Math 151 Exam 1 B  Summer 2014

Neatly Print Name  Key

"An Aggie does not lie, cheat, or steal or tolerate those who do"

"On my honor, as an Aggie, I have neither given nor received unauthorized aid on this academic work."

Your signature ________________________________

There are 9 work out questions. Show all your work. Attempt each problem as far as you can go with it. Partial credit will be given.

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1. For problem 1, use the points P(0, 4) and Q(2, 9).

a) Find a vector equation of the line through P and Q.
\[ \vec{v} = \vec{PQ} = <2-0, 9-4> = <2, 5> \]
\[ \vec{r}(t) = \vec{OP} + t\vec{v} = <0, 4> + t<2, 5> \]
or \[ <0 + 2t, 4 + 5t> \]

b) Find the work done when the force vector \( \vec{F} = 3\hat{i} + 2\hat{j} \) is applied along the vector \( \vec{PQ} \).
\[ W = \vec{F} \cdot \vec{PQ} = (3\hat{i} + 2\hat{j}) \cdot (2\hat{i} + 5\hat{j}) \]
\[ = 6 + 10 = 16 \] assuming appropriate units

c) Find the scalar component of \( \vec{F} = 3\hat{i} + 2\hat{j} \) in the direction of \( \vec{PQ} \).
\[ \frac{\vec{F} \cdot \vec{PQ}}{||\vec{PQ}||} = \frac{16}{\sqrt{2^2 + 5^2}} = \frac{16}{\sqrt{29}} \]

d) Find two vectors orthogonal to \( \vec{PQ} \) that have length 5.
\[ \frac{\vec{PQ}}{||\vec{PQ}||} = \frac{<2, 5>}{\sqrt{29}} \] and its negative \( \frac{-<2, 5>}{\sqrt{29}} \) have length 1.
\[ \frac{5\vec{PQ}}{||\vec{PQ}||} = \frac{<25, 10>}{\sqrt{29}} \] and its negative \( \frac{<25, -10>}{\sqrt{29}} \)
2. Two lines have vector equations \( \vec{n}(t) = t \vec{i} + (6+t) \vec{j}, \quad \vec{n}(s) = -2s \vec{i} + (-3+4s) \vec{j} \).

a) Find the intersection point.

\[
\begin{align*}
\vec{x}_1(t) &= \vec{x}_2(s) \\
\vec{y}_1(t) &= \vec{y}_2(s) \\
t &= -2s \\
6 + t &= -3 + 4s
\end{align*}
\]

Substituting

\[
\begin{align*}
6 - 2s &= -3 + 4s \\
9 &= 6s \\
3 &= s \\
t &= -3
\end{align*}
\]

\( \vec{r}_1(-3) = \langle -3, 3 \rangle \) so the intersection point is \( P(-3, 3) \).

b) Find the cosine of the smaller angle between the lines.

\[
\begin{align*}
\vec{v}_1 \parallel l_1 & \implies \vec{v}_1 = \langle 1, 1 \rangle \\
\vec{v}_2 \parallel l_2 & \implies \vec{v}_2 = \langle -2, 4 \rangle \\
\cos \theta &= \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{-2 + 4}{\sqrt{2} \sqrt{20}} = \frac{2}{\sqrt{40}} \quad \text{or} \quad \frac{1}{\sqrt{10}}
\end{align*}
\]

\( \theta \) is positive so this is it.
12 pts

3. \( f(x) = \begin{cases} \frac{3x^2 - 12}{x^2 - x - 6} & x \neq 3, \ x \neq -2 \\ 2 & x = 3, \ x = -2 \end{cases} \)

\[ \frac{3(x-2)(x+2)}{(x-3)(x+2)} \]

Evaluate each limit or write DNE if the limit does not exist.

a) \( \lim_{x \to -2} f(x) = \frac{3(-2 - 2)}{-2 - 3} = \frac{-12}{-5} = \frac{12}{5} \)

b) \( \lim_{x \to 3} f(x) \quad \text{DNE} \quad \vee \text{A. } x = 3 \)

c) \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x^2 - 12}{x^2 - x - 6} = 3 \)
4. Evaluate \( \lim_{x \to \infty} \left( \frac{\sqrt{4x^2 + 10x} - 2x}{\sqrt{4x^2 + 10x} + 2x} \right) \)

\[
= \frac{1}{x} \frac{10x}{(\sqrt{4x^2 + 10x} + 2x)} = \frac{10}{\sqrt{4x + 10} + 2} \quad x \to \infty \quad 2 + 2 = \frac{15}{2}
\]

5. \( f(x) = (x-1)^{5/3} + 2(x-1)^{2/3} \)

a) Find \( f'(x) \) using the power rule, not the limit definition.

\[
\frac{5}{3} (x-1)^{2/3} + \frac{2}{3} (x-1)^{-1/3}
\]

b) Find an equation of the tangent line to the graph of \( f(x) \) at \( (9, f(9)) \).

Find the pt. on the graph. \( f(9) = 8^{5/3} + 2(8^{2/3}) \)

\[ f(9) = \frac{5}{3} (8^{2/3}) + \frac{4}{3} (8^{-1/3}) = \frac{5}{3} (2^2) + \frac{4}{3} \left( \frac{1}{2} \right) = \frac{20}{3} + \frac{2}{3} \]

Pt. Slope: \( y = \frac{22}{3} (x-9) + 40 \)
6. a) Find the derivative of \( f(x) = \frac{1}{x^2} \) using only the limit definition of the derivative.

The difference quotient is

\[
\frac{f(x+h) - f(x)}{h} = \left( \frac{1}{(x+h)^2} - \frac{1}{x^2} \right) \frac{1}{h} = \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}
\]

\[
= \frac{x^2 - (x^2 + 2xh + h^2)}{hx^2(x+h)^2} = \frac{-2xh - h^2}{hx^2(x+h)^2} = \frac{-2x - h}{x^2(x+h)^2}
\]

\[
\lim_{h \to 0} \frac{-2x - h}{x^2(x+h)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3} \quad \text{or} \quad \frac{-2}{x^3}
\]

b) Evaluate the limit by recognizing it as the derivative of a familiar function.

\[
\lim_{h \to 0} \frac{3(x+h)^{1/4} - 4(x+h)^2 - 3x^{1/4} + 4x^2}{h}
\]

Inside the limit is the diff. quot. for \( f(x) = 3x^{1/4} - 4x^2 \).

The limit is \( f'(x) = \frac{3}{4} x^{-3/4} - 8x \).
12 pts 7. \( f(x) = \begin{cases} 
10 & x = -2, x = -3 \\
\frac{x(x^2 - 9)}{(x + 2)(x + 3)} & x \leq 0, x \neq -3, x \neq -2 \\
x \cos \left( \frac{1}{x} \right) & 0 < x 
\end{cases} \)

For each value of \( a \), determine if the function above is continuous at \( x = a \). Show why or why not? You may give either a graphical reason or a reason using the definition of continuity.

i) \( a = -3 \) when \( x \) is near \(-3\), \( f(x) = \frac{x(x-3)(x+3)}{(x+2)(x+3)} \neq -3 \)

\[
\lim_{x \to -3} f(x) = \frac{-3(-3-3)}{-3+2} = \frac{18}{-1} = -18
\]

but \( f(-3) = 10 \) so \( f \) is not continuous at \( x = -3 \).

ii) \( a = -2 \) \( f \) has V.A. \( x = -2 \).

\[
\lim_{x \to -2} f(x) \text{ DNE}
\]

so \( f \) is not continuous at \( x = -2 \).

iii) \( a = 0 \) \( \lim_{x \to 0} f(x) = \frac{0(0-3)}{0+2} = 0 \) and \( f(0) = 0 \)

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x \cos \left( \frac{1}{x} \right) = 0 \text{ by the squeeze thm and}
\]

\( -1 \leq \cos \left( \frac{1}{x} \right) \leq 1 \)

so \( f \) is continuous at \( x = 0 \).
12 pts 8. a) Find $\frac{d}{dx} \left( \frac{5}{\sqrt{x}} \right)$ using the power rule.

$$f(x) = \frac{5}{\sqrt{x}} = 5x^{-\frac{1}{2}} \quad \Rightarrow \quad f'(x) = -\frac{5}{2}x^{-\frac{3}{2}}$$

b) Find $\frac{d}{dx} (7x^2 - 2x)$ using the power rule and the linear property.

$$f'(x) = 14x - 2$$

c) Is the function above continuous at $x = 1$?

$$\lim_{{x \to 1^-}} \frac{5}{\sqrt{x}} = 5 = \lim_{{x \to 1^+}} f(x)$$

so $\boxed{\text{yes}}$

d) Is $f(x)$ differentiable at $x = 1$? Why or why not?

From a), the slope on the left is $\frac{-5}{2}(1^{-\frac{3}{2}}) = -\frac{5}{2}$

but the slope on the right is $14 - 2 = 12$

so $f$ has a corner at $x = 1$ and $f$ is not differentiable at $x = 1$. 

9. The path of a traveling object is given by \( \vec{r}(t) = (3t^2 - t) \hat{i} + t^3 \hat{j}, \ t \geq 0 \).

a) Find the average velocity vector for the time interval \( t = 1 \) to \( t = 3 \).

\[
\frac{\vec{r}(3) - \vec{r}(1)}{3 - 1} = \frac{\langle 24, 27 \rangle - \langle 2, 1 \rangle}{2} = \langle 11, 13 \rangle
\]

b) Find the instantaneous velocity vector at time \( t = 2 \).

\[
\vec{r}'(t) = (6t - 1) \hat{i} + 3t^2 \hat{j}
\]

\[
\vec{r}'(2) = 11 \hat{i} + 12 \hat{j}
\]

c) Find any equation of the tangent line to the path of motion at \( t = 2 \).

\[
\vec{r}(t) = \vec{r}(2) + t (11 \hat{i} + 12 \hat{j})
\]

\[
\vec{r}(2) = \langle 12 - 2, 8 \rangle = \langle 10, 8 \rangle
\]

so \( \vec{r}(t) = 10 \hat{i} + 8 \hat{j} + t (11 \hat{i} + 12 \hat{j}) \) or \( (10 + 11t) \hat{i} + (8 + 12t) \hat{j} \) or \( \langle 10, 8 \rangle + t \langle 11, 12 \rangle \). For the slope-intercept form \( y = mx + b \), \( m = \frac{12}{11} \).