Math 151 Exam 2 Review

1. Evaluate each limit as a number, infinity, or -infinity or state DN E.

   a) \( \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x - \frac{\pi}{2}} \)
   b) \( \lim_{x \to 0} \frac{\sin(\frac{\pi}{6} + x) - \frac{1}{2}}{x} \)
   c) \( \lim_{x \to 2^-} \frac{1}{\sin(x - 2)} \)
   d) \( \lim_{x \to 2} \frac{x^2 - 2x}{\sin(x - 2)} \)
   e) \( \lim_{x \to 0} \frac{e^{x+2} - e^2}{x} \)
   f) \( \lim_{t \to 0} \frac{\tan^2(4t)}{\tan(5t) \sin t} \)

2. Find \( f'(x) \). For b you can do it with the chain rule or the product rule. Compare the answers.

   a) \( f(x) = \sin(x^2) \)
   b) \( f(x) = \sin(2x) = 2 \sin x \cos x \)
   c) \( f(x) = \sqrt{\tan(3x)} \)
   d) \( f(x) = \sec^n(x) \)

3. a) Show that the tangent to a circle is perpendicular to a radial line through the center and the point of tangency using implicit differentiation on the equation for the unit circle.

   b) Is the same true for the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \)?

4. A curve is described by \( \tan x \sec y + xy = A \). Use implicit differentiation to find \( y' \).

5. A curve has equation \( x^2 = \frac{x + 2y}{x - 2y} \). Find the tangent line to the curve at the point (1, 0).

6. \( x = 2 \cos^2 t + \sin(4t) \quad y = \tan^2 t \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \) Find the equations of the tangent lines to this curve at the point (1, 1).

7. Find the 127th derivative with respect to \( x \) of \( f(x) = \cos(5x) \).

8. Show that \( y = Ae^{-x} + Bxe^{-x} \) solves the differential equation \( y'' + 2y' + y = 0 \)

9. For what values of \( a \) does \( y = e^{ax} \) solve the differential equation \( y'' = y' + y \)?

10. \( f(x) = p(x)e^x \) where \( p(x) \) is differentiable three times. Find the third derivative of \( f(x) \) with respect to \( x \).
11. A tank is in the shape of an inverted cone having height 16 ft. and top radius of 4 ft. Water is filling the tank at the rate of 2 cubic feet per minute. How fast is the depth of the water rising when the water is 5 feet deep?

12. Car A is traveling due east toward an intersection at 40 mph. Car B is traveling due north toward the same intersection at 30 mph. How fast is the distance between them decreasing when car A is 300 feet away and car B is 400 feet away from the intersection. Note, the conversion from feet to miles will cancel out.

13. \( h(x) = f(g(x)) \). \( L_1(x) = 5x-3 \) is the linear approximation to \( g(x) \) near \( x=1 \). \( L_2(u) = 4u+5 \) is the linear approximation to \( f(u) \) near \( u=2 \). Find the linear approximation to \( h(x) \) near \( x=1 \).

14. Use linear approximation to approximate \( \sqrt[3]{30} \).

15. Find the inverse function to \( f(x) = \frac{x+1}{x-2} \). Find the asymptotes of \( f \) and the asymptotes of the inverse.

16. \( f(x) = e^x \) Find the derivative of \( f^{-1}(u) \) at any \( u \).

17. The tangent line to a function, \( f \), at \( x=1 \) is \( y = -3x + 5 \). \( f \) is one to one near \( x = 1 \). Find the tangent line to the inverse function at \( x=f(1) \).

18. The path of an object is a curve with the given vector equation. Find the velocity, speed and acceleration.

\[ \vec{r}(t) = \sqrt{t^2 + 1} \hat{i} + te^t \hat{j} \]

\[ \vec{r}(t) = \frac{t}{e^t} \hat{i} + e^{2t} \hat{j} \]

19. \( f(x) = e^{-x^2} \) a) Find an equation of the tangent line at \( x=0 \).

b) Find the 2nd derivative of \( f(x) \).

20. Find a formula for the nth derivative of \( f(x) \).

\[ a) \quad f(x) = xe^{-x} \quad b) \quad f(x) = xe^x \quad c) \quad g(x) = 2xe^{2x} \]

21. A curve is given parametrically by \( x(t) = t^2e^{t+2} \quad y(t) = t^2e^{-t^2} \).

Find the points where the tangent line is horizontal and the points where it is vertical.
22. Find the two tangent lines to \( y = x^2 - x \) that pass through the point (5, 19).

23. Evaluate each limit.

\[ a) \lim_{x \to 0^+} \frac{1}{e^x} \quad b) \lim_{x \to 0^-} \frac{1}{e^x} \quad c) \lim_{x \to \infty} \frac{7e^x + 2}{14e^x + 3} \quad d) \lim_{x \to -\infty} \frac{7e^x + 2}{14e^x + 3} \]

24. Determine the values of \( a \) and \( b \) so that the function is continuous and differentiable for all \( x \).

\[ a) \quad f(x) = \begin{cases} x^2 + 5x + 7 & x < 1 \\ ax + b & 1 \leq x \end{cases} \quad b) \quad f(x) = \begin{cases} x^3 - bx & x < 1 \\ ax^2 + bx & 1 \leq x \end{cases} \]