In Class Exam 2 Review Solutions

1. a) Factor out 3 and divide by 3.

\[ x^2 + y^2 - 2x + 4y = 5 \]
\[ a = \frac{2}{3} = 1 \quad b = \frac{-4}{3} = -2 \]
\[
\frac{x^2 - 2x}{(x-1)^2} + \frac{y^2 + 4y}{(y+2)^2} = \frac{5 + 1 + 4}{5 + 1 + 4} = \frac{10}{10} = 1
\]
\[
(x-1)^2 + (y+2)^2 = 10 \quad \text{C}(1, -2) \ r = \sqrt{10}
\]

b) Divide by 2. Move \( \frac{4}{2} \) to right of \( = \).

(Subtract \( \frac{4}{2} \) from both sides)

\[ x^2 + y^2 - 6x + 5y = -4 \quad \text{Take half of} \ 6 \]
\[
\frac{b}{2} = \frac{3}{2} = a \quad \text{Take half of} \ 5 \]
\[
(x-3)^2 + (y + \frac{5}{2})^2 = \frac{-8 + 9 + \frac{25}{4}}{4} = \frac{45}{4}
\]
\[
\text{C}(3, -\frac{5}{2}) \ r = \frac{3\sqrt{5}}{2}
\]

2. Find the center and radius of the circle:

\[ (x - 2)^2 + (y + 4)^2 = -12 + 4 + 16 = 8 \]
\[
\text{C}(2, -4) \ r = \sqrt{8} = 2\sqrt{2}
\]

Find \( d(P, C) = \sqrt{(3-2)^2 + (-2+4)^2} = \sqrt{1 + 4} = \sqrt{5} \leq r \)

P is inside the circle.
6) Find the center = mid pt. of \( \overline{AB} \). 
\[ C\left( \frac{4+3}{2}, \frac{5+9}{2} \right) = C\left(2, 7\right) \]
The radius is \( \frac{1}{2}d(A, B) = \frac{1}{2}\sqrt{(3-1)^2 + (9-5)^2} \)
\[ r = \frac{1}{2}\sqrt{4 + 16} = \frac{1}{2}\sqrt{20} = \sqrt{5} \]
Find \( d(P, C) = \sqrt{(5-2)^2 + (6-7)^2} = \sqrt{10} \)
\( \sqrt{10} > \sqrt{5} \) so \( P \) is outside the circle.

3. Find the slope of \( \overline{AB} \):
\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{9 - 5} = \frac{2}{4} = \frac{1}{2} \]
m = \( \frac{1}{2} \) so \( m^{-1} = -2 \)
Find the midpoint of \( \overline{AB} = \left( \frac{4+5}{2}, \frac{4+9}{2} \right) \)
= \((7, 3)\)
Pt. slope for slope -2 and Pt \((7,3)\):
\[ y - 3 = -2(x - 7) \]
\[ y = -2x + 17 \]
4. a) \((a, f(a)) = (8, f(8)) = (8, 6)\) \(m = \frac{1}{4}\)

Pt. \(\text{Slope:} \quad y - 6 = \frac{1}{4}(x - 8)\)

\[y = \frac{1}{4}x + 8\]

b) \(f(1) = \frac{1+5}{1+1} = 3\) \((1, 3)\) \(m = -5\)

\[y - 3 = -5(x - 1)\]

\[y = -5x + 8\]

5. Rational functions are defined where the denominator is non-zero.

For \(x < 3\), \(x^2 - 1 = 0\) if \(x = -1, 1\). Since \(-1, 1\) are both less than 3, \(f\) is not defined at \(-1\) or 1.

For \(x \geq 3\), \(x^2 - 4 = 0\) if \(x = 2, -2\) but these are less than 3 so \(x^2 - 4\)

\(x\) is defined for \(x \geq 3\).

\[
\text{Domain of } f = \text{all reals except } -1 \text{ and } 1.
\]

\[
= (-\infty, -1) U (-1, 1) U (1, \infty)
\]

b) \(f(2) = \frac{10-5}{4-1} = \frac{5}{3}\)

\(f(3) = \frac{3}{9-4} = \frac{3}{5}\)

\(f(5) = \frac{5}{21}\)
6. a) $x$ must be in domain $f$ and $f(x)$ must be in domain $g$.

$\text{domain } f = [0, \infty)$, $\text{domain } g = (-\infty, \infty)$.

So $\text{domain } (g \circ f) = [0, \infty)$.

$(g \circ f)(x) = 4 - \left(\frac{\sqrt{x}}{x+5}\right)^2 = 4 - \frac{x}{(x+5)^2}$

$\text{Domain of } (4 - \frac{x}{(x+5)^2}) = (-\infty, -5) \cup (-5, \infty)$

But $\text{domain } (g \circ f)$ is only $[0, \infty)$.

b) $\text{domain } g = (-\infty, \infty)$

We need $g(x)$ to be in domain $f$.

So $4-x^2$ must be non-negative, or $0$.

$4-x^2$ must be $\geq 0$.

Sign chart:

$\begin{array}{cccc}
\text{Sign Chart} & -4 & + & - \\
\text{at } x & -2 & 2 & \infty \\
\end{array}$

So $\text{domain } f \circ g = [-2, 2]$

$f \circ g (x) = \frac{\sqrt{4-x^2}}{4-x^2+5} = \frac{\sqrt{4-x^2}}{9-x^2}$
7a) Find the vertex \((h,k)\).

\[ h = -\frac{b}{2a} = -\frac{-21}{-6} = \frac{7}{2} \]

\[ k = p\left(\frac{7}{2}\right) = -3\left(\frac{7}{2}\right)^2 + 21\left(\frac{7}{2}\right) \]

\[ = 36.75 \text{ is the maximum}. \]

b) \[ h = -\frac{10}{-10} = 1 \]

\[ q(1) = -5 + 10 - 12 = -7 \]

\[ 1 \text{ is the maximum} \]

8. a) \[ h = -\frac{8}{4} = -2 \]

\[ k = p(-2) = 2(-2)^2 + 8(-2) - 20 \]

\[ = 8 - 16 - 20 = -28 \text{ is the minimum} \]

b) The zeros of \(q(x)\) are \(2x - 10 = 0\)

\[ x = 5 \]

and \(3x + 15 = 0\)

\[ x = -5 \]

The midpoint of the zeros is \(h\),

so \(h = 0\).

\[ k = (-10)(15) = 150 \text{ is the minimum} \]
9. a) \(x^4 + 5y^3 - x^2y = 0\)

about \(x\)-axis:
\[x^4 + 5(-y)^3 - x^2(-y) = 0\]
Not even in \(y\)
\[\text{No}\]
\[x^4 - 5y^3 + x^2 = 0\]

about \(y\)-axis:
\[(-x)^4 + 5y^3 - (-x)^2y = 0\]
Even in \(x\)
\[\text{Yes}\]
\[x^4 + 5y^3 + x^2y = 0\]

about origin:
\[\text{No}\]
\[x^4 - 5y^3 + x^2 = 0\] is different

b) \(xy^5 + x^3 = 0\)

\(x\)-axis:
\[x(-y)^5 + x^3 = 0\]
\[-xy^5 + x^3 = 0\]
\[\text{No}\]
\[\text{not even in } y\]

\(y\)-axis:
\[-xy^5 - x^3 = 0\]
\[\text{No}\]

origin:
\[(-x)^4 + (-x)^3 = 0\]
\[\text{No}\]
\[xy^5 - x^3 = 0\] is different
10 a) \( f(x) = \sqrt{x^2 - 5x + 6} \)  \[ \text{Domain} f = (-\infty, 2) \cup [3, \infty) \]
\( f(x) \) is defined if \( x^2 - 5x + 6 \geq 0 \).
\[
x^2 - 5x + 6 = (x-2)(x-3) = 0 \text{ if } x=2 \text{ or } x=3
\]

The graph shows \( x^2 - 5x + 6 \geq 0 \) on \((-\infty, 2) \cup [3, \infty)\)
so that it is the domain of \( f \).

10 b) \( \sqrt{2x - x^2} = g(x) \)  \[ \text{Domain} g = [0, 2] \]
Solve \( 2x - x^2 \geq 0 \) \( x(2-x) = 0 \) at \( x=0 \) or \( x=2 \)
and this parabola opens downward.

10 c) \( 4(x+2)^2 + 9(y-3)^2 = 36 \)
[ellipse]

The extreme values of \( x \) occur when
\[ 4(x+2)^2 = 36 \rightarrow (x+2)^2 = 9 \rightarrow x+2 = \pm 3 \]
\( x = -3 - 2 \text{ to } x = +3 - 2 \) Domain = \([-5, 1] \)

The extreme of \( y \) occur when
\[ 9(y-3)^2 = 36 \rightarrow (y-3)^2 = 4 \rightarrow y - 3 = \pm 2 \]
\( y = -2 + 3 \text{ to } y = +2 + 3 \) Range = \([1, 5] \)
i) The axis of symmetry is $x = h$ and $h$ is the midpoint of the zeros of $f(x)$.

$$f(x) = 0 \text{ if } x = -10 \text{ or } x = 8, \quad h = \frac{-10 + 8}{2} = -1$$

$x = -1$ is the axis.

ii) This parabola opens upward so has a minimum value of $K = f(h)$.

$$K = f(-1) = 3(9)(-9) = -243 \text{ is the min.}$$

iii) Put the quadratic in vertex form.

$$f(x) = a(x-h)^2 + k = 3(x+1)^2 - 243$$

$y = x^2$ is shifted left 1 unit, stretched vertically by a factor of 3, and shifted down 243 units.

b) $f(x) = -5x^2 + 15x - 8$

i) Find $h = \frac{-b}{2a} = -\frac{15}{-10} = \frac{3}{2}$

Axis is $x = -\frac{3}{2}$

ii) This parabola opens downward so has a maximum of $K = f(-3)$.

$$f(\frac{3}{2}) = -5\left(\frac{9}{4}\right) + 15\left(\frac{-3}{2}\right) - 8 = -41.75$$
iii) Put $f(x)$ in vertex form.

$f(x) = -5(x + \frac{3}{2})^2 - 41.75$

$y = x^2$ is shifted left $\frac{3}{2}$ units,
stretched vertically by a factor of 5,
reflected across the $x$-axis,
and shifted down 41.75 units.

12. a) $p(x) = -6x^5 + 20x + 400$
\[\text{degree}(p) = 5\]
\[\text{leading coeff.} = -6\]
$p(x) \rightarrow -\infty$ as $x \rightarrow +\infty$

$b) p(x) = 6x^3 - 8x^2 + 7x$
\[\text{degree}(p) = 3\]
\[\text{leading coeff.} = 6\]
$p \rightarrow +\infty$ as $x \rightarrow +\infty$
$p \rightarrow -\infty$ as $x \rightarrow -\infty$

$c) p(x) = 2x - 8x^2 + 3x^6$
\[\text{degree}(p) = 6\]
\[\text{leading coefficient} = 3\]
$p \rightarrow +\infty$ as $x \rightarrow \infty$
$p \rightarrow +\infty$ as $x \rightarrow \infty$ The even power is always nonneg.
13. a) \( f(x) = 2x^2 - 6x \) on \([1.5, \infty)\)

Put \( f \) in vertex form.

\[ f(x) = 2x(x - 3) \text{ zeros are } x=0 \text{ and } x=3 \]

so \( h = \frac{0 + 3}{2} = 1.5 \) or \( \frac{3}{2} \)

\[ k = f(1.5) = 2\left(\frac{9}{4}\right) - 6\left(\frac{3}{2}\right) \]

\[ f(x) = 2(x - \frac{3}{2})^2 - 4.5 \]

We see \( f \) is one-to-one on this domain so it has an inverse. Switch \( x \) and \( y \), then solve for \( y \).

\[ f^{-1}: \quad y = 2(x - \frac{3}{2})^2 - 4.5 \longrightarrow x \geq 1.5 \]

\[ f^{-1}: \quad x = 2(y - \frac{3}{2})^2 - 4.5 \quad y \geq 1.5 \]

\[ x + 4.5 = (y - \frac{3}{2})^2 \]

\[ \sqrt{\frac{x + 4.5}{2} + \frac{3}{2}} = y \]

Take the positive sign on the radical since \( y \geq 1.5 \).

We have \( \text{range}(f^{-1}) = [1.5, \infty) = \text{domain } f \)

\( \text{domain } f^{-1} = \text{range } f = (-4.5, \infty) \)
b) \( f(x) = -3x^2 + 12x - 8 \) on \((-\infty, 2]\)

To find the inverse, put \( f \) in vertex form.

\[
\begin{align*}
h &= \frac{-b}{2a} = \frac{-12}{-6} = 2 \\
k &= f(2) = -3(4) + 12(2) - 8 \\
    &= 4
\end{align*}
\]

We see \( f \) is one-one on \((-\infty, 2]\) so \( f \) has an inverse.

\[
\begin{align*}
f &: y = -3(x-2)^2 + 4 \quad x \leq 2 \\
f^{-1} &: x = -3(y-2)^2 + 4 \quad y \leq 2 \\
\frac{x-4}{-3} &= (y-2)^2 \\
-\sqrt{\frac{x-4}{-3}} &= y - 2 \quad \text{Take the negative sign on the radical since } y \leq 2 \\
2 - \sqrt{\frac{x-4}{-3}} &= y \\
or \quad f^{-1}(x) &= 2 - \sqrt{\frac{4-x}{3}} \left(\frac{x-4 = 4-\sqrt{3}}{3}\right)
\end{align*}
\]

domain \( f^{-1} = (-\infty, 4] \)

range \( f^{-1} = (-\infty, 2]\)
13 c) \[ f' : y = (x+8)^{1/3} \]
\[ \text{domain of } f = (-\infty, \infty) \]
\[ \text{range of } f = (-\infty, \infty) \]
\[ f^{-1} : x = (y+8)^{1/3} \]
\[ x^3 = y+8 \]
\[ y = \sqrt[3]{x^3-8} = f^{-1}(x) \]

Domain and range of \( f^{-1} \) are \((-\infty, \infty)\).

13 d) \[ f(x) = \frac{x}{2x-7} \]
\[ \text{domain of } f = (-\infty, \frac{7}{2}) \cup (\frac{7}{2}, \infty) \]
all reals except \( \frac{7}{2} \).

\[ f : y = \frac{x}{2x-7} \]
\[ f^{-1} : x = \frac{y}{2y-7} \]
\[ x(2y-7) = y \]
\[ 2xy - 7x = y \]
\[ 2xy - y = 7x \]
\[ (2x-1)y = 7x \]
\[ f^{-1} : y = \frac{7x}{2x-1} \]
\[ \text{domain of } f^{-1} = (-\infty, \frac{7}{2}) \cup (\frac{7}{2}, \infty) \]
all reals except \( \frac{7}{2} \).

So this is the range of \( f \).

\[ \text{range of } f^{-1} = \text{domain of } f = (-\infty, \frac{7}{2}) \cup (\frac{7}{2}, \infty) \]
14. a) \[ f: \quad y = \frac{3x+2}{x+5} = \frac{3x+15-13}{x+5} = 3 - \frac{13}{x+5} \]

\( y = \frac{1}{x} \) is shifted left 5 units, stretched vertically by a factor of 13, reflected across the x-axis, and lastly shifted up 3.

b) \[ f: \quad y = \frac{x+1}{3x+9} = \frac{x+1}{3(x+3)} = \frac{1}{3} \left( \frac{x+1}{x+3} \right) \]

\[ = \frac{1}{3} \left( \frac{x+3-3+1}{x+3} \right) = \frac{1}{3} \left( 1 - \frac{2}{x+3} \right) \]

\( y = \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{x+3} \)

or \[ y = \frac{x + \frac{9}{3} - \frac{9}{3} + 1}{3x+9} = \frac{1}{3} - \frac{2}{3x+9} \]

\[ = \frac{1}{3} - \frac{2}{3(x+3)} \]

\[ = \frac{1}{3} - \frac{2}{3} \cdot \frac{1}{x+3} \]

\( y = \frac{1}{x} \) is shifted left 3 units, shrunk vertically by a factor of \( \frac{2}{3} \), reflected across the x-axis, and lastly shifted up \( \frac{1}{3} \).
15. Revenue = quantity sold \times price/unit

Let \( x = \#$ weeks from now,

price = p = 60 - 2x \) since \( \downarrow \) by 2/week

and is currently $60.

quantity = q = 3 + x \) since \( \uparrow \) by 1/wk

and is currently 3-

\[ R(x) = (3+x)(60-2x) \] is a parabola

opening downward so has a maximum

at the midpoint of its zeros.

\[ R(\text{max}) \text{ at } h = \frac{-3+30}{2} = \frac{27}{2} = 13.5 \text{ weeks} \]

16. Area = \( xy \) is to be maximized.

\[ 3x + 2y = \text{total fence material} \]

so \( 3x + 2y = 1600 \)

Solve for \( y \), substitute into \( A = xy \),

and find the maximum of the parabola.

\[ 3x + 2y = 1600 \]

\[ 2y = 1600 - 3x \]

\[ y = 800 - \frac{3}{2}x \]

\[ A = x(800 - \frac{3}{2}x) \]

\( A \) is a parabola opening downward so

has a maximum at \( h = \text{midpoint of the zeros of } A(x) \).
The zeros of \( A(x) \) are \( x=0 \) and
\[
800 - \frac{3}{2}x = 0 \quad \text{or} \quad x = \frac{3}{2}(800) = \frac{1600}{3}
\]
The midpoint is \( h = \frac{800}{3} \) so
maximum area is at \( x = \frac{800}{3} \).

\[
A\left(\frac{800}{3}\right) = \frac{800}{3} \left(800 - \frac{3}{2}\left(\frac{800}{3}\right)\right)
= \frac{800}{3}(400) = \frac{320000}{3} \text{ sq. ft}.
\]

16 b) \( x \quad \text{building} \quad x \\
\hspace{2cm} y \\
\hspace{4cm} A = xy \\
3x + y = 1600 \\
y = 1600 - 3x
\]

\( A(x) = x(1600 - 3x) \) has zeros
at 0 and \( \frac{1600}{3} \) so \( h = \frac{800}{3} \)

max. \( A = \frac{800}{3} \left(1600 - 3\left(\frac{800}{3}\right)\right) = \frac{800}{3}(800) = \frac{640000}{3} \text{ sq. ft} \).

16 c) \( 2x + y = 1600 \quad A = xy \\
A = x(1600 - 2x) \\
h = \frac{0 + 800}{2} = 400 \\
A_{\max} = 400(800) = 320000 \text{ sq. ft} \).
\[ \text{deg}(p(x)) = 3 \quad \text{lead. coeff.} = 4 \quad \text{zeros at} \quad -1, 2, \text{and} \quad 7 \]

17) \[ p(x) = 4(x+1)(x-2)(x-7) \]

\[ |p(x)| \]

\[ p(x) = 2(x-3)^2(x-5) \]

\[ |p(x)| \]
18. a) \[
\frac{1}{x+h} - \frac{1}{x} = \frac{x - (x+h)}{x(x+h)} \cdot \frac{1}{h} = \frac{-h}{h(x(x+h))} = \frac{-1}{x(x+h)}
\]

b) \[
\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) = \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}
\]

c) \[
\frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = h(3x^2 + 3xh + h^2) = \frac{3x^2 + 3xh + h^2}{h}
\]