151 Exam 3 Review
Solutions

1 a) $\frac{500e^{.04x}}{500} = \frac{750}{500} = 1.5$

$e^{.04x} = 1.5$

$.04x = \ln(1.5)$

$$\boxed{x = \frac{\ln(1.5)}{.04}}$$

b) $\ln\left[\frac{x-2}{x-1}\right] - \ln\left[\frac{x-3}{x-1}\right] = 1$

$\ln\left[\frac{x-2}{\cancel{x-1}} \cdot \frac{\cancel{x-1}}{x-3}\right] = 1$

$\frac{x-2}{x-3} = e^1 \qquad x-2 = e(x-3)$

$x(1-e) = 2-3e$

$$\boxed{x = \frac{2-3e}{1-e}}$$

c) $\frac{1}{2}\log_3(x-4) + \frac{1}{2}\log_3(x+4) = 1 \qquad x>4$

$\log_3(x-4) + \log_3(x+4) = 2$

$\log_3(x^2-16) = 2$

$x^2-16 = 3^2 = 9$

$x^2 = 25$ so $x=5$

($x = -5$ cannot be used)

2. a) Expand first:

$$2\ln(2x+4) + \ln(3x^2+12) - \frac{1}{2}\ln(x^3+7) = f(x)$$

$$2 \cdot \frac{2}{2x+4} + \frac{6x}{3x^2+12} - \frac{1}{2} \cdot \frac{3x^2}{x^3+7} = f'(x)$$

$$= \boxed{\frac{2}{x+2} + \frac{2x}{x^2+4} - \frac{3x^2}{2x^3+14} = f'(x)}$$

b) $\ln f(x) =$ the function in 2a)

$$\frac{f'(x)}{f(x)} = \frac{2}{x+2} + \frac{2x}{x^2+4} - \frac{3x^2}{2x^3+14}$$

$$f'(x) = \frac{(2x+4)^2(3x^2+12)}{\sqrt{x^3+7}}\left[\frac{2}{x+2} + \frac{2x}{x^2+4} - \frac{3x^2}{2x^3+14}\right]$$

c) $$\left[\log_2(x^2+7)\right]' = \frac{2x}{(x^2+7)\ln 2}$$

d) $$\frac{d}{dx}\left(\log_2(x^2+1)\right)^2 = \frac{2x}{(x^2+1)\ln 2} \cdot 2\log_2(x^2+1)$$

$$= \frac{4x}{(x^2+1)\ln 2}\log_2(x^2+1)$$

2e) $\frac{d}{dx}\left[\ln\left(x+\sqrt{x^2+1}\right)\right] = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot\cancel{2}x\right)$

$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$

$= \boxed{\frac{1}{\sqrt{x^2+1}}}$

f) $\frac{1}{x+y}(1+y') = 2x\ln y + x^2\frac{y'}{y}$

$\frac{1}{x+y} + \frac{y'}{x+y} = 2x\ln y + \frac{x^2}{y}y'$

$y'\left(\frac{1}{x+y} - \frac{x^2}{y}\right) = 2x\ln y - \frac{1}{x+y}$

$y' = \frac{2x\ln y - \frac{1}{x+y}}{\frac{1}{x+y} - \frac{x^2}{y}}$

g) $\frac{d}{dx}\left(\ln|\sec x|\right) = \frac{1}{\sec x}\cdot\sec x\tan x = \tan x$

h) $(x^2+1)^x = e^{x\ln(x^2+1)}$

$\frac{d}{dx}\left((x^2+1)^x\right) = e^{x\ln(x^2+1)}\cdot\left(\ln(x^2+1) + x\cdot\frac{2x}{x^2+1}\right)$

$= (x^2+1)^x\left[\ln(x^2+1) + \frac{2x^2}{x^2+1}\right]$

3. The rate of cooling is proportional to difference between the temp. of fudge and $70°$ = temp of surroundings.

If y = temp. of fudge then

$y' = k(y-70)$. Since $y' = (y-70)'$,

$(y-70)' = k(y-70)$ So $y - 70 = Ce^{kt}$

$y = Ce^{kt} + 70$

Initially $y = 234$.

$234 = Ce^{0} + 70 = C + 70 \quad C = 164$

At $t = 30$ min. $y = 110 = 164e^{30k} + 70$

$\frac{1}{30}\ln\frac{10}{41} = \frac{1}{30}\ln\frac{110-70}{164} = k$

$y = 164e^{\frac{1}{30}\left(\ln\frac{10}{41}\right)t} + 70$ at t min.

In Short:

$y - 70 = 164\left(\frac{40}{164}\right)^{t/30}$ $\quad \frac{t}{\#\text{min to go from } 164 \text{ to } 40}$

$= y(0)\left(\frac{\text{end difference}}{\text{initial diff}}\right)$

4a) In continuous compound interest, the growth rate of the amount is the int. rate times the amount.

$$y'(t) = r\,y(t)$$

$$y = Pe^{rt} \qquad P = 1500 = y(0)$$

$$y = 1500e^{rt} \qquad \text{Since } 2000 = 1500e^{r(4)},$$

$$4/3 = \frac{2000}{1500} = e^{4r}$$

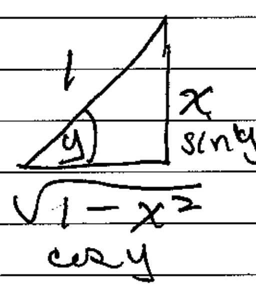
so $r = \frac{1}{4}\ln(4/3) \approx .07192$ or 7.192%

b) $W(t) = 12\left(\frac{60}{12}\right)^{t/4}$ $\quad t =$ # hours since it weighed 12 g.

$$= 12\left(5^{t/4}\right)$$

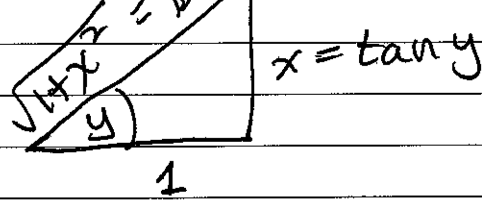
5. a) $\sin y = x$ $\quad$ Differentiating implicitly:

$$(\cos y)\,y' = 1 \qquad y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

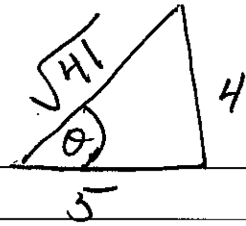


b) $(\sec^2 y)\,y' = 1$

$$y' = \frac{1}{\sec^2 y}$$

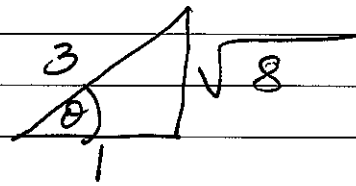


$$y' = \frac{1}{1+x^2}$$

6. a) 

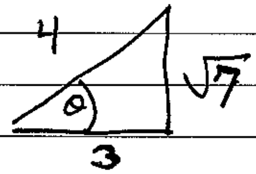
Make the triangle so $\tan\theta = \frac{4}{5}$

$\cos\theta = \frac{5}{\sqrt{41}}$

b) $\cos\theta = \frac{1}{3}$ 

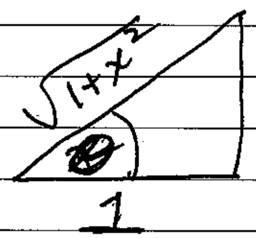
Find $\sin(2\theta)$ Recall the identity $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 2\theta = 2\left(\frac{\sqrt{8}}{3}\right)\left(\frac{1}{3}\right) = \frac{2\sqrt{8}}{9} = \frac{4\sqrt{2}}{9}$

c) 

Find $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= \frac{9}{16} - \frac{7}{16} = \frac{2}{16} = \frac{1}{8}$

d) $\sec(\arctan x)$ 

$x = \tan\theta$

$\sec\theta = \boxed{\sqrt{1+x^2}}$

e) $\sqrt{1+\tan^2 x} = \sec x$ $\arccos\left(\frac{1}{\sec x}\right) = x$

$\arccos(\cos x)$

so this is just $\sin x$.

7a) $y' = \arctan x + \frac{x}{1+x^2}$ $\quad y'(1) = \frac{\pi}{4} + \frac{1}{2}$

b) $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

Using the chain rule:

$$\frac{d}{dx}\arcsin(\sqrt{x}) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}}$$

8 a) Substitution gives "∞^0" which is indeterminate. We must have "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$" to apply L'Hospital's rule.

$$\ln[(\cot x)^{\sin x}] = \sin x \ln(\cot x)$$

$$= \frac{\ln(\cot x)}{\frac{1}{\sin x}} = \frac{\ln(\cot x)}{\csc x}$$

Now substitution gives $\frac{\infty}{\infty}$ so applying L'H rule:

$$\lim_{x\to 0^+} \ln[(\cot x)^{\sin x}] = \lim_{x\to 0^+} \frac{\frac{-\csc^2 x}{\cot x}}{-\csc x \cot x}$$

$$\frac{\frac{-\csc^2 x}{\cot x}}{-\csc x \cot x} = \frac{\csc x}{\cot^2 x} = \frac{\sin x}{\cos^2 x} \xrightarrow[x\to 0^+]{} 0 = \lim_{x\to 0^+} \ln f(x)$$

So $\lim_{x\to 0^+} f(x) = e^0 = 1$

7. a) $y' = \arctan x + \frac{x}{1+x^2}$ $\quad y'(1) = \frac{\pi}{4} + \frac{1}{2}$

b) $y' = \frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right)$ $\quad x > 0$

(Chain Rule)

8. a) $\lim_{x \to 0^+} (\cot x)^{\sin x} = ?$

Substitution gives "∞^0" which is indeterminate. We must have "$\frac{\infty}{\infty}$" or "$\frac{0}{0}$" to use L'Hospital's rule:

$\ln\left[(\cot x)^{\sin x}\right] = (\sin x)\ln(\cot x)$

$= \frac{\ln(\cot x)}{\frac{1}{\sin x}}$ Now subst. gives $\frac{\infty}{\infty}$

Applying L'Hospital's rule:

$$\frac{\frac{-\csc^2 x}{\cot x}}{\frac{-\cos x}{\sin^2 x}} = \frac{1}{\sin x} \xrightarrow[x \to 0^+]{} +\infty$$

Since $\ln f(x) \to +\infty$

$f(x) \to +\infty$ also

8b) The limit must be 1 since the reciprocal was found to have a limit of 1 in 8a).

Without using 7a:

$\ln\left((\tan x)^{\sin x}\right) = (\sin x)\ln(\tan x)$

$= \dfrac{\ln(\tan x)}{\frac{1}{\sin x}}$ Using L'Hospital's rule:

The limit is the same as the limit of

$\dfrac{\frac{\sec^2 x}{\tan x}}{\frac{-\cos x}{\sin^2 x}} = \dfrac{1}{\cos x \sin x} \cdot \dfrac{(-\sin^2 x)}{\cos x}$

$= -\dfrac{\sin x}{\cos^2 x} \xrightarrow[x\to 0^+]{} 0$

$\lim_{x\to 0^+} \ln(f(x)) = 0$ so $\lim_{x\to 0^+} f(x) = e^0 = 1$

8c) $\arcsin 0 = 0$ so $\lim(2x - \arcsin x) = 0$

$\arccos 0 = \pi/2$ so $\lim(2x + \arccos x) = \pi/2$

so $\lim_{x\to 0} \dfrac{2x - \arcsin x}{2x + \arccos x} = \dfrac{0}{\pi/2} = 0$

8d) Subst. gives "$\frac{0}{0}$" so L'H. rule can be applied.

$$\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$$

e) Subst. gives "$\infty - \infty$" which is indeterminate

Conjugating: $\left(\sqrt{x^2+5x} - \sqrt{x^2+7x}\right)\dfrac{\left(\sqrt{x^2+5x} + \sqrt{x^2+7x}\right)}{\sqrt{x^2+5x} + \sqrt{x^2+7x}}$

$$= \frac{-2x}{\sqrt{x^2+5x} + \sqrt{x^2+7x}} = \frac{-2}{\sqrt{1+5/x} + \sqrt{1+7/x}} \xrightarrow[x\to\infty]{} -1$$

f) $\lim_{x\to 0^+} (x^2+2x)^x$ Substitution results in "0^0".

$$\ln\left[(x^2+2x)^x\right] = x\ln(x^2+2x) = \frac{\ln(x^2+2x)}{\frac{1}{x}}$$

Now substitution results in "$\frac{-\infty}{\infty}$". Applying L'Hospital's rule:

$$\lim_{x\to 0^+} \frac{\ln(x^2+2x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{2x+2}{x^2+2x}}{-1/x^2} \overset{\text{do the algebra}}{=} \lim_{x\to 0^+} \frac{-(2x^3+2x^2)}{x^2+2x}$$

$$= \lim_{x\to 0^+} \frac{-(2x^2+2x)}{x+2} = 0$$

So $\lim_{x\to 0^+} (x^2+2x)^x = e^0 = 1$

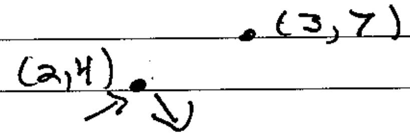
8g) $\lim_{x \to \infty} \left(\frac{1}{x}\right)^x$ Substitution results in "0^{∞}".

$\ln\left(\frac{1}{x}\right)^x = x \ln\left(\frac{1}{x}\right) = -x \ln x \underset{x \to \infty}{\Longrightarrow} -\infty$

So $\lim_{x \to \infty} \left(\frac{1}{x}\right)^x = 0$

$\left(= \lim_{x \to \infty} e^{-x \ln x} = 0 \right)$

9. If f is continuous then there must a pt. in the interval [2, 3] at which $f' = 0$ or f' DNE.



10. $\frac{f(2) - f(0)}{2 - 0} = \frac{12}{2} = 6 = f'(c)$ for some c in [0,2].

$f'(c) = 3c^2 + 2 = 6 \qquad 3c^2 = 4 \quad c = \pm \frac{2}{\sqrt{3}}$

$\boxed{c = \frac{2}{\sqrt{3}}}$

11. $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

C.V.'s $x = -1, x = 2$

a)

x	$f(x)$
-5	-265
-1	7
2	-20
7	455

The absolute max on $[-5, 7]$ is 455 at $x = 7$.

The abs. min is -265 at $x = -5$

11 b)

x	$f(x)$
0	0
2	-20
4	32

Abs. max on $[0,2]$ is 32 at $x=$

Abs. min is -20 at $x=2$

c)

x	$f(x)$
-3	-45
-1	7
0	0

Abs max on $[-3,0]$ is 7 at $x=-1$,

Abs min is -45 at $x=-3$

12. $f(x) = 2x^3 - 3x^2 - 12x$

$f'(x) = 6x^2 - 6x - 12 \quad = 6(x-2)(x+1)$

$f''(x) = 12x - 6 \quad = 12(x - \frac{1}{2})$

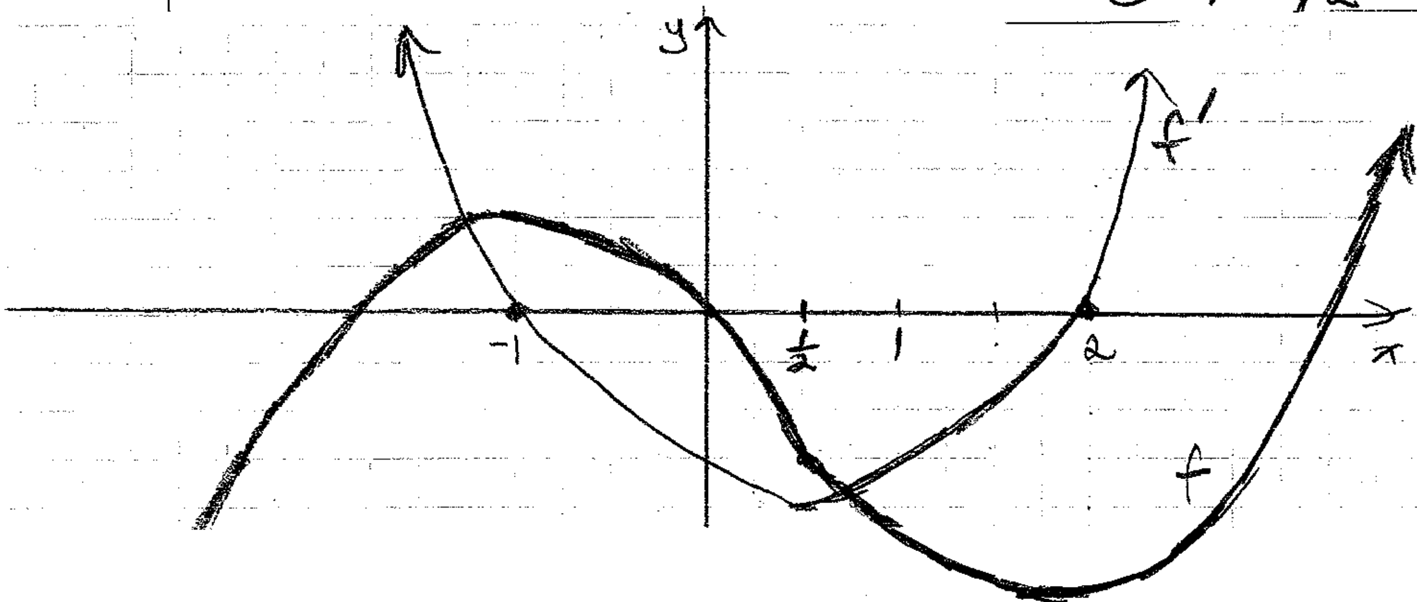
sign f': $+$ | -1 | $-$ | 2 | $+$ → x

f: $\nearrow$ $\searrow$ $\nearrow$

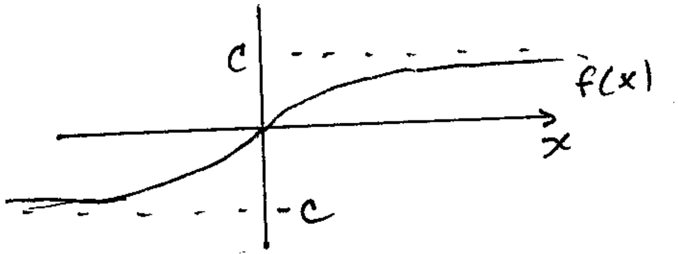
sign f'': $-$ | $\frac{1}{2}$ | $+$ → x

f

inflection @ $x = \frac{1}{2}$



13.



$f'(x) = e^{-x^2}$

$f''(x) = -2x e^{-x^2}$

sign f'' + | −

f ∪ 0 ∩

inflection at $x=0$

14. $f(t) = 400 - 400e^{-0.1t^2}$

$v(t) = f'(t) = -400e^{-0.1t^2}(-0.2t)$

$= 80te^{-0.1t^2}$

If $v(t)$ has a max, $a(t)$ must be 0. (since $a(t)$ exist)

$a(t) = 80e^{-0.1t^2} + 80te^{-0.1t^2}(-0.2t)$

$= 80e^{-0.1t^2}(1 - 0.2t^2)$ is 0 if and

only if $1 - 0.2t^2 = 0$ $t = \sqrt{\frac{1}{.2}} = \sqrt{5}$ hr

(since t cannot be negative)

$v(\sqrt{5}) = 80\sqrt{5}\,e^{-0.5} \approx 108.5$ mph

15.

$V = \pi r^2 h$ Since $\frac{dr}{dt} = 2$ is constant, and

$r(0) = 20$, $r(t) = 20 + 2t$ cm t in minutes.

Since $\frac{dh}{dt} = -3$ and $h(0) = 30$,

$h(t) = 30 - 3t$

Then

$V = \pi(20+2t)^2(30-3t)$ so

$V'(t) = \pi \cdot 4(20+2t)(30-3t) - 3\pi(20+2t)^2$

$= (20+2t)(3\pi)[4(10-t) - (20+2t)]$

$40 - 4t - 20 - 2t$

$20 - 6t$

$V' = 0$ at $t = \frac{10}{3}$ min (ignoring $t = -10$)

sign V' $\quad + \quad \frac{10}{3} \quad - \quad \rightarrow$ $\quad$ V has a max. at $t = \frac{10}{3}$ min.

$V \quad \nearrow \quad \searrow$

$V\left(\frac{10}{3}\right) = \frac{128\pi}{9} \times 10^3$ cubic cm.

16. At $x=1$, $f'(1) \neq 0$ so the 2nd derivative test does not apply.

At $x=2$, $f'(2)=0$, $f''(2)>0$ so f has a local min.

At $x=3$, $f'(3)=0$, $f''(3)=0$ no conclusión can be made.

At $x=4$, $f'(4)=0$, $f''(4)<0$, f has a local max,

17. a) $f(x) = x^3 e^x \qquad f(0)=0 \qquad \lim_{x\to-\infty} f(x) = 0 \qquad \lim_{x\to\infty} f(x) = \infty$

$f'(x) = 3x^2e^x + x^3e^x$
$\quad = x^2e^x(3+x)$

sign f' $\quad - \quad -3 \quad + \quad 0 \quad + \quad \rightarrow x$

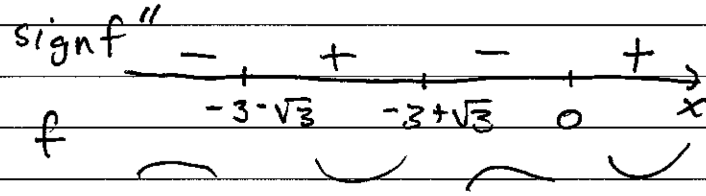
$f \quad \searrow \quad \nearrow \quad \nearrow$

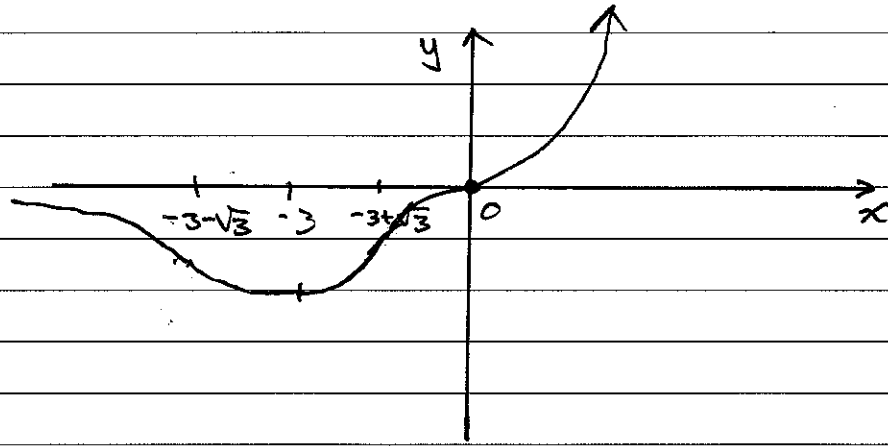
local min @ $x=-3$

$f''(x) = 6xe^x + 3x^2e^x + 3x^2e^x + x^3e^x$
$\quad =$
$\quad = xe^x(6 + 6x + x^2)$

f'' is 0 at $-3-\sqrt{3}$, $-3+\sqrt{3}$ and 0

Since f'' changes sign at each of its roots, f has 3 inflection pts.

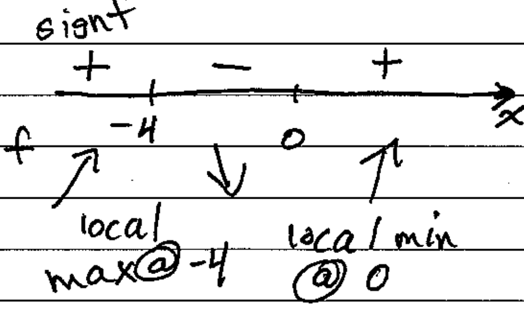




17 b) $f(x) = x^4 e^x \quad f(0)=0 \quad \lim_{x\to-\infty} f(x)=0 \quad \lim_{x\to+\infty} f(x) = \infty$

$f'(x) = (4x^3 + x^4)e^x$

$= x^3 e^x (4+x)$



$f''(x) = (12x^2 + 8x^3 + x^4)e^x = x^2 e^x (12 + 8x + x^2)$

is 0 at −6, −2 and 0 but does not change sign @ $x=0$, −6 and −2 are the only inflections pts.