Solving $|ax+b| + c = |ex+f|$

Example

$|2x+3| + 1 = |5x+10|$

$|ax+b| = \begin{cases} -(ax+b) & \text{if } ax+b<0 \\ ax+b & \text{if } ax+b \geq 0 \end{cases}$

We need to see where $2x+3$ and $5x+10$ can change sign.

solve

$2x+3 = 0$ and $5x+10 = 0$

$x = -\frac{3}{2}$ and $x = -2$

$\text{case 1}$  $\text{case 2}$  $\text{case 3}$

both $-2$  $-\frac{3}{2}$  $x$ both pos,  $2x+3 \leq 0$  $5x+10 > 0$

Case 1: $x < -2$

$-(2x+3) + 1 = -(5x+10)$

$-2x - 2 = -5x - 10$

$3x = -8$

$x = -\frac{8}{3}$

Case 2: $-2 < x < -\frac{3}{2}$

$-(2x+3) + 1 = -(5x+10)$

$-2x - 2 = 5x + 10$

$-12 = 7x$

$x = -\frac{12}{7}$

Case 3: $x > -\frac{3}{2}$
case 3: \[ x > -\frac{3}{2} \]

\[
\begin{align*}
2x+3 + 1 &= 5x+10 \\
\quad -9 &= 3x
\end{align*}
\]

\[ x = -3 \]

The 3 solutions are \(-\frac{8}{3}\), \(-\frac{12}{7}\) and \(-3\).
Polynomial and Rational Function Sign Charts

A product or quotient of factors can change sign only if a factor changes sign.

A linear factor, \( x-a \), can only change sign at \( a \).

Sign Chart for \( x-a \):

\[
\begin{array}{c|ccc}
  x-a & - & 0 & + \\
  a & + & + & + \\
\end{array}
\]

means \( x-a < 0 \) if \( x < a \), \( x-a > 0 \) if \( x > a \).

\((x-a)^3\) has the same sign chart.

\((x-a)^{\text{odd}}\) has the same sign chart.

\((x-a)^2 \geq 0\) for all \( x \) so its chart is

\[
\begin{array}{c|ccc}
  (x-a)^2 & - & 0 & + \\
  a & + & + & + \\
\end{array}
\]

An odd product of negatives is negative.

An even product of negatives is positive.

Example: \((x-1)(x-2) < 0\) if exactly one factor is negative. The sign of this product can only change at \( x=1 \) and \( x=2 \).
Make the chart for \((x-1)(x-2)\):

Method I
\[
\begin{array}{c|c|c|c}
(x-1) & -1 & + & - \\
(x-2) & -1 & + & - \\
\hline
(x-1)(x-2) & + & - & + \\
\hline
x & 2 & 1 & 2 \\
\end{array}
\]

Method II
\[
\begin{array}{c|c|c|c}
(x-1)(x-2) & - & + \\
\hline
x & 1 & 2 & x \\
\end{array}
\]

Test the sign at an easy point. I chose \(x=0\).

\[(0-1)(0-2) = 2 > 0\] Put a + sign in the interval containing 0.

Since \(x-1\) changes sign at 1 and \(x-2\) does not, the sign must change across 1.

So far:
\[
\begin{array}{c|c|c|c}
& + & - & + \\
\hline
1 & 2 & x \\
\end{array}
\]

Since \(x-2\) changes sign at 2 but \(x-1\) does not, change the sign again across 2.

We have:
\[
\begin{array}{c|c|c|c}
& + & - & + \\
\hline
1 & 2 & x \\
\end{array}
\]
Make the chart for $(x-1)^2(x-2)$:

\[
\begin{array}{c|c|c|c|c|c}
- & - & q & + & - & + \\
\hline
l & 1 & 2 & 3 & 4 & u \\
\end{array}
\]

This is the same as the chart for $x-2$ since $(x-1)^2 \geq 0$ for all $x$ and cannot change the sign.

Make the chart for \(\frac{(x-1)^2(x+3)}{(x-4)^3(x+5)^2}\)

Set up:

\[
\begin{array}{c|c|c|c|c|c}
- & - & u & + & - & + \\
\hline
-5 & -3 & 1 & 4 & u \\
\end{array}
\]

Test an easy pt. like $x=0$:

At 0 the value is negative so put $-$ between $-3$ and 1.

Then change the sign across each number whose factor has an odd exponent:

\[
\begin{array}{c|c|c|c|c|c}
+ & u & + & - & q & + \\
\hline
-5 & -3 & 1 & 4 & u & x \\
\end{array}
\]

The solution set to \(\frac{(x-1)^2(x+3)}{(x-4)^3(x+5)^2} < 0\)

is \((-3, 1) \cup (1, 4)\).