151 WIR 3.4 – 3.9 Solutions

3.4

1. a) $\lim\limits_{x\to 0} \dfrac{2x^2}{\sin^2 x} = 2\lim\limits_{x\to 0}\left(\dfrac{x}{\sin x}\right)^2 = 2$

b) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{\sin 2x \sin 3x} = \lim\limits_{x\to 0}\left(\dfrac{\sin^2 x}{x^2}\cdot\dfrac{2x}{\sin 2x}\cdot\dfrac{3x}{\sin 3x}\cdot\dfrac{1}{2\cdot 3}\right)$

$= \boxed{1/6}$ since $\dfrac{\sin^2 x}{x^2} \xrightarrow[x\to 0]{} 1$

$\dfrac{2x}{\sin 2x} \xrightarrow[x\to 0]{} 1$

and $\dfrac{3x}{\sin 3x} \xrightarrow[x\to 0]{} 1$

2. c) $\lim\limits_{x\to 0}\left(\dfrac{\tan 4x}{4x}\cdot\dfrac{5x}{\tan 5x}\cdot\dfrac{4}{5}\right)$

$= \dfrac{4}{5}$ since $\dfrac{\tan u}{u} = \dfrac{\sin u}{u}\cdot\cos u \xrightarrow[u\to 0]{} 1$

2. a) $(\sec x \tan x)' = \sec x \tan^2 x + \sec^3 x$

$f'(\frac{\pi}{3}) = 2\cdot(\sqrt{3})^2 + 2^3 = 14$

$f(\pi/3) = \sec \pi/3 \tan \pi/3 = 2\sqrt{3}$

Tangent line: $y = 14(x - \frac{\pi}{3}) + 2\sqrt{3}$

b) $(\sin x - \cos x)' = \cos x + \sin x$

$f'(\pi/2) = 1$

$f(\pi/2) = 1 \qquad y = (x - \pi/2) + 1$

or $\quad y = x - \pi/2 + 1$

c) $(\tan x)' = \sec^2 x$

$f'(\pi/6) = \left(\frac{2}{\sqrt{3}}\right)^2 = 4/3$

$f(\pi/6) = \frac{1}{\sqrt{3}} \qquad y = \frac{4}{3}\left(x - \frac{\pi}{6}\right) + \frac{1}{\sqrt{3}}$

3.5

3 a) $f'(x) = \frac{1}{2}(\sin^2 x + 1)^{-1/2}(\sin^2 x + 1)'$

$= \frac{1}{2}(\sin^2 x + 1)^{-1/2}(2 \sin x \cos x)$

$= \dfrac{\sin x \cos x}{\sqrt{\sin^2 x + 1}}$

b) $f'(x) = 6x \sec^2(3x^2)$

c) $f'(x) = 3(\sin^2 x + 1)^2 (2 \sin x \cos x)$

$= 6(\sin^2 x + 1)^2 \sin x \cos x$

4. By the chain Rule $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$.

$f(u) = u^2+5$ so $\frac{df}{du} = 2u$

We know $\left.\frac{du}{dx}\right|_{x=1} = u'(1) = -2$

Then $\left.\left(\frac{df}{du}\frac{du}{dx}\right)\right|_{x=1} = 2u(1)(-2) = 2(3)(-2)$

$= \boxed{-12}$

3.6

5. Treating x as a function of y and y as the independent variable:

$$2(x^2+y^2)(2xx' + 2y) = 2axx'y + ax^2$$

$$4x(x^2+y^2)x' + 4y(x^2+y^2) = 2axyx' + ax^2$$

$$[4x(x^2+y^2) - 2ax]x' = ax^2 - 4y(x^2+y^2)$$

$$x' = \frac{ax^2 - 4y(x^2+y^2)}{4x(x^2+y^2) - 2ax}$$

6. Verify (3,1) is on the curve.

Then differentiating with respect to x :

$$y^3 + 3xy^2 y' + y + xy' = 0$$

$$y + y^3 + (3xy^2 + x)y' = 0$$

$$y' = \frac{-y - y^3}{3xy^2 + x}$$

$x = 3$ and $y = 1$ so

$$y'(3) = \frac{-1-1}{9+3} = -1/6$$

tangent line: $y = -1/6(x-3) + 1$

$$x^2 + y^2 = 5x \qquad\qquad x^2 + y^2 = 10y$$

7. $2x + 2yy' = 5$ $\qquad$ $2x + 2yy' = 10y'$

$$m_1 = y' = \frac{5-2x}{2y} \qquad (2y-10)y' = -2x$$

$$m_2 = y' = \frac{-2x}{2y-10}$$

Find the intersection pts:

$$x^2 + y^2 = 5x = 10y$$

$$\text{so} \quad x = 2y$$

Substituting into $x^2+y^2 = 5x$

$$4y^2 + y^2 = 10y$$

$$5y^2 = 10y \qquad y=0 \text{ or } y=2$$

If $y=0$, $x=0$. If $y=2$, $x=4$. (Since $x=2y$)
$(0,0)$ and $(4,2)$.

At $(0,0)$: $m_1 = \dfrac{5-2x}{2y}$ has a nonzero numerator and a zero denominator.
The line is vertical.

$$m_2 = \left.\frac{-2x}{2y-10}\right|_{\substack{x=0\\y=0}} = 0$$ The line is horizontal.

At $(4,2)$: $m_1 = \dfrac{5-8}{4} = -3/4$ $\qquad$ $m_2 = \dfrac{-8}{-6} = 4/3$

shows the tangents are orthogonal.

3.7

8. $\vec{r}\,'(t) = (t\sin t)'\,\vec{i} + (t\cos t)'\,\vec{j}$

$= (\sin t + t\cos t)\,\vec{i} + (\cos t - t\sin t)\,\vec{j}$

9. a) Solve $\vec{r_1}(t) = \vec{r_2}(s)$

$1 - t = s - 2$ and $3 + t^2 = s^2$

$3 - t = s \longrightarrow 3 + t^2 = (3 - t)^2$

$3 + t^2 = 9 - 6t + t^2$

$-6 = -6t$

$1 = t$

and $s = 2$

Check:

$\vec{r_1}(1) = \langle 0, 4 \rangle \qquad \vec{r_2}(2) = \langle 0, 4 \rangle$ ✓

Find the ∡ between their tangent vectors:

$\vec{r_1}\,'(t) = \langle -1, 2t \rangle \qquad \vec{r_2}\,'(s) = \langle 1, 2s \rangle$

$\vec{r_1}\,'(1) = \langle -1, 2 \rangle \qquad \vec{r_2}\,'(2) = \langle 1, 4 \rangle$

$$\frac{\langle -1, 2 \rangle}{\sqrt{1+4}} \cdot \frac{\langle 1, 4 \rangle}{\sqrt{1+16}} = \frac{7}{\sqrt{85}}$$

$\theta = \arccos\left(\frac{7}{\sqrt{85}}\right)$

9b. $\vec{r}(t) = \langle t\cos t,\ t\sin t \rangle$

$\vec{r}\,'(t) = \langle \cos t - t\sin t,\ \sin t + t\cos t \rangle$

$\vec{r}\,'(\frac{\pi}{4}) = \langle \frac{\sqrt{2}}{2} - \frac{\pi}{4}\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2} + \frac{\pi}{4}\frac{\sqrt{2}}{2} \rangle$

$= \frac{\sqrt{2}}{2} \langle 1 - \pi/4,\ 1 + \frac{\pi}{4} \rangle$

A tangent vector of unit length is

$$\vec{u} = \frac{\langle 1 - \pi/4,\ 1 + \pi/4 \rangle}{\sqrt{(1-\pi/4)^2 + (1+\pi/4)^2}}$$

$$= \left\langle \frac{1 - \pi/4}{\sqrt{2 + \pi^2/16}},\ \frac{1 + \pi/4}{\sqrt{2 + \pi^2/16}} \right\rangle$$

$$= \left\langle \frac{4 - \pi}{\sqrt{32 + \pi^2}},\ \frac{4 + \pi}{\sqrt{32 + \pi^2}} \right\rangle$$

3.8

1a) $4 \overline{)\,83}$ $20 \text{ r} 3$ so $\frac{d^{83}}{dx^{83}} \sin x = \frac{d^3}{dx^3} \sin x$

$= -\sin x$

$4 \overline{)\,125}$ $31 \text{ r} 1$ so $\frac{d^{125} \cos x}{dx^{125}} = \frac{d}{dx} \cos x = -\sin x$

3.8

2. $\vec{r}'(t) = \vec{v}(t) = \langle \cos t - t\sin t, \sin t + t\cos t \rangle$

$\vec{r}''(t) = \vec{a}(t) = \langle -\sin t - \sin t - t\cos t,$

$\cos t + \cos t - t\sin t \rangle$

$\vec{a}(t) = \langle -2\sin t - t\cos t, 2\cos t - t\sin t \rangle$

3. 9

1. A parallel vector is $\vec{v} = \langle -1, 3 \rangle$.

A normal vector is $\vec{v}^{\perp} = \langle 3, 1 \rangle$ and

$$\frac{\vec{v}^{\perp}}{|\vec{v}^{\perp}|} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$$

2 a) $x' = 3t^2 - 6t$ $\quad y' = 3t^2 - 3$ We have a horizontal tangent when $y' = 0$ and $x' \neq 0$.

so $3t^2 - 3 = 0$ $\quad t = 1$ or $t = -1$

and $x'(1) \neq 0$, $x'(-1) \neq 0$.

$(x(1), y(1)) = (-2, -2)$ $\quad (x(-1), y(-1)) = (-4, 2)$

We have vertical tangents where $x' = 0$, $y' \neq 0$:

so $3t^2 - 6t = 0$

$t = 0$ or $t = 2$ ay $y'(0) \neq 0$ $\quad y'(2) \neq 0$

$(x(0), y(0)) = (0, 0)$ and $(x(2), y(2)) = (-4, 2)$

The curve passes through $(-4, 2)$ at $t=-1$ with a horizontal tangent and also at $t=2$ with a vertical tangent.

3.9

2b) $x'(t) = a(-\sin t + 2\cos t \sin t) = a(-\sin t + \sin 2t)$

$y'(t) = a(\cos t - \cos^2 t + \sin^2 t) = a(\cos t - \cos 2t)$

To solve $y'(t) = 0$:

One way is to solve $\cos t - \cos^2 t + 1 - \cos^2 t$

$= -2\cos^2 t + \cos t + 1$

as a quadratic in $\cos t$.

Another way is to solve $\cos t = \cos 2t$

$t = 2k\pi$ for k any whole number or

$t = \pm\frac{2\pi}{3} + 2k\pi$ (The solutions in $[0, 2\pi)$ are $0, 2\pi/3, 4\pi/3$)

At $t = 2k\pi$, $x'(t)$ is also 0 $\left[\lim_{t\to 0} \frac{y'}{x'} = 0 \text{ can be shown} \right]$

At $\frac{2\pi}{3}$, $x' \neq 0$.

At $\frac{4\pi}{3}$, $x' \neq 0$.