(x^2+1)(x-4)^3}\right] = \frac{1}{2}\log x - \log(x^2+1) - 3\log(x-4)$$

$$f'(x) = \frac{1}{2x} - \frac{2x}{x^2+1} - \frac{3}{x-4}$$

d) $$\left((x+1)^3(x-2)^4\right)' = 3(x+1)^2(x-2)^4 + (x+1)^3\cdot 4(x-2)^3$$

Where is the tangent line horizontal?
Solve $f'(x) = 0$.
Factor $f'(x)$: Factor out $(x+1)^2(x-2)^3$:

$$(x+1)^2(x-2)^3\left[3(x-2)+4(x+1)\right]$$

$$3x-6+4x+4$$

$$7x-2$$

$$= (x+1)^2(x-2)^3(7x-2)$$

is 0 at $x=-1$, $x=2$ and $x=\frac{2}{7}$

5. The tangent line is horizontal where $f' = 0$ so at
$x = -3.5$, -1, ~ 1.75, 3.5 and 6.5

f is increasing where $f' > 0$
so on $(-3.5, -1)$ and
$(1.75, 2.5)$ and $(3.5, 6.5)$

f is decreasing where $f' < 0$
so on $(-\infty, -4)$ and $(-1, 1.75)$
and $(6.5, \infty)$

f has a relative min where
f' changes from neg. to pos.
so at $x = -3.5$, 1.75 and 3.5.

f has a relative max where f'
changes from pos. to neg.
so at $x = -1$ and $x = 6.5$

At $x = 3$, if f is continuous at $x = 3$ it
has a relative max. at $x = 3$.

f has inflection pts. where f' changes
from increasing to decreasing or
from decreasing to increasing, so at
$x = -2.25$, $.5$, and 5.

6) Solve for $x = f(p)$:

$$x^2 = 1458 - p^2$$

$$f(p) = x = \sqrt{1458 - p^2} \quad \text{since } x > 0.$$

$$E(p) = \frac{-p\,f'(p)}{f(p)}$$

$$f'(p) = \frac{1}{2}(1458 - p^2)^{-1/2}(-2p)$$

$$= \frac{-p}{\sqrt{1458 - p^2}}$$

$$E(p) = \frac{-p}{\sqrt{1458-p^2}} \cdot \frac{-p}{\sqrt{1458-p^2}} = \frac{p^2}{1458 - p^2}$$

Rev. is a max. when $E(p) = 1$.

$$\frac{p^2}{1458-p^2} = 1 \qquad p^2 = 1458 - p^2$$

$$2p^2 = 1458 \qquad p^2 = 729$$

$$p = \sqrt{729} = 27$$

b) Revenue will increase as p increases from \$20 to \$25 since $E < 1$ on $(0, 27)$.

c) $\text{Approx}\left(\%\text{ Change in } f(p)\right) = E(p_0)\dfrac{p_1 - p_0}{p_0} \cdot 100\%$

$$= \frac{20^2}{1458 - (20)^2} \cdot \frac{5}{20} \cdot 100\% = 9.45\%$$

7. a) $f(x) = x^3 - 6x^2 + 9x$ on $[-1, 2]$

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$= 3(x-3)(x-1)$

critical values are $x=1$, $x=3$.

Only $x=1$ is in $[-1, 2]$

x	$f(x)$	
-1	$-1-6-9 = -16$	$\leftarrow$ absolute min
1	$1-6+9 = 4$	$\leftarrow$ absolute max
2	$8-24+18 = 2$	

b) $f(x) = 4x^5 - 5ax^4$ on $[-a, 2a]$ $a>0$

$f'(x) = 20x^4 - 20ax^3 = 20x^3(x-a)$

Critical values $x=0$, $x=a$ are both in $[-a, 2a]$.

x	$f(x)$	
$-a$	$-9a^5$	$\leftarrow$ absolute min
0	0	
a	$-a^5$	
$2a$	$48a^5$	$\leftarrow$ absolute max

8. $f(x) = \dfrac{2x^2+5x+18}{x} = 2x + 5 + 18x^{-1}$

$f'(x) = 2 - 18x^{-2} = 0$ if $2 = \dfrac{18}{x^2}$

$2x^2 = 18$ $x^2 = 9$ $x = \pm 3$

$x=3$ is in $(0, \infty)$

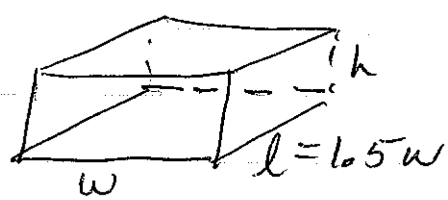
$f''(x) = 36x^{-3} > 0$ if $x=3$, local min @ $x=3$

3 is the only c.V. in $(0,\infty)$ so

$f(3)$ is an absolute minimum.

9. This is a 2nd derivative test question.

x	1	2	3	4
$f'(x)$	0	0	0	1
$f''(x)$	3	-2	0	-1
	local min at $x=3$	local max at $x=2$	No conclusion at $x=3$	just increasing, concave down. 2nd deriv. test does not apply since $f'(4) \neq 0$

10 

$V = 1800 \text{ ft}^3$

$V = wlh = w(1.5w)h = 1.5w^2h = 1800$

Surface Area $= S = \underbrace{2(1.5w^2)}_{\text{top + bottom}} + \underbrace{2wh + 2(1.5wh)}_{\text{sides}}$

$S = 3w^2 + 5wh$ to be minimized

Given $1.5w^2h = 1800$ so $h = \dfrac{1800}{1.5w^2}$

$h = \dfrac{1200}{w^2}$

Substitute into S:

$S(w) = 3w^2 + 5w\left(\dfrac{1200}{w^2}\right)$

$S(w) = 3w^2 + \dfrac{6000}{w} = 3w^2 + 6000w^{-1}$

Find minimum:

$S'(w) = 6w - 6000w^{-2} = 0$ if $6w - 6000w^{-2} = 0$

$6w^3 = 6000$ $\boxed{w = 10}$ so $\boxed{l = 15}$ $\boxed{h = 12}$

11.

Find Revenue as a function of price, p.

$R(p) = p \cdot x(p)$

$x(p)$ is a line: $(\overset{p}{40}, \overset{x}{500})$ slope $= -20$

$x(p) = -20(p-40) + 500$ pt slope

$= -20p + 1300$

$R(p) = -20p^2 + 1300p$

$R'(p) = -40p + 1300 = 0$ if $p = \frac{1300}{40} \doteq \$32.50$

$R'' = -40 < 0$ so it's a max pt.

12. Similar problem but looking for # months. Note: In #11, we do not know the monthly rates of change.

m = # months from now.

$p = 40 - m \qquad q = 500 + 20m$

$R(m) = (40-m)(500+20m)$

$R'(m) = -(500+20m) + 20(40-m)$

$= -500 - 20m + 800 - 20m$

$= 300 - 40m = 0$ if

$m = \frac{300}{40} = 7.5$ months

Note:
After 7.5 months, $p = 40 - 7.5 = 32.50$

$x(p) = 500 + 20(7.5)$

$= 650$

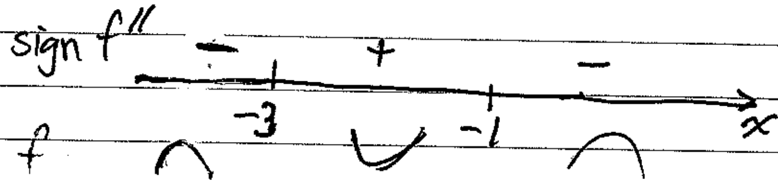
a) $f'(x) = (x+3)^2 e^{-x}$

13a) $f'(x) > 0$ for all x so f is always increasing, no relative extrema.

$$f''(x) = 2(x+3)e^{-x} + (x+3)^2 e^{-x}(-1)$$

$$= (x+3)e^{-x}(2-(x+3))$$

$$= (x+3)e^{-x}(-x-1)$$

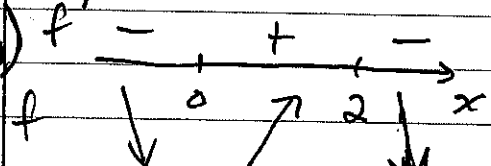


Inflection pts. at $x = -3$ and $x = -1$.

f is concave up on $(-3, -1)$

f is concave down on $(-\infty, -3)$ and $(-1, \infty)$

b)



$f'(x) = x(2-x)e^{-x}$

relative min @ $x = 0$

relative max @ $x = 2$

f is increasing on $(0, 2)$

f is decreasing on $(-\infty, 0)$ and $(2, \infty)$

$$f'(x) = (2x - x^2)e^{-x}$$

$$f''(x) = (2-2x)e^{-x} - (2x - x^2)e^{-x}$$

$$= (2 - 4x + x^2)e^{-x}$$

$$x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

sign f'' $+$ $-$ $+$ x

f $\cup$ $2-\sqrt{2}$ $\cap$ $2+\sqrt{2}$ $\cup$

test $x = 2$

$f''(2) = (2-8+4)e^{-2} < 0$

the sign changes at each root here or do more test pts.

inflection pts at $x = 2-\sqrt{2}$ and $2+\sqrt{2}$.

concave down on $(2-\sqrt{2}, 2+\sqrt{2})$

concave up on $(-\infty, 2-\sqrt{2})$ and $(2+\sqrt{2}, \infty)$

c)

$$f'(x) = \frac{(x-2)(x-1)}{(x-4)^2}$$

sign f' $+$ 1 $-$ 2 $+$ 4 $+$ x

f $\nearrow$ $\downarrow$ $\nearrow$ $\nearrow$

test $x = 0$ $\dfrac{(0-2)(0-1)}{(0-4)^2} > 0$

The sign changes at 1 and 2 but not across 4.

relative max @ $x = 1$

relative min @ $x = 2$

increasing on $(-\infty, 1)$, $(2, 4)$ and $(4, \infty)$

decreasing on $(1, 2)$.

For f'', go back to the unfactored form of f':

$$f'(x) = \frac{x^2-3x+2}{(x-4)^2}$$

$$f''(x) = \frac{(2x-3)(x-4)^2 - (x^2-3x+2)\cdot 2(x-4)}{(x-4)^4}$$

Factor out $x-4$

$$= \frac{(x-4)\left[(2x-3)(x-4) - 2(x^2-3x+2)\right]}{(x-4)^4}$$

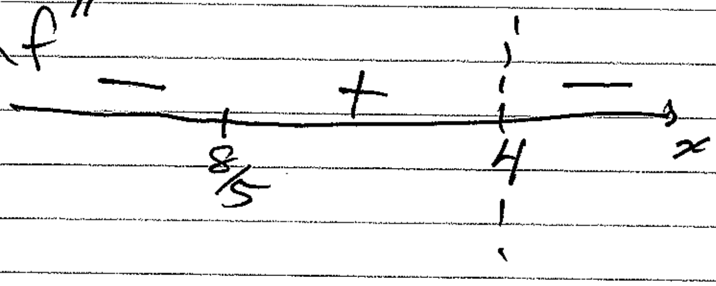
$$\cancel{2x^2} - 8x - 3x + 12 - \cancel{2x^2} + 6x - 4$$

$$= \frac{-5x+8}{(x-4)^3}$$

test $x=0$

$f''(0) = \frac{8}{(-4)^3} < 0$

sign f''



f has an inflection pt. at $x = 8/5$

f is concave down on $(-\infty, 8/5)$ and on $(4, \infty)$.

f is concave up on $(8/5, 4)$.